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Research Article

Periodic Solutions in Shifts $\delta_{\pm}$ for a Nonlinear Dynamic Equation on Time Scales

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Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$. We use a fixed point theorem due to Krasnosel'skiĭ to show that nonlinear delay in dynamic equations of the form $x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x^{\Delta}(\delta_{-}(k,t))\delta_{-}^{\Delta}(k,t) + q(t,x(t),x(\delta_{-}(k,t)))$, $t \in \mathbb{T}$, has a periodic solution in shifts $\delta_{\pm}$. We extend and unify periodic differential, difference, h-difference, and q-difference equations and more by a new periodicity concept on time scales.

1. Introduction

The time scales approach unifies differential, difference, h-difference, and q-differences equations and more under dynamic equations on time scales. The theory of dynamic equations on time scales was introduced by Hilger in this Ph.D. thesis in 1988 [1]. The existence problem of periodic solutions is an important topic in qualitative analysis of ordinary differential equations. There are only a few results concerning periodic solutions of dynamic equations on time scales such as in [2, 3]. In these papers, authors considered the existence of periodic solutions for dynamic equations on time scales satisfying the condition

$$\text{“there exists a } \omega > 0 \text{ such that } t \pm \omega \in \mathbb{T} \quad \forall t \in \mathbb{T}.\text{”} \tag{1.1}$$

Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy condition (1.1). Adıvar and Raffoul introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed

$\omega > 0$. He defined a new periodicity concept with the aid of shift operators $\delta_{\pm}$ which are first defined in [4] and then generalized in [5].

Let $\mathbb{T}$ be a periodic time scale in shifts $\delta_{\pm}$ with period $P \in (t_0, \infty)_{\mathbb{T}}$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. We are concerned with the existence of periodic solutions in shifts $\delta_{\pm}$ for the nonlinear dynamic equation with a delay function $\delta_{-}(k,t)$:

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x^{\Delta}(\delta_{-}(k,t))\delta_{-}^{\Delta}(k,t) + q(t, x(t), x(\delta_{-}(k,t))), \quad t \in \mathbb{T}, \tag{1.2}$$

where k is fixed if $\mathbb{T} = \mathbb{R}$ and $k \in [P, \infty)_{\mathbb{T}}$ if $\mathbb{T}$ is periodic in shifts $\delta_{\pm}$ with period P.

Kaufmann and Raffoul in [2] used Krasnosel'skiĭ fixed point theorem and showed the existence of a periodic solution of (1.2) and used the contraction mapping principle to show that the periodic solution is unique when $\mathbb{T}$ satisfies condition (1.1). Similar results were obtained concerning (1.2) in [6, 7] in the case $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, respectively. Currently, Adıvar and Raffoul used Lyapunov's direct method to obtain inequalities that lead to stability and instability of delay dynamic equations of (1.2) when $q = 0$ on a time scale having a delay function δ_{-} in [8] and also using the topological degree method and Schaefers fixed point theorem, they deduce the existence of periodic solutions of nonlinear system of integrodynamic equations on periodic time scales in [9].

Hereafter, we use the notation $[a,b]_{\mathbb{T}}$ to indicate the time scale interval $[a,b] \cap \mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and $(a,b)_{\mathbb{T}}$ are similarly defined.

In Section 2, we will state some facts about the exponential function on time scales, the new periodicity concept for time scales, and some important theorems which will be needed to show the existence of a periodic solution in shifts $\delta_{\pm}$. In Section 3, we will give some lemmas about the exponential function and the graininess function with shift operators. Finally, we present our main result in Section 4 by using Krasnosel'skiĭ fixed point theorem and give an example.

2. Preliminaries

In this section, we mention some definitions, lemmas, and theorems from calculus on time scales which can be found in [10, 11]. Next, we state some definitions, lemmas, and theorems about the shift operators and the new periodicity concept for time scales which can be found in [12].

Definition 2.1 (see [10]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^{+}$ is given by $\mathcal{R}^{+} = \{\varphi \in \mathcal{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $\varphi \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_{\varphi}(t,s) = \exp\left(\int_{s}^{t} \zeta_{\mu(r)}(\varphi(r))\Delta r\right), \tag{2.1}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(\varphi(r)) := \begin{cases} \dfrac{1}{\mu(r)}\operatorname{Log}(1+\mu(r)\varphi(r)), & \text{if } \mu(r) > 0, \\ \varphi(r), & \text{if } \mu(r) = 0. \end{cases} \tag{2.2}$$

Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y, y(s) = 1$. Other properties of the exponential function are given in the following lemma [10, Theorem 2.36].

Lemma 2.2 (see [10]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $1/e_p(t,s) = e_{\ominus}(t,s)$, *where*, $\ominus p(t) = -(p(t))/(1+\mu(t)p(t))$;

(iv) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$;

(viii) $(1/e_p(\cdot,s))^{\Delta} = -p(t)/e_p^{\sigma}(\cdot,s)$.

The following definitions, lemmas, corollaries, and examples are about the shift operators and new periodicity concept for time scales which can be found in [12].

Definition 2.3 (see [12]). Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_{\pm} : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties.

(P.1) The function $\delta_{\pm}$ are strictly increasing with respect to their second arguments, that is, if

$$(T_0,t),(T_0,u) \in \mathfrak{D}_{\pm} := \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\mp}(s,t) \in \mathbb{T}^*\}, \tag{2.3}$$

then

$$T_0 \leq t < u \ \text{ implies } \ \delta_{\pm}(T_0,t) < \delta_{\pm}(T_0,u). \tag{2.4}$$

(P.2) If $(T_1,u),(T_2,u) \in \mathfrak{D}$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in D_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$.

