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Research Article

A Banach Algebraic Approach to the Borsuk-Ulam Theorem

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Using methods from the theory of commutative graded Banach algebras, we obtain a generalization of the two-dimensional Borsuk-Ulam theorem as follows. Let $\phi : S^2 \to S^2$ be a homeomorphism of order n, and let $\lambda \neq 1$ be an nth root of the unity, then, for every complex valued continuous function f on S^2, the function $\sum_{i=0}^{n-1} \lambda^i f(\phi^i(x))$ must vanish at some point of S^2. We also discuss some noncommutative versions of the Borsuk-Ulam theorem.

"To my Children Tarannom and Pouya"

1. Introduction

The classical Borsuk-Ulam theorem states that for every continuous function $g : S^n \to \mathbb{R}^n$, there always exist a point $x \in S^n$ such that $g(-x) = g(x)$. If we define $f(x) = g(x) - g(-x)$, we obtain an equivalent statement as follows. For every odd continuous function $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $f(x) = 0$.

We consider the case $n = 2$ and identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$. Let $A = C(S^2)$ be the Banach algebra of all continuous complex-valued functions on S^2 with the Z_2-graded structure:

$$A = A_{\text{ev}} \bigoplus A_{\text{odd}}, \tag{1.1}$$

where A_{ev} is the space of all even functions and A_{odd} is the space of all odd functions, and the decomposition is the standard decomposition of functions to even and odd functions.

Then the two-dimensional Borsuk-Ulam theorem says that a homogenous element of A with nonzero degree, namely, an odd function, is not invertible.

In this paper we are mainly interested in invertible elements of a graded unital Banach algebra which are homogenous of nontrivial degree. Some natural questions about such elements are as follows. If any such element is invertible, can it be connected to the identity in the space of invertible elements? What can be said about the relative position of their spectrum with respect to the origin?

As we will see in the main theorem of this paper, for a commutative Banach algebra without nontrivial idempotent, which is graded by a finite Abelian group, a nontrivial homogenous element cannot be connected to the identity. On the other hand, using an standard lifting lemma in the theory of covering spaces, we conclude that an invertible element of $C(S^2)$ has a logarithm, then it lies in the same connected component of the identity. This shows that an odd element of $C(S^2)$ cannot be invertible, so it would give us a Banach algebraic proof of the Borsuk-Ulam theorem, in dimension two. The purpose of this paper is to translate the classical Borsuk-Ulam theorem into the language of noncommutative geometry.

We also give a concrete example of an S_3-graded structure for $C^*_{\mathrm{red}}(F_2)$, the reduced C^* algebra of the free group on two generators, such that a nontrivial homogenous element lies in the same component of the identity. This would show that the commutativity of grading group and graded algebra are necessary conditions in our main theorem. Finally we give a question as a weak version of the Kaplansky-Kadison conjecture. This question naturally arises from our main result; see Question 5 at the end of the paper.

2. Preliminaries

Let A be a unital complex Banach algebra, and let G be a finite group with neutral element e. A G-graded structure for A is a decomposition

$$A = \bigoplus_{g\in G} A_g, \tag{2.1}$$

where each A_g is a Banach subspace of A and $A_gA_h \subseteq A_{gh}$. An element $a \in A_g, g \in G$ is called a homogenous element; it is called nontrivial homogenous if $a \in A_g$ where $g \neq e$. When $G = Z_2$ and $A = A_0 \bigoplus A_1$, an element of A_1 is called an odd element. A morphism $\alpha : A \to A$ is called a graded morphism provided that $\alpha(A_g) \subseteq A_g$ for all $g \in G$. Let A be a G-graded Banach algebra, and let H be a normal subgroup of G. Then there is an induced G/H-graded structure for A with

$$A_{g/H} = \bigoplus_{h\in H} A_{gh}. \tag{2.2}$$

The following proposition is used in the proof of our main theorem.