(P.3) If $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in D_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in D_+$ and $\delta_+(t_0,t) = t$ holds.

(P.4) If $(s,t) \in D_{\pm}$, then $(s,\delta_{\pm}(s,t)) \in D_{\mp}$ and $\delta_{\mp}(s,\delta_{\pm}(s,t)) = t$, respectively.

(P.5) If $(s,t) \in D_{\pm}$ and $(u,\delta_{\pm}(s,t)) \in D_{\pm}$, then $(s,\delta_{\mp}(u,t)) \in D_{\pm}$ and $\delta_{\mp}(u,\delta_{\pm}(s,t)) = \delta_{\pm}(s,\delta_{\mp}(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_\pm(s,t)$ is called the shift size. The values $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathfrak{D}_\pm$ are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ exist.

Example 2.4 (see [12]).

(i) $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{R}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.

(ii) $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{Z}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.

(iii) $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $t_0 = 1$, $\mathbb{T}^* = q^{\mathbb{Z}}$, $\delta_-(s,t) = t/s$ and $\delta_+(s,t) = ts$.

(iv) $\mathbb{T} = \mathbb{N}^{1/2}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{N}^{1/2}$, $\delta_-(s,t) = \sqrt{t^2 - s^2}$ and $\delta_+(s,t) = \sqrt{t^2 + s^2}$.

Definition 2.5 (periodicity in shifts [12]). Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shift $\delta_\pm$ if there exists a $p \in (t_0, \infty)_{\mathbb{T}^*}$ such that $(p,t) \in D_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\left\{p \in (t_0, \infty)_{\mathbb{T}^*} : (p,t) \in D_\pm, \quad \forall t \in \mathbb{T}^*\right\} \neq t_0, \tag{2.5}$$

then P is called the period of the time scale $\mathbb{T}$.

Example 2.6 (see [12]). The following time scales are not periodic in the sense of condition (1.1) but periodic with respect to the notion of shift operators given in Definition 2.5:

(i) $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}, \delta_\pm(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2, & t>0; \\ \pm P, & t=0; \\ -(\sqrt{-t} \pm \sqrt{P})^2, & t<0; \end{cases}$, $P = 1, t_0 = 0$,

(ii) $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1} t$, $P = q, t_0 = 1$,

(iii) $\mathbb{T}_3 = \overline{\cup_{n \in \mathbb{Z}} [2^{2n}, 2^{2n+1}]}$, $\delta_\pm(P,t) = P^{\pm 1} t$, $P = 4$, $t_0 = 1$,

(iv) $\mathbb{T}_4 = \{q^n/(1+q^n) : q > 1 \ \text{ is constant and } n \in \mathbb{Z}\} \cup \{0, 1\}$,

$$\delta_\pm(P,t) = \frac{q^{((\ln(t/(1-t)) \pm \ln(P/(1-P)))/\ln q)}}{1 + q^{((\ln(t/(1-t)) \pm \ln(P/(1-P)))/\ln q)}}, \quad P = \frac{q}{1-q}. \tag{2.6}$$

Notice that the time scale $\mathbb{T}_4$ in Example 2.6 is bounded above and below and $\mathbb{T}_4^* = \{q^n/(1+q^n) : q > 1$ is constant and $n \in \mathbb{Z}\}$.

Remark 2.7 (see [12]). Let $\mathbb{T}$ be a time scale, that is, periodic in shifts with the period P. Thus, by $(P.4)$ of Definition 2.3 the mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, by $(P.1)$ of Definition 2.3 shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) = \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $(\delta_+^P)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.

We assume that $\mathbb{T}$ is a periodic time scale in shift $\delta_\pm$ with period P. The operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ given by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. That is, $(\delta_\pm^P \circ \sigma)(t) = (\sigma \circ \delta_\pm^P)(t)$ for all $t \in \mathbb{T}^*$.

Lemma 2.8 (see [12]). *The mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ preserves the structure of the points in $\mathbb{T}^*$. That is,*

$$\sigma(t) = t \quad \textit{implies } \sigma(\delta_+(P,t)) = \delta_+(P,t) \textit{ and } \sigma(t) > t \quad \textit{implies } \sigma(\delta_+(P,t)) > \delta_+(P,t). \tag{2.7}$$

Corollary 2.9 (see [12]). *$\delta_+(P,\sigma(t)) = \sigma(\delta_+(P,t))$ and $\delta_-(P,\sigma(t)) = \sigma(\delta_-(P,t))$ for all $t \in \mathbb{T}^*$.*

Definition 2.10 (periodic function in shift $\delta_\pm$ [12]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that

$$(T,t) \in D_\pm, \qquad f\left(\delta_\pm^T(t)\right) = f(t) \quad \forall t \in \mathbb{T}^*, \tag{2.8}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (5) holds is called the period of f.

Definition 2.11 (Δ-periodic function in shifts $\delta_\pm$ [12]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that

$$\begin{gathered} (T,t) \in D_\pm \quad \forall t \in \mathbb{T}^*, \\ \text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with } rd\text{-continuous derivatives,} \\ f\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T} = f(t) \quad \forall t \in \mathbb{T}^*, \end{gathered} \tag{2.9}$$

where $\delta_\pm^T := \delta_\pm(T,t)$. The smallest number $T \in [P,\infty)_{\mathbb{T}^*}$ such that (2.9) hold is called the period of f.

Notice that Definitions 2.10 and 2.11 give the classic periodicity definition on time scales whenever $\delta_\pm^T := t \pm T$ are the shifts satisfying the assumptions of Definitions 2.10 and 2.11.

Now, we give two theorems concerning the composition of two functions. The first theorem is the chain rule on time scales [10, Theorem 1.93].

Theorem 2.12 (chain rule [10]). *Assume that $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\tilde{\Delta}}$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \upsilon)^\Delta = \left(w^{\tilde{\Delta}} \circ \nu\right)\nu^\Delta. \tag{2.10}$$

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$. If one takes $\nu(t) = \delta_\pm(T,t)$, then one has $\nu(\mathbb{T}) = \mathbb{T}$ and $[f(\nu(t))]^\Delta = (f^\Delta \circ \nu(t))\nu^\Delta(t)$.

The second theorem is the substitution rule on periodic time scales in shifts $\delta_\pm$ which can be found in [12].

Theorem 2.13 (see [12]). *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in C_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} f(s)\Delta s. \tag{2.11}$$

This work is mainly based on the following theorem [13].

Theorem 2.14 (Krasnosel'skiĭ). *Let M be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that A and B map M into $\mathbb{B}$ such that*

(i) *$x, y \in M$ imply $Ax + By \in M$,*

(ii) *A is completely continuous,*

(iii) *B is a contraction mapping.*

Then there exists $z \in M$ with $z = Az + Bz$.

3. Some Lemmas

In this section, we show some interesting properties of the exponential functions $e_p(t, t_0)$ and shift operators on time scales.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P and the shift $\delta_\pm^T$ is Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Then the graininess function $\mu : \mathbb{T} \to [0, \infty)$ satisfies*

$$\mu\left(\delta_\pm^T(t)\right) = \delta_\pm^{\Delta T}(t)\mu(t). \tag{3.1}$$

Proof. Since $\delta_\pm^T$ is Δ-differentiable at t we can use Theorem 1.16 (iv) in [10]. Then we have

$$\mu(t)\delta_\pm^{\Delta T}(t) = \delta_\pm^T(\sigma(t)) - \delta_\pm^T(t). \tag{3.2}$$

Then by using Corollary 2.9 we have

$$\begin{aligned} \mu(t)\delta_\pm^{\Delta T}(t) &= \sigma\left(\delta_\pm^T(t)\right) - \delta_\pm^T(t) \\ &= \mu\left(\delta_\pm^T(t)\right). \end{aligned} \tag{3.3}$$

Thus, the proof is complete. □

Lemma 3.2. *Let $\mathbb{T}$ be a time scale, that is, periodic in shifts $\delta_\pm$ with the period P and the shift $\delta_\pm^T$ is Δ-differentiable on $t \in \mathbb{T}^*$, where $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Then,*

$$e_p\left(\delta_\pm^T(t), \delta_\pm^T(t_0)\right) = e_p(t, t_0) \quad for\ t, t_0 \in \mathbb{T}^*. \tag{3.4}$$

Proof. Assume that $\mu(\tau) \neq 0$. Set $f(\tau) = (1/\mu(\tau))\mathrm{Log}(1 + p(\tau)\mu(\tau))$. Using Lemma 3.1 and Δ-periodicity of p in shifts $\delta_{\pm}$ we get

$$\begin{aligned}
f\left(\delta_{\pm}^{T}(\tau)\right)\delta_{\pm}^{\Delta T}(\tau) &= \frac{\delta_{\pm}^{\Delta T}(\tau)}{\mu(\delta_{\pm}^{T}(\tau))}\mathrm{Log}\left(1 + p\left(\delta_{\pm}^{T}(\tau)\right)\mu\left(\delta_{\pm}^{T}(\tau)\right)\right) \\
&= \frac{\delta_{\pm}^{\Delta T}(\tau)}{\mu(\delta_{\pm}^{T}(\tau))}\mathrm{Log}\left(1 + p\left(\delta_{\pm}^{T}(\tau)\right)\delta_{\pm}^{\Delta T}\frac{1}{\delta_{\pm}^{\Delta T}}\mu\left(\delta_{\pm}^{T}(\tau)\right)\right) \\
&= \frac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau)) \\
&= f(\tau).
\end{aligned} \tag{3.5}$$

Thus, f is Δ−periodic in shifts $\delta_{\pm}$ with the period T. By using Theorem 2.13 we have

$$\begin{aligned}
e_p\left(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(t_0)\right) &= \begin{cases} \exp\left(\int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} \dfrac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{for } \mu(\tau) \neq 0, \\ \exp\left(\int_{\delta_{\pm}^{T}(t_0)}^{\delta_{\pm}^{T}(t)} p(\tau)\Delta\tau\right), & \text{for } \mu(\tau) = 0, \end{cases} \\
&= \begin{cases} \exp\left(\int_{t_0}^{t} \dfrac{1}{\mu(\tau)}\mathrm{Log}(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{for } \mu(\tau) \neq 0, \\ \exp\left(\int_{t_0}^{t} p(\tau)\Delta\tau\right), & \text{for } \mu(\tau) = 0, \end{cases} \\
&= e_p(t, t_0).
\end{aligned} \tag{3.6}$$