Proposition 2.1. *Let G be Abelian, and let a be a nontrivial homogenous element of a G-graded Banach algebra, then for some positive integer n, there is a Z_n-graded structure for A such that a is a nontrivial homogenous element of A as a Z_n-graded algebra.*

Proof. This can be proved by induction on order of G as follows. The first step of the induction is obvious since the only group of order 2 is Z_2. Let G be a finite Abelian group and $g \in G$ where $g \neq e$. If G is not a cyclic group, then there is a subgroup H which does not contain g, so a is a nontrivial homogenous element of the induced G/H-graded structure for A. Now the order of G/H is strictly less than the order of G. So an induction argument on order of G completes the proof. □

Note that the existence of a Z_n-graded structure for a Banach algebra A is equivalent to existence of a bounded multiplicative operator $T : A \to A$ with $T^n = Id$. For any such operator, we choose a root of unity $\lambda \neq 1$ and observe that the decomposition

$$A = \bigoplus_{i=0}^{n-1} \ker\left(T - \lambda^i\right) \tag{2.3}$$

is a Z_n-graded structure. Conversely for the grading $A = \bigoplus_{i=0}^{n-1} A_i$, the following multiplicative operator T satisfies $T^n = Id$:

$$T\left(\sum_{i=0}^{n-1} a_i\right) = \sum_{i=0}^{n-1} \lambda^i a_i. \tag{2.4}$$

Example 2.2. Let X be a compact topological space and ϕ be an order-n homeomorphism of X, that is $\phi^n = Id$. Define $T : C(X) \to C(X)$ with $T(f) = f \circ \phi$, where $C(X)$ is the space of all continuous functions on X. Then T is a bounded multiplicative operator on Banach algebra $A = C(X)$ with $T^n = Id$, so we would have a Z_n-graded structure for A. As a particular example, the rotation of circle by $2\pi/n$ is an order n homeomorphism of circle. Then we naturally obtain a Z_n-graded structure for $C(S^1)$.

For a group G, a Z_n-partition for G is a partition of G into disjoint subsets G_i, $i = 0, 1, \ldots n-1$ such that $G_i G_j \subseteq G_{i+j \pmod n}$. This is equivalent to say that G_0 is a normal subgroup of G whose quotient is isomorphic to Z_n. This is an obvious consequence of the normal property for subgroups; see [1]. So every group G which has a normal subgroup H such that G/H is isomorphic to Z_n posses a Z_n-graded structure. For example, for every group K, the group $G = K \times Z_n$ has $K \times \{0\}$ as a normal subgroup, and $G/K \times \{0\}$ is isomorphic to Z_n. In this case the Z_n partition for G is $G = \coprod_{i \in Z_n} K \times \{i\}$.

For a discrete group G, we denote by $\mathbb{C}G$ the group algebra of G with complex coefficients, namely, the space of all linear combinations $\sum a_g g$ where a_g's are complex numbers. let $l^2(G)$ be the Hilbert space of all $\gamma : G \to \mathbb{C}$ such that $|\gamma|_2^2 = \sum_{g \in G} |\gamma(g)|^2 < \infty$. G acts on $l^2(G)$ with $g \cdot \gamma(h) = \gamma(g^{-1}h)$. This action defines a unitary representation of G on $l^2(G)$. We extend this action by linearity to $\mathbb{C}G$. So each element of $\mathbb{C}G$ can be considered as an element of $B(l^2(G))$, the space of all bounded operators on $l^2(G)$, and every element of G can be considered as a unitary operator on $l^2(G)$. The reduced C^* algebra of G, $C^*_{\text{red}}(G)$, is the closure of $\mathbb{C}G$, with respect to operator norm $|\cdot|_{\text{op}}$ defined on $B(l^2(G))$.

In Proposition 2.1, we shall prove that a Z_n-partition for a Group G gives a Z_n-graded structure for $C^*_{\text{red}}(G)$. For the proof of the Proposition 2.1, we need to the following technical lemma.

Lemma 2.3. *Let $(E, |\cdot|)$ be a Banach space with a dense linear subspace F. Assume $F = \bigoplus_{i=0}^{n} F_i$ is a direct sum of its linear subspaces F_i's. Suppose that the original norm of F is equivalent to the direct limit norm $|\sum f_i| = \sum |f_i|$. Then $E = \bigoplus_{i=0}^{n} \overline{F_i}$.*

Proof of Lemma. Let $\sum_{i=0}^{n} \widetilde{f}^i = 0$ where $\widetilde{f}^i$ is in $\overline{F_i}$, the closure of F_i. Then there are sequences $\{f_k^i\}_{k\in\mathbb{N}}$ of elements of F_i which converges to $\widetilde{f}^i$. So $\lim_{k\to\infty} |\sum_{i=0}^{n} f_k^i| = 0$. Since the original norm of F is equivalent to the direct sum norm, we conclude that each sequence $\{f_k^i\}$ converges to zero. So $\widetilde{f}^i = 0$ for all i. From continuity of norm, we have the equivalency of norm and direct sum norm on the space $\bigoplus_{i=0}^{n} \overline{F_i}$. So this direct sum is a topological direct sum. let $x \in E$ be given, there is a sequence $\sum_{i=0}^{n} f_i^k$, $k \in \mathbb{N}$, which converges to x. This shows that each sequence $\{f_i^k\}$ is a cauchy sequence which converges to an element $\widetilde{f}_i \in \overline{F_i}$. So $x = \sum_{i=0}^{n} \widetilde{f}_i$. This completes the proof of lemma. □