The proof is complete. □

Lemma 3.3. *Let $\mathbb{T}$ be a time scale, that is, periodic in shifts $\delta_{\pm}$ with the period P and the shift $\delta_{\pm}^{T}$ is Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_{\pm}$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Then*

$$e_p\left(\delta_{\pm}^{T}(t), \sigma\left(\delta_{\pm}^{T}(s)\right)\right) = e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(t)p(t)} \quad \textit{for } t, s \in \mathbb{T}^*. \tag{3.7}$$

Proof. From Corollary 2.9, we know $\sigma(\delta_{\pm}^{T}(s)) = \delta_{\pm}^{T}(\sigma(s))$. By Lemmas 3.2 and 2.2 we obtain

$$e_p\left(\delta_{\pm}^{T}(t), \sigma\left(\delta_{\pm}^{T}(s)\right)\right) = e_p\left(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(\sigma(s))\right) = e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(t)p(t)}. \tag{3.8}$$

The proof is complete. □

4. Main Result

We will state and prove our main result in this section. We define

$$P_T = \left\{ x \in C(\mathbb{T}, \mathbb{R}) : x\left(\delta_+^T(t)\right) = x(t) \right\}, \tag{4.1}$$

where $C(\mathbb{T}, \mathbb{R})$ is the space of all real valued continuous functions. Endowed with the norm

$$\|x\| = \max_{t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}} |x(t)|, \tag{4.2}$$

P_T is a Banach space.

Lemma 4.1. *Let $x \in P_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

Proof. Since $x \in P_T$, then $x(\delta_+^T(t_0)) = x(t_0)$, and by Corollary 2.9, we have $x(\sigma(\delta_+^T(t_0))) = x(\sigma(t_0))$. For all $t \in [t_0, \delta_+^T(t_0)]_{\mathbb{T}}$,$|x(\sigma(t))| \le \|x\|$. Hence $\|x^\sigma\| \le \|x\|$. Since $x \in C(\mathbb{T}, \mathbb{R})$, there exists $t_1 \in [t_0, \delta_+^T(t_0)]$ such that $\|x\| = |x(t_1)|$. If t_1 is left scattered, then $\sigma(\rho(t_1)) = t_1$. And so, $\|x^\sigma\| \ge |x^\sigma(\rho(t_1))| = x(t_1) = \|x\|$. Thus, we have $\|x^\sigma\| = \|x\|$. If t_1 is dense, $\sigma(t_1) = t_1$ and $\|x^\sigma\| = \|x\|$.

Assume that t_1 is left dense and right scattered. Note that if $t_1 = t_0$ then we work $t_1 = \delta_+^T(t_0)$. Fix $\epsilon > 0$ and consider a sequence $\{a_n\}$ such that $a_n \uparrow t_1$. Note that $\sigma(a_n) \le t_1$ for all n. By the continuity of x, there exists N such that for all $n > N$, $|x(t_1) - x^\sigma(a_n)| < \epsilon$. This implies that $\|x\| - \epsilon \le \|x^\sigma\|$. Since $\epsilon > 0$ was arbitrary, then $\|x\| = \|x^\sigma\|$ and the proof is complete. □

In this paper we assume that $a(t) \in \mathcal{R}^+$ is a continuous function with $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a\left(\delta_+^T(t)\right)\delta_+^{\Delta T}(t) = a(t), \quad b\left(\delta_+^T(t)\right) = b(t), \tag{4.3}$$

where $b^\Delta(t)$ is continuous. We further assume that $q(t, x, y)$ is continuous and periodic with $\delta_\pm$ in t and Lipschitz continuous in x and y. That is,

$$q\left(\delta_+^T(t), x, y\right)\delta_+^{\Delta T}(t) = q(t, x, y), \tag{4.4}$$

and there are some positive constants L and E such that

$$\left|q(t, x, y) - q(t, z, w)\right| \le L\|x - z\| + E\left\|y - w\right\|. \tag{4.5}$$

Lemma 4.2. *Suppose that* (4.3)–(4.5) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of* (1.2) *if and only if*

$$\begin{aligned} x(t) &= b(t)x(\delta_-(k,t)) + \frac{1}{1 - e_{\ominus a(t)}\left(t, \delta_-^T(t)\right)} \\ &\quad \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s, \end{aligned} \tag{4.6}$$

where

$$r(s) = a(s)b^\sigma(s) + b^\Delta(s). \tag{4.7}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.2). We can rewrite (1.2) as

$$x^\Delta(t) + a(t)x^\sigma(t) = b(t)x^\Delta(\delta_-(k,t))\delta_-^\Delta(k,t) + q(t, x(t), x(\delta_-(k,t))). \tag{4.8}$$

Multiply both sides of the above equation by $e_{a(t)}(t, t_0)$ and then integrate from $\delta_-^T(t)$ to t to obtain

$$\begin{aligned} &\int_{\delta_-^T(t)}^{t} \left[x(s)e_{a(s)}(s,t_0)\right]^\Delta \Delta s \\ &\quad = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t_0)\Delta s. \end{aligned} \tag{4.9}$$

We arrive at

$$\begin{aligned} &x(t)\left[e_{a(t)}(t,t_0) - e_{a(t)}\left(\delta_-^T(t), t_0\right)\right] \\ &\quad = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t_0)\Delta s. \end{aligned} \tag{4.10}$$

Dividing both sides of the above equation by $e_{a(t)}(t, t_0)$ and using $x(\delta_+^T(t)) = x(t)$ and Lemma 2.2, we have

$$\begin{aligned} &x(t)\left(1 - e_{a(t)}\left(\delta_-^T(t), t\right)\right) \\ &\quad = \int_{\delta_-^T(t)}^{t} \left[b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s) + q(s, x(s), x(\delta_-(k,s)))\right] e_{a(s)}(s,t)\Delta s. \end{aligned} \tag{4.11}$$

Now, we consider the first term of the integral on the right-hand side of (4.11)

$$\int_{\delta_-^T(t)}^{t} b(s)x^\Delta(\delta_-(k,s))\delta_-^\Delta(k,s)e_{a(s)}(s,t)\Delta s. \tag{4.12}$$

Using integration by parts from rule [10] we obtain

$$\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s$$
$$= \int_{\delta_-^T(t)}^{t} \left[b(s)e_{a(s)}(s,t)x(\delta_-(k,s))\right]^{\Delta}\Delta s - \int_{\delta_-^T(t)}^{t} \left[b(s)e_{a(s)}(s,t)\right]_s^{\Delta} x^{\sigma}(\delta_-(k,s))\Delta s \tag{4.13}$$

$$\begin{aligned}
&\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s \\
&\quad = b(t)e_{a(t)}(t,t)x(\delta_-(k,t)) - b\left(\delta_-^T(t)\right)e_{a(s)}\left(\delta_-^T(t),t\right)x\left(\delta_-\left(k,\delta_-^T(t)\right)\right) \\
&\qquad - \int_{\delta_-^T(t)}^{t} \left[b^{\sigma}(s)a(s)e_{a(s)}(s,t) + b^{\Delta}(s)e_{a(s)}(s,t)\right]x^{\sigma}(\delta_-(k,s))\Delta s.
\end{aligned} \tag{4.14}$$

Since $b(\delta_-^T(t)) = b(t)$ and $x(\delta_-^T(t)) = x(t)$, the above equality reduces to

$$\begin{aligned}
&\int_{\delta_-^T(t)}^{t} b(s)x^{\Delta}(\delta_-(k,s))\delta_-^{\Delta}(k,s)e_{a(s)}(s,t)\Delta s \\
&\quad = b(t)x(\delta_-(k,t))\left(1 - e_{a(s)}\left(\delta_-^T(t),t\right)\right) \\
&\qquad - \int_{\delta_-^T(t)}^{t} \left[a(s)b^{\sigma}(s) + b^{\Delta}(s)\right]x^{\sigma}(\delta_-(k,s))e_{a(s)}(s,t)\Delta s.
\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.11) we get

$$\begin{aligned}
x(t) &= b(t)x(\delta_-(k,t)) \\
&\quad + \frac{1}{1 - e_{\ominus a(t)}(t,\delta_-^T(t))} \\
&\quad \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^{\sigma}(\delta_-(k,s)) + q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s.
\end{aligned} \tag{4.16}$$

Thus the proof is complete. □

Define the mapping $H : P_T \to P_T$ by

$$\begin{aligned}
Hx(t) &:= b(t)x(\delta_-(k,t)) \\
&\quad + \frac{1}{\left(1 - e_{\ominus a(t)}(t,\delta_-^T(t))\right)} \\
&\quad \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^{\sigma}(\delta_-(k,s)) + q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s.
\end{aligned} \tag{4.17}$$

To apply Theorem 2.14 we need to construct two mappings: one map is a contraction and the other map is compact and continuous. We express (4.17) as

$$Hx(t) = Bx(t) + Ax(t), \tag{4.18}$$

where A, B are given

$$Bx(t) = b(t)x(\delta_-(k,t)), \tag{4.19}$$

$$\begin{aligned} Ax(t) = {} & \frac{1}{1 - e_{\ominus a(t)}\left(t, \delta_-^T(t)\right)} \\ & \times \int_{\delta_-^T(t)}^{t} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}(t,s)\Delta s, \end{aligned} \tag{4.20}$$

and $r(s)$ is defined in (4.7).

Lemma 4.3. *Suppose that* (4.3)–(4.5) *hold. Then* $A : P_T \to P_T$*, as defined by* (4.20)*, is compact and continuous.*

Proof. We show that $A : P_T \to P_T$. Evaluate (4.20) at $\delta_+^T(t)$,

$$\begin{aligned} Ax\left(\delta_+^T(t)\right) = {} & \frac{1}{1 - e_{\ominus a(t)}\left(\delta_+^T(t), \delta_-^T\left(\delta_+^T(t)\right)\right)} \\ & \times \int_{\delta_-^T\left(\delta_+^T(t)\right)}^{\delta_+^T(t)} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}\left(\delta_+^T(t), s\right)\Delta s \\ = {} & \frac{1}{1 - e_{p(t)}\left(\delta_+^T(t), t\right)} \\ & \times \int_{\delta_+^T\left(\delta_-^T(t)\right)}^{\delta_+^T(t)} \left[-r(s)x^\sigma(\delta_-(k,s)) + q(s, x(s), x(\delta_-(k,s)))\right] e_{\ominus a(s)}\left(\delta_+^T(t), s\right)\Delta s. \end{aligned} \tag{4.21}$$

Now, since (4.3) and Corollary 2.9 hold, then we have

$$\begin{aligned} r\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) &= a\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) b^\sigma\left(\delta_+^T(s)\right) + b^\Delta\left(\delta_+^T(s)\right)\delta_+^{T\Delta}(s) \\ &= a(s)b^\sigma(s) + b^\Delta(s) = r(s). \end{aligned} \tag{4.22}$$