Proposition 2.4. *A Z_n-partition structure for a group $G = \coprod_{i=0}^{n-1} G_i$ gives a Z_n-graded structure for $C^*_{red}(G) = \bigoplus_{i\in Z_n} C^*_{red}(G_i)$.*

Proof. Assume that $G = \coprod_{i\in Z_n} G_i$ is a Z_n-partition for G, then $\mathbb{C}G = \bigoplus \mathbb{C}G_i$ and $l^2(G) = \bigoplus l^2(G_i)$ where

$$l^2(G_i) = \left\{\gamma \in l^2(G) : \gamma|_{G_j} = 0,\ j \neq i\right\}. \tag{2.5}$$

For $x_i \in \mathbb{C}G_i$ and $\gamma_j \in l^2(G_j)$, we have $x_i \cdot \gamma_j$ belongs to $l^2(G_{j+i \pmod n})$. Assume $x = \sum_{i=0}^{n-1} x_i$ and $\gamma = \sum_{i=0}^{n-1} \gamma_i$. Then

$$\left|x \cdot \gamma_j\right|_2^2 = \sum_{i=0}^{n-1} \left|x_i.\gamma_j\right|_2^2, \qquad \left|x_i \cdot \gamma\right|_2^2 = \sum_{j=0}^{n-1} \left|x_i \cdot \gamma_j\right|_2^2. \tag{2.6}$$

Let $x_i \in \mathbb{C}G_i$ and $\varepsilon > 0$ be given. We apply the definition of norm operator on operator x_i, then we conclude that there is a $\gamma = \sum_{i=0}^{n-1} \gamma_i$ with $|\gamma|_2^2 = \sum_{i=0}^{n} |\gamma_i|_2^2 = 1$ such that

$$\sum_{j=0}^{n-1} \left|x_i \cdot \gamma_j\right|_2^2 = \left|x_i \cdot \gamma\right|_2^2 \geq \left(|x_i|_{\mathrm{op}} - \varepsilon\right)^2. \tag{2.7}$$

This shows that there is a $j \in Z_n$ such that

$$\frac{\left(|x_i|_{\mathrm{op}} - \varepsilon\right)^2}{n} \leq \left|x_i \cdot \gamma_j\right|_2^2 \leq \left|x \cdot \gamma_j\right|^2 \leq |x|_{\mathrm{op}}^2. \tag{2.8}$$

Since $|\gamma_j|_2 \leq 1$, so $|x|_{\mathrm{op}} \geq |x_i|_{\mathrm{op}}/\sqrt{n}$, for $i = 0, 1, \ldots n-1$. We conclude that the operator norm on $\mathbb{C}G$ is equivalent to direct sum norm $|\sum x_i|_{\mathrm{op}} = \sum |x_i|_{\mathrm{op}}$. Now if we put $E = C^*_{\mathrm{red}}(G)$, $F = \mathbb{C}G$, $F_i = \mathbb{C}G_i$ and apply the above lemma to these spaces, the proof of the proposition would be completed. □

Remark 2.5. Let G and H be two groups where H is finite. Similar to above we can define an H-partition for G. It is a decomposition $G = \coprod_{h\in H} G_h$ such that $G_{h_1}G_{h_2} \subseteq G_{h_1h_2}$.This is equivalent to say that G_e is a normal subgroup of G whose quotient is isomorphic to H. In the same manner as above we can prove that an H-partition for a group G, gives an H-graded structure for $C^*_{\mathrm{red}}(G)$. Moreover it is clear from the definition that an element $g \in G$ which is not in G_e, can be considered as a nontrivial homogenous element of H-graded algebra $C^*_{\mathrm{red}}(G)$.