That is, $r(s)$ is Δ-periodic in $\delta_\pm$ with period T. Using the periodicity of $r,\ x,\ q$, and Lemma 3.2 we get

$$\begin{aligned}
&\left[-r\left(\delta_+^T(s)\right)x^\sigma\left(\delta_-\left(k,\delta_+^T(s)\right)\right)+q\left(\delta_+^T(s),x\left(\delta_+^T(s)\right),x\left(\delta_-\left(k,\delta_+^T(s)\right)\right)\right)\right]\delta_+^{T\Delta}(s)\\
&\quad\times e_{\ominus a(s)}\left(\delta_+^T(t),\delta_+^T(s)\right)\\
&\quad=\left[-r(s)x^\sigma(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s).
\end{aligned}\tag{4.23}$$

That is, inside the integral of (4.21) is Δ-periodic in $\delta_\pm$ with period T. By Theorem 2.13 and Lemma 3.2 we have

$$\begin{aligned}
Ax\left(\delta_+^T(t)\right)&=\frac{1}{1-e_{\ominus a(t)}\left(t,\delta_-^T(t)\right)}\\
&\quad\times\int_{\delta_-^T(t)}^t\left[-r(s)x^\sigma(\delta_-(k,s))+q(s,x(s),x(\delta_-(k,s)))\right]e_{\ominus a(s)}(t,s)\Delta s\\
&=Ax(t).
\end{aligned}\tag{4.24}$$

That is, $A:P_T\to P_T$.

To see that A is continuous, we let $\varphi,\ \psi\in P_T$ with $\|\varphi\|\le C$ and $\|\psi\|\le C$ and define

$$\begin{gathered}
\eta:=\max_{t\in\left[t_0,\delta_+^T(t_0)\right]_{\mathbb{T}}}\left|\left(1-e_{\ominus a(t)}\left(t,\delta_-^T(t)\right)\right)^{-1}\right|,\qquad\gamma:=\max_{u\in\left[\delta_-^T(t),t\right]_{\mathbb{T}}}e_{\ominus a(t)}(t,u),\\
\beta:=\max_{t\in\left[t_0,\delta_+^T(t_0)\right]_{\mathbb{T}}}|r(t)|.
\end{gathered}\tag{4.25}$$

Given that $\epsilon>0$, take $\delta=\epsilon/M$ such that $\|\varphi-\psi\|<\delta$. By making use of the Lipschitz inequality (4.5) in (4.20), we get

$$\begin{aligned}
\|A\varphi-A\psi\|&\le\gamma\eta\int_{\delta_-^T(t)}^t\beta\|\varphi-\psi\|+L\|\varphi-\psi\|+E\|\varphi-\psi\|\Delta s\\
&=\eta\gamma[\beta+L+E]\left(t_0-\delta_-^T(t_0)\right)\|\varphi-\psi\|\\
&\le M\|\varphi-\psi\|<\epsilon,
\end{aligned}\tag{4.26}$$

where $L,\ E$ are given by (4.5) and $M=\eta\gamma[\beta+L+E](t_0-\delta_-^T(t_0))$. This proves that A is continuous.

We need to show that A is compact. Consider the sequence of periodic functions in $\delta_{\pm}\{\varphi_n\} \subset P_T$ and assume that the sequence is uniformly bounded. Let $R > 0$ be such that $\|\varphi_n\| \le R$, for all $n \in \mathbb{N}$. In view of (4.5) we arrive at

$$\begin{aligned} |q(t,x,y)| &= |q(t,x,y) - q(t,0,0) + q(t,0,0)| \\ &\le |q(t,x,y) - q(t,0,0)| + |q(t,0,0)| \\ &\le L\|x\| + E\|y\| + \alpha, \end{aligned} \tag{4.27}$$

where $\alpha := \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} |q(t,0,0)|$. Hence,

$$\begin{aligned} |A\varphi_n| &= \left| \frac{1}{1-e_{\ominus a(t)}(t,\delta_-^T(t))} \int_{\delta_-^T(t)}^{t} [-r(s)\varphi_n^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))] e_{\ominus a(s)}(t,s)\Delta s \right| \\ &\le \eta\gamma[(\beta+L+E)\|\varphi_n\| + \alpha]\left(t_0 - \delta_-^T(t_0)\right) \\ &\le \eta\gamma[(\beta+L+E)R + \alpha]\left(t_0 - \delta_-^T(t_0)\right) := D. \end{aligned} \tag{4.28}$$

Thus, the sequence $\{A\varphi_n\}$ is uniformly bounded. If we find the derivative of $A\varphi_n$, we have

$$\begin{aligned} (A\varphi_n)^{\Delta}(t) = {} & a(t)A\varphi_n(t) \\ & + \frac{-a(t) + \ominus a(t)}{1 - e_{\ominus a(t)}(\sigma(t),\delta_-^T(\sigma(t)))} \\ & \times \int_{\delta_-^T(t)}^{t} [-r(s)\varphi_n^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))] e_{\ominus a(s)}(t,s)\Delta s \\ & + \frac{1}{1+\mu(t)a(t)} [-r(t)\varphi^{\sigma}(\delta_-(k,s)) + q(s,\varphi_n(s),\varphi_n(\delta_-(k,s)))]. \end{aligned} \tag{4.29}$$

Consequently,

$$\left|(A\varphi_n)^{\Delta}(t)\right| \le D\|a\| + [(\beta+E+L)R + \alpha]\left[2\|a\|\gamma\eta\left(t_0 - \delta_-^T(t_0)\right)\right] := F. \tag{4.30}$$

for all n. That is, $|(A\varphi_n)^{\Delta}(t)| \le F$, for some positive constant F. Thus the sequence $\{A\varphi_n\}$ is uniformly bounded and equicontinuous. The Arzela-Ascoli theorem implies that $\{A\varphi_{n_k}\}$ uniformly converges to a continuous T-periodic function φ^* in $\delta_{\pm}$. Thus A is compact. □

Lemma 4.4. *Let B be defined by* (4.19) *and*

$$\|b(t)\| \le \xi < 1. \tag{4.31}$$

Then $B : P_T \to P_T$ is a contraction.

Proof. Trivially, $B : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$\begin{aligned}\|B\varphi - B\psi\| &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \left|B\varphi(t) - B\psi(t)\right| \\ &= \max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}} \left\{|b(t)|\left|\varphi(\delta_-(k,s)) - \psi(\delta_-(k,s))\right|\right\} \\ &\leq \xi\|\varphi - \psi\|.\end{aligned} \tag{4.32}$$

Hence B defines a contraction mapping with contraction constant ξ. □

Theorem 4.5. *Let $\alpha := max_{t\in[t_0,\delta_+^T(t_0)]_{\mathbb{T}}}|q(t,0,0)|$. Let β, η, and γ be given by* (4.39)*. Suppose that* (4.3)–(4.5) *and* (4.31) *hold and that there is a positive constant G such that all solutions $x(t)$ of* (1.2)*, $x \in P_T$, satisfy $|x(t)| \leq G$, the inequality*

$$\left\{\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right\}G + \gamma\eta\alpha\left(t_0 - \delta_-^T(t_0)\right) \leq G \tag{4.33}$$

holds. Then (1.2) *has a T-periodic solution in $\delta_\pm$.*

Proof. Define $M := \{x \in P_T : \|x\| \leq G\}$. Then Lemma 4.3 implies that $A : P_T \to P_T$ is compact and continuous. Also, from Lemma 4.4, the mapping $B : P_T \to P_T$ is contraction.

We need to show that if $\varphi, \psi \in M$, we have $\|A\varphi - B\psi\| \leq G$. Let $\varphi, \psi \in M$ with $\|\varphi\|, \|\psi\| \leq G$. From (4.19) and (4.20) and the fact that $|q(t,x,y)| \leq L\|x\| + E\|y\| + \alpha$, we have

$$\begin{aligned}\|A\varphi + B\psi\| &\leq \gamma\eta\int_{\delta_-^T(t)}^{t} [L\|\varphi\| + E\|\varphi\| + \beta\|\varphi\| + \alpha]\,\Delta s + \xi\|\psi\| \\ &\leq \left\{\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right\}G + \gamma\eta\alpha\left(t_0 - \delta_-^T(t_0)\right) \\ &\leq G.\end{aligned} \tag{4.34}$$

We see that all the conditions of Krasnosel'skiĭ theorem are satisfied on the set M. Thus there exists a fixed point z in M such that $z = Bz + Az$. By Lemma 4.2, this fixed point is a solution of (2) has a T-periodic solution in $\delta_\pm$. □

Theorem 4.6. *Suppose that* (4.3)–(4.5) *and* (4.31) *hold. Let β, η, and γ be given by* (4.39)*. If*

$$\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right) \leq 1, \tag{4.35}$$

then (1.2) *has a unique T-periodic solution in $\delta_\pm$.*

Proof. Let the mapping H be given by (4.17). For $\varphi, \psi \in P_T$ we have

$$\begin{aligned} \|H\varphi - H\psi\| &\le \xi\|\varphi - \psi\| + \gamma\eta\int_{\delta_-^T(t)}^{t} [L\|\varphi - \psi\| + E\|\varphi - \psi\| + \beta\|\varphi - \psi\|]\,\Delta s \\ &\le \left[\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right)\right]\|\varphi - \psi\|. \end{aligned} \tag{4.36}$$

This completes the proof. □

Example 4.7. Let $\mathbb{T} = \{2^n\}_{n\in N_0} \cup \{1/4, 1/2\}$ be a periodic time scale in shift $\delta_\pm(P,t) = P^{\pm 1}t$ with period $P = 2$. We consider the dynamic equation (1.2) with $a(t) = 1/5t$, $b(t) = (1/500)(-1)^{\ln t/\ln q}$ and $q(t,x,y) = (\sin x + \arctan x + 1)/1000t$.

The operators $\delta_-(s,t) = t/s$ and $\delta_+(s,t) = st$ are backward and forward shift operators for $(s,t) \in D_\pm$. Here $\mathbb{T}^* = \mathbb{T}$, the initial point $t_0 = 1$ and $\delta_-(k,t) = t/k$ for $k \in [2,\infty)_\mathbb{T}$. If we consider conditions (4.3)-(4.4) we find $T = 4$. Then $a(t)$, $b(t)$ satisfy condition (4.3), $a(t) \in \mathcal{R}^+$ and $q(t,x,y)$ satisfies the condition (4.4) for all $t \in \mathbb{T}$. Also, $q(t,x,y)$ is Lipschitz continuous in x and y for $L = E = 1/250$. Since $\|b(t)\| = \|(1/500)(-1)^{\ln t/\ln q}\| = (1/500) = \xi < 1$, then the condition (4.31) holds.