A part of the philosophy of noncommutative geometry is to translate the classical facts about compact topological spaces into language of (noncommutative) Banach or C^* algebras; see [2] or [3]. According to the Gelfand-Naimark theorem, there is a natural contravariant functor from the category of compact Hausdorff topological space to the category of unital complex C^* algebras. This functor assigns to X the commutative C^* algebra $C(X)$ of all complex-valued continuous functions on X. It also assigns to a continuous function $f : X \to Y$, the C^* algebra morphism $f^* : C(Y) \to C(X)$ by $f^*(\phi) = \phi \circ f$. Conversely every unital commutative C^* algebra A is isomorphic to $C(X)$ for some compact topological space X, and every morphism from $C(Y)$ to $C(X)$ is equal to f^* for some continuous function $f : X \to Y$. In particular constant maps from X to Y correspond to morphism from $C(Y)$ to $C(X)$ with one-dimensional range. Most of the statements about topological spaces have an algebraic analogy in the world of C^* algebras. For example it can be easily shown that a topological space X is connected if and only if the algebra $C(X)$ has no nontrivial idempotent, where a nontrivial idempotent a in an algebra A is an element $a \in A$ such that $a^2 = a$ and $a \neq 0, 1$. So nonexistence of nontrivial idempotent for a (noncommutative) Banach algebra A is interpreted as "noncommutative connectedness."

For a topological space X, considering the above functor, it is natural to identify $C(I \times X)$, with $C(I, C(X))$, where I is the unit interval. So in order to obtain a homotopy theory, for a Banach algebra A, we define the Banach algebra AI as follows.

Definition 2.6. Let A be a Banach algebra, we denote by AI, the Banach algebra of all continuous $\gamma : I \to A$ with the standard operations and norm. We define $\pi_i; AI \to A$ with $\pi_i(\gamma) = \gamma(i)$, for $i = 0, 1$.

We say that two morphisms $\alpha, \beta : A \to B$ are homotopic if there is a morphism $\Theta : A \to BI$ such that $\Theta_0 = \alpha$, $\Theta_1 = \beta$, where $\Theta_i = \pi_i \circ \Theta$. A morphism $\alpha : A \to B$ is called null homotopic if it is homotopic to a morphism with one-dimensional range. Obviously this is a natural Banach algebraic analogy of classical null homotopicity.

For a unital Banach algebra A with unit element 1, the element $\lambda.1$ is simply shown by λ where $\lambda \in \mathbb{C}$ is an scalar. For an element $a \in A$, we denote $\mathrm{sp}(a)$ for all $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible. The spectral radius of a, which is denoted by $\mathrm{spr}(a)$, is defined as the infimum of all r such that the disc of radius r around the origin contains $\mathrm{sp}(a)$.

3. Main Result

In the next theorem, which is the main result of this paper, k is an arbitrary positive integer, G is a finite Abelian group, and A is a unital complex Banach algebra.

Main Theorem 1. Let A be a G-graded Banach algebra with no nontrivial idempotents. Let $a \in A$ be a nontrivial homogenous element. Then 0 belongs to the convex hull of the spectrum $\mathrm{sp}(a^k)$.

Further, if A is commutative and a is invertible, then a^k and 1 do not lie in the same connected component of the space of invertible elements $G(A)$.

Proof. Without lose of generality we assume that $G = Z_n$. So we have a multiplicative operator $T : A \to A$ with $T^n = Id$. Let a be a nontrivial homogenous element so $T(a) = \lambda a$ where $\lambda \neq 1$ is a root of the unity. Assume for contrary that the convex hull of $\mathrm{sp}(a^k)$ does not contain 0. Then $\mathrm{sp}(a^k)$ can be included in a disc with center z_0, for some $z_0 \in \mathbb{C}$, which is a subset of a branch of logarithm. Using holomorphic functional calculus as described in [4, Chapter 10], we can find a convergent series $b = \sum_{i=0}^{\infty} c_i (a^k - z_0)^i$ where $\exp b = a^k$ where c_i's are complex numbers.

So $T(b) = \sum_{i=0}^{\infty} c_i (\lambda^k a^k - z_0)^i$. Thus $T(b)$ and b are power series in a. In particular $ab = ba$ and $T^i(b)T^j(b) = T^j(b)T^i(b)$, for all i and j. So we have $(a^{-1}\exp(b/k))^k = 1$. Then $\mathrm{sp}(a^{-1}\exp(b/k))$ is a subset of the set of all kth roots of unity. On the other hand, since A has no nontrivial idempotent, the spectrum of each element must be connected. So we can assume that $\mathrm{sp}(a^{-1}\exp(b/k)) = \{1\}$; otherwise we multiply a with an appropriate root of unity. Put $q = a^{-1}\exp(b/k) - 1$. Then q is a quasinilpotent element of A; that is, its spectral radius $\mathrm{spr}(q) = 0$. We have

$$\exp \frac{b}{k} = a + qa = a + q', \tag{3.1}$$

where q' is a quasinilpotent too because $aq = qa$ and note that $\mathrm{spr}(aq) \leq \mathrm{spr}(a)\mathrm{spr}(q)$ for commuting elements a and q; see [4, page 302]. Moreover, we have

$$\exp \frac{T(b)}{k} = T(a) + T(qa) = \lambda a + T(q') = \lambda a + q_1, \tag{3.2}$$

where q_1 is a quasinilpotent.