If we compute η, γ, and β, we have

$$\eta = \max_{t\in[t_0,\delta_+^T(t_0)]_\mathbb{T}} \left|\left(1 - e_{\ominus a(t)}\left(t, \delta_-^T(t)\right)\right)^{-1}\right| \cong 3,45, \qquad \alpha = \max_{t\in[t_0,\delta_+^T(t_0)]_\mathbb{T}} |q(t,0,0)| = \frac{1}{250} \tag{4.37}$$

$$\gamma = \max_{u\in[\delta_-^T(t),t]_\mathbb{T}} e_{\ominus a(t)}(t,u) \cong 1,5, \qquad \beta = \max_{t\in[t_0,\delta_+^T(t_0)]_\mathbb{T}} |r(t)| = \frac{11}{2500}. \tag{4.38}$$

If we take $G = 1$, then inequality (4.33) satisfies.

Let $x(t) \in P_T$. We show that $\|x(t)\| \le G = 1$. Integrate (1.2) from 1 to 4, we get

$$x(4) - x(1) = \int_1^4 \left[-a(t)x^\sigma(t) + b(t)x^\Delta\left(\frac{t}{k}\right)\frac{1}{k} + q\left(t, x(t), x\left(\frac{t}{k}\right)\right)\right]\Delta t. \tag{4.39}$$

Since $x(t) \in P_T$, then $x(4) = x(1)$ and so after integration by parts (23) becomes

$$\int_1^4 a(t)x^\sigma(t)\Delta t = \int_1^4 q\left(t, x(t), x\left(\frac{t}{k}\right)\right) - b^\Delta(t)x\left(\frac{t}{k}\right)\Delta t. \tag{4.40}$$

Claim 1. There exist $t^* \in [1,4]_\mathbb{T}$ such that $3a(t^*)x^\sigma(t^*) \le \int_1^4 a(t)x^\sigma(t)\Delta t$.

Suppose that the claim is false. Define $S := \int_1^4 a(t)x^\sigma(t)\Delta t$. Then there exists $\epsilon > 0$ such that

$$3a(t)x^\sigma(t) > S + \epsilon, \tag{4.41}$$

for all $t \in [1,4]_{\mathbb{T}}$. So,

$$S = \int_1^4 a(t)x^\sigma(t)\Delta t > \frac{1}{3}\int_1^4 (S+\epsilon)\Delta T = S+\epsilon. \tag{4.42}$$

That is, $S > S+\epsilon$, a contradiction.

As a consequence of the claim, we have

$$\begin{aligned} 3|a(t^*)||x^\sigma(t^*)| &\le \int_1^4 \left|q\left(t, x(t), x\left(\frac{t}{k}\right)\right)\right| + \left|b^\Delta(t)x\left(\frac{t}{k}\right)\right|\Delta t \\ &\le \int_1^4 [(L+E)\|x\| + \alpha + \delta\|x\|]\Delta t \\ &= 3\left[\left(\frac{2}{250}+\frac{1}{250}\right)\|x\| + \frac{1}{250}\right] = 3\left[\frac{3}{250}\|x\| + \frac{1}{250}\right], \end{aligned} \tag{4.43}$$

where $\delta = \max_{[1,4]_{\mathbb{T}}}|b^\Delta(t)| = 1/250$.

So, $|a(t^*)||x^\sigma(t^*)| \le (3/250)\|x\| + (1/250)$, which implies $|x^\sigma(t^*)| \le 20[(3/250)\|x\| + (1/250)]$. Since for all $t \in [1,4]_{\mathbb{T}}$,

$$x^\sigma(t) = x^\sigma(t^*) + \int_{t^*}^t x^\Delta(\sigma(s))\Delta s, \tag{4.44}$$

we have

$$|x^\sigma(t)| \le |x^\sigma(t^*)| + \int_1^t \left|x^\Delta(\sigma(s))\right|\Delta s \le 20\left[\frac{3}{250}\|x\| + \frac{1}{250}\right] + 3\left\|x^\Delta\right\|. \tag{4.45}$$

This implies that

$$\|x\| \le \frac{2}{19} + \frac{1500}{19}\left\|x^\Delta\right\|. \tag{4.46}$$

Taking the norm in (1.2) yields

$$\left\|x^\Delta\right\| \le \frac{(\|a\| + L + E)\|x\| + \alpha}{1 - \|b\|} = \frac{(1/5 + 2/250)\|x\| + 1/250}{1 - 1/500} = \frac{104\|x\| + 2}{250.499}. \tag{4.47}$$

Substitution of (4.47) into (4.46) yields that for all $x(t) \in P_T$, $\|x(t)\| \le G = 1$. Then by Theorem 4.5, (1.2) has a 4-periodic solution in shifts $\delta_\pm$.

In this example, if we take $q(t,x,y) = (\sin x + \arctan x)/1000t$, we have

$$\xi + \gamma\eta(\beta + L + E)\left(t_0 - \delta_-^T(t_0)\right) = \frac{1}{500} + \frac{96255}{2.10^6} < 1. \tag{4.48}$$

So, all the conditions of Theorem 4.6 are satisfied. Therefore, (1.2) has a unique 4-periodic solution in shifts $\delta_{\pm}$.

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