On the other hand, $(a+q)^{-1} - a^{-1} = -(a+q)^{-1}qa^{-1}$, then $(a+q)^{-1} = a^{-1} + q''$ for some quasinilpotent q''. Thus we have $\exp(-b/k) = a^{-1} + q''$. We obtain from (3.1) and (3.2) and the latest equation $\exp(T(b) - b)/k = \lambda + q_2$ for some quasinilpotent q_2. Then $\mathrm{sp}(T(b) - b)$ is a single point $\{\mu\}$ different from zero, since $\lambda \neq 1$, so $T(b) = b + \mu + y$ for some quasinilpotent y. Then $y = T(b) - b - \mu$ is a quasinilpotent element that can be expanded as a power series in a. Note that we emphasize on this power series expansion only for commutativity purpose. We have $T(b) = b + \mu + y$, so by induction we obtain

$$b = T^n(b) = b + n\mu + y + T(y) + T^2(y) + \cdots + T^{n-1}(y). \tag{3.3}$$

Since T is multiplicative and y has a power series expansion in a, we conclude that $T^j(y)$'s are commuting quasinilpotent elements so their sum is quasinilpotent too; see [4, page 302]. This implies that $n\mu$ is quasinilpotent which is a contradiction. This completes the first part of the theorem.

Now assume that A is commutative and a is a nontrivial homogenous element such that a^k is in the same connected component as the identity. Then there exists an element $b \in A$ with $\exp b = a^k$. Obviously the same argument as above, but without needing to expansion of b as a power series in a, leads to a contradiction. So the proof is complete.

The following corollaries are immediate consequence of the above theorem. □

Corollary 3.1. *Let X be a compact locally path connected and simply connected space, and let $\phi : X \mapsto X$ be a homeomorphism of order n. Assume that $\lambda \neq 1$ is an nth root of the unity. Then, for every continuous function $f : X \to \mathbb{C}$, there is a point $x \in X$ such that $\sum_{i=0}^{p-1} \lambda^i f(\phi^i(x)) = 0$.*

Proof. Let A be the commutative Banach algebra of continuous functions $f : X \to \mathbb{C}$ with the usual structures. Since X is connected, A has no nontrivial idempotent. Define the continuous automorphism $T : A \mapsto A$ by $T(f) = f \circ \phi$. T satisfies $T^n = 1$, so we have a Z_n-graded structure for A in the form $A = \bigoplus_{i=0}^{n-1} \ker(A - \lambda^i)$. Put $g = \sum_{i=0}^{p-1} \lambda^i f(\phi^i(x))$, then $T(g) = \lambda^{n-1} g$, so g is a nontrivial homogenous element of A. If $g(x) \neq 0$ for all $x \in X$, then g is an invertible element of A which is not in the same connected component as the identity, by the above theorem. On the other hand, consider the covering space $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$. Since X is simply connected and locally path connected, there is a lifting h of f; that is, $h \in A$ with $\exp h = g$, using the standard lifting lemma in the theory of covering space; see [5, proposition 1.33]. So g is a logarithmic element and must be in the same connected component as the identity. This contradicts to the fact that g cannot be connected to the identity. This completes the proof. □

Putting $X = S^2$, $\phi(x) = -x$, and $\lambda = -1$, we obtain the classical two-dimensional Borsuk Ulam theorem as follows.

Corollary 3.2 (Borsuk-Ulam Theorem). *For a continuous function $f : S^2 \to \mathbb{C}$, there must exist a point $x \in S^2$ with $f(x) = f(-x)$.*

Example 3.3. Put $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and let $f : S^2 \to \mathbb{R}^2$ *be a* continuous function. Then there is a point $(x_0, y_0, z_0) \in S^2$ *such that* $f(y_0, -x_0, -z_0) - f(-x_0, -y_0, z_0) + f(-y_0, x_0, -z_0) - f(x_0, y_0, z_0) = 0$. To prove this, consider the fourth-order homeomorphism ϕ *of* S^2 *with* $\phi(x, y, z) = (-y, x, -z)$. Now apply Corollary 3.1 with $\lambda = -1$.

The following corollary is an obvious consequence of the last part of the main theorem. In this corollary and its sequel, $G(A)_0$ is the connected component of the identity.

Corollary 3.4. *Let A be an idempotentless commutative Banach algebra which is graded by a finite Abelian group such that a nontrivial homogenous element is invertible. Then $G(A)/G(A)_0$ is an infinite group.*

4. Further Questions and Remarks

In this section we present some questions which naturally arise from the main theorem and the corollaries of the previous sections.

First we discuss about a pure algebraic analogy of Corollary 3.4. For this purpose, we need some elements of stable rank theory and K-theory for both Banach algebras and complex algebras. We say that a commutative Banach algebra A has topological stable rank one if $G(A)$ is dense in A; see [6]. Let A be a commutative complex algebra. We say that a pair (a_1, a_2) is invertible if there is a pair (b_1, b_2) such that $a_1 b_1 + a_2 b_2 = 1$. We say A has Bass stable rank one if, for every invertible pair $(a, b) \in A^2$, there is an element $x \in A$ such that $a + bx$ is invertible; see [7]. It is mentioned in [6] that, for a commutative C^* algebra, the Bass stable rank coincide with the topological stable rank.

For a complex Banach algebra A, let $\mathrm{GL}_n(A)$ be the space of all n by n invertible matrices with entries in A. There is a natural topology on this space, and there is a natural embedding of $\mathrm{GL}_n(A)$ into $\mathrm{GL}_{n+1}(A)$ which sends a matrices B to diag(B,1), so we have

an inductive system with $\mathrm{GL}_n(A)$'s. Put $GL(A) = \varinjlim \mathrm{GL}_n(A)$. Then $\mathrm{GL}(A)$ has a natural inductive limit topology and algebraic structure. Define $K_1(A) = \mathrm{GL}(A)/\mathrm{GL}(A)_0$ where $\mathrm{GL}(A)_0$ is the connected component of the identity. This Abelian group $K_1(A)$ is the standard K_1 functor defined on the category of the Banach algebras. For a complex algebra A, there is a pure algebraic K_1^{alg} functor defined as the quotient of $\mathrm{GL}(A)$ by its commutator. On the other hand, it is well known that for an stable rank one Banach algebra A, $G(A)/G(A)_0$ is naturally isomorphic to $K_1(A)$; see [8].

So considering this isomorphism, the equality of the Bass and topological stable rank for commutative C^* algebras, and the above corollary, it is natural to ask the next pure algebraic question.

Question 1. Let A be an idempotentless involutive and commutative complex algebra with Bass stable rank one which is graded by a finite Abelian group. Assume that a nontrivial homogenous element of A is invertible. Does this imply that $K_1^{\mathrm{alg}}(A)$ is an infinite group?

In the following example, we drop simultaneously the commutativity of both grading group G and the graded idempotent less algebra A in the main theorem. We observe that in this case the theorem is no longer valid.

Example 4.1. Let F_2 be the free group on two generators x and y. We shall see that there is a normal subgroup G_0 of F_2 which does not contain $yxy^{-1}x^{-1}$ and its quotient is isomorphic to S_3, the permutation group on three elements. Assuming the existence of such subgroup G_0, we obtain an S_3-graded structure for $C^*_{\mathrm{red}}(F_2)$ for which $yxy^{-1}x^{-1}$, as an element of $C^*_{\mathrm{red}}(F_2)$, is a nontrivial homogenous element. On the other hand, it is well known that this algebra has no nontrivial idempotent and $yxy^{-1}x^{-1} \in C^*_{\mathrm{red}}(F_2)$, is in the same connected component as the identity; see [8]. To prove the existence of such subgroup G_0, we first note that F_4, the free group on four generators, can be considered as an index three subgroup of F_2 which does not contain $yxy^{-1}x^{-1}$. This can be proved using certain covering space of figure-8 space as illustrated in [5, page 58]. Let $h : \tilde{X} \to X$ be a covering space with $h(q) = p$. Then the induced map $h_* : \pi_1(\tilde{X}, q) \to \pi_1(X, p)$ is an injective map. $h_*(\pi_1(\tilde{X}, q))$ is isomorphic to $\pi_1(\tilde{X}, q)$ whose index, as a subgroup of $\pi_1(X, p)$, is equal to the cardinal of a fibre of the covering space. Moreover, a loop $\gamma \in \pi_1(X, p)$ lies in the range of h_* if and only if the unique lifting $\tilde{\gamma}$ of γ with starting point q is a loop with base point q, see [5, proposition 1.31].

Now consider the 3-fold covering space which is illustrated in the Figure 1. The fundamental group of the total space is F_4, and the fundamental group of the base space is F_2. The loop $yxy^{-1}x^{-1}$ is not in the range of projecting map of the covering because its lifting with base point q ends to a different point O.

Put $G = F_2$ and $b = yxy^{-1}x^{-1}$. Then G has a subgroup H of index 3 which does not contain a commutator element b. We obtain a morphism $\beta : G \to S_3$ with the standard action of G on the set of left cosets of H. Then $\ker\beta$ is a normal subgroup of G which is contained in H and does not contain b. $G/\ker\beta$ is isomorph to a subgroup of S_3; on the other hand, $G/\ker\beta$ is not Abelian since $\ker\beta$ does not contain a commutator element b. This shows that $\ker\beta$ is a normal subgroup of $G = F_2$ in which quotient is isomorphic to S_3 and does not contain the commutator element $b = yxy^{-1}x^{-1}$. So the above construction gives an example of a noncommutative Banach algebra $C^*_{\mathrm{red}}(F_2)$, without nontrivial idempotent, which is graded by a non-Abelian finite group S_3, such that a nontrivial homogenous element $yxy^{-1}x^{-1}$ lies in the same connected component as the identity.

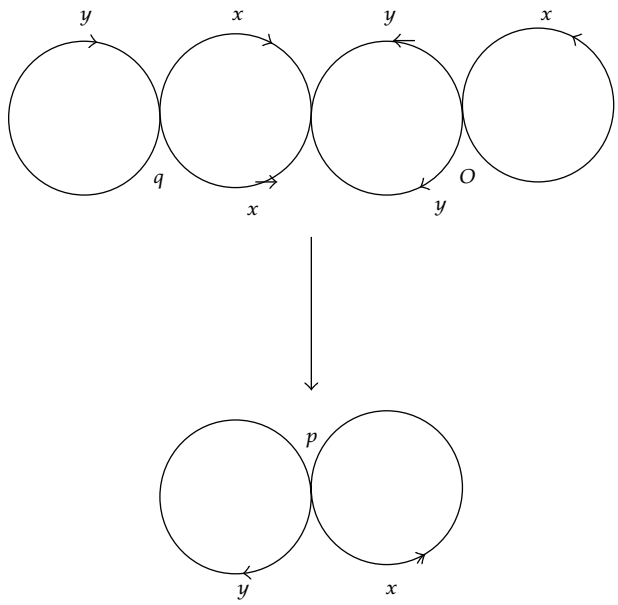

Figure 1

But could we give any such example with a finite Abelian group G? In other words, can we drop the hypothesis of commutativity of Banach algebra A from the second part of the main theorem. This is a motivation for the next question which can be considered as a noncommutative analogy of the two-dimensional Borsuk-Ulam theorem.

Question 2. Let A be a Banach algebra without nontrivial idempotent which is equipped with a G-graded structure, where G is a finite Abelian group. Can one prove that the connected component of the identity has null intersection with nontrivial homogenous elements? As a particular case, put $A = C^*_{\mathrm{red}}(F_2)$, with the Z_2-graded structure corresponding to the Z_2-partition of F_2 to the union of odd and even words. Can a linear combination of odd words be connected to the identity? Is the decomposition of this algebra to even and odd words, the only Z_2-graded structure for A, up to graded isomorphism?

There is an affirmative answer to a particular case of the second part of the above question. We thank professor Valette for his affirmative answer in this case. This affirmative answer is mentioned in the following proposition.

Proposition 4.2. *Every linear combination of two odd words a and b in $C^*_{red}(F_2)$ cannot be connected to the identity.*

Proof. For a contradiction assume that $ra + sb$ can be connected to the identity in the space of invertible elements, where r and s are two complex numbers. With a small perturbation, we can assume that $|r| < |s|$. (Note that with this perturbation the connected component does not change.) Now put $z_0 = r/s$, so $z_0 a + b$ lies in the same connected component as the identity with $|z_0| < 1$. Then the curve $tz_0 a + b,\ t \in [0,1]$ is a curve which lies in the space of invertible elements of $C^*_{\mathrm{red}}(F_2)$. Because ba^{-1} is a unitary element, its spectrum does not contain an element tz_0 where $t \in [0,\ 1]$. So b as an odd element lies in the same connected component as the identity. But the only words which can be connected to the identity are members of

the commutator subgroup of F_2. This contradiction shows that a linear combination of two odd words can not be connected to the identity. □

What is a Banach algebraic formulation of the higher dimensional Borsuk-Ulam theorem? In order to obtain a noncommutative version of this theorem, we restate the classical case as follows.

Let $f_1, f_2, \ldots, f_n$ be n odd continuous real-valued functions on S^n, then the function $\sum_{i=1}^{n} f_i^2$ is not an invertible element of $C(S^n)$ or equivalently is not in the same connected component as the identity (since every invertible element can be connected to the identity). In fact f_i's are self-adjoint elements of $C(S^n)$ which are odd elements of the standard Z_2-graded structure of $C(S^n)$. Now a relevant noncommutative version of this statement can be presented as the next question. So it seems natural to ask that for what type of noncommutative spheres the answer to the next question is affirmative?

Question 3. Assume that A is a Z_2-graded noncommutative n-sphere and $a_1, a_2, \ldots a_n$ are self-adjoint elements of A which are odd elements of this graded algebra. Is it true to say that $\sum_{i=1}^{n} a_i^2$ is either non-invertible or is not in the same connected component as the identity?

Remark 4.3. A family of noncommutative spheres is a family of C^* algebras A_θ, $\theta \in \mathbb{R}$ with some relations as a natural generalization of the algebra of continuous functions on n-sphere. For more information on noncommutative spheres, see [9] or [10].

Another candidate for the noncommutative analogy of the Borsuk-Ulam theorem can be presented as follows.

Consider the equivalent statement of the Borsuk-Ulam theorem which says that an odd continuous map $f : S^n \to S^n$ is not null homotopic; namely, it is not homotopic to a constant map. We try to translate this statement into the language of Banach or C^* algebras. The antipodal map $\phi(x) = -x$ defines an order two automorphism $T; C(S^n) \to C(S^n)$ with $T(g) = g \circ \phi$ which naturally gives a Z_2-graded structure for $C(S^n)$. Similarly an odd map $f : S^n \to S^n$ defines a morphism $\alpha : C(S^n) \to C(S^n)$ which satisfies $T\alpha = \alpha T$. This means that α is a graded morphism. So we ask, for what type of noncommutative spheres (spaces), the answer to the next question is affirmative?

Question 4. Let A be a noncommutative sphere with a nontrivial Z_2-graded structure, and let $\alpha : A \to A$ be a graded morphism. Is it true to say that α is not a null homotopic morphism?

Final Remark on the Main Theorem

We explain that the main theorem gives us a weaker version of the Kaplansky-Kadison conjecture. This conjecture says that, for a torsion free group Γ, $C^*_{\mathrm{red}}(\Gamma)$ has no nontrivial idempotent; see [11]. Now as a consequence of our theorem, we present the following question as a weaker version of the Kaplansky-Kadison conjecture.

Question 5 (A Weak Version of the Kaplansky-Kadison Conjecture). Let Γ be a torsion-free group and $C^*_{\mathrm{red}}(\Gamma)$ equipped with a Z_n-graded structure. Is it possible that the convex hull of the spectrum of a nontrivial homogenous element does not contain the origin?

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References

[1] I. N. Herstein, *Topics in Algebra*, Wiley, 1975.
[2] A. Connes, "Noncommutative differential geometry," *Institut des Hautes Études Scientifiques*, no. 62, pp. 257–360, 1985.
[3] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of Non-Commutative Geometry*, Birkhäuser, Boston, Mass, USA, 2001.
[4] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, NY, USA, 2nd edition, 1991.
[5] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
[6] M. A. Rieffel, "Dimension and stable rank in the K-theory of C^*-algebras," *Proceedings of the London Mathematical Society*, vol. 46, no. 2, pp. 301–333, 1983.
[7] H. Bass, "K-theory and stable algebra," *Institut des Hautes Études Scientifiques*, no. 22, pp. 5–60, 1964.
[8] K. Dykema, U. Haagerup, and M. Rørdam, "The stable rank of some free product C^*-algebras," *Duke Mathematical Journal*, vol. 90, no. 1, pp. 95–121, 1997.
[9] O. Bratteli, G. A. Elliott, D. E. Evans, and A. Kishimoto, "Noncommutative spheres," *International Journal of Mathematics*, vol. 2, no. 2, pp. 139–166, 1991.
[10] T. Natsume, "Some non-commutative spheres," in *Quantum Field Theory and Noncommutative Geometry*, vol. 662 of *Lecture Notes in Physics*, pp. 57–66, Springer, Berlin, Germany, 2005.
[11] A. Valette, *Introduction to the Baum-Connes Conjecture*, Birkhäuser, Basel, Switzerland, 2002.