Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 696283, 21 pages
doi:10.1155/2012/696283

Research Article

Positive Solutions for Second-Order Singular Semipositone Differential Equations Involving Stieltjes Integral Conditions

Jiqiang Jiang,[1] Lishan Liu,[1] and Yonghong Wu[2]

[1] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China*
[2] *Department of Mathematics and Statistics, Curtin University of Technology, Perth, WA 6845, Australia*

Correspondence should be addressed to Lishan Liu, lls@mail.qfnu.edu.cn

Received 18 March 2012; Accepted 3 May 2012

Academic Editor: Shaoyong Lai

By means of the fixed point theory in cones, we investigate the existence of positive solutions for the following second-order singular differential equations with a negatively perturbed term: $-u''(t) = \lambda[f(t,u(t)) - q(t)]$, $0 < t < 1$, $\alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s)$, $\gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s)$, where $\lambda > 0$ is a parameter; $f : (0,1) \times (0,\infty) \to [0,\infty)$ is continuous; $f(t,x)$ may be singular at $t = 0$, $t = 1$, and $x = 0$, and the perturbed term $q : (0,1) \to [0,+\infty)$ is Lebesgue integrable and may have finitely many singularities in $(0,1)$, which implies that the nonlinear term may change sign.

1. Introduction

In this paper, we are concerned with positive solutions of the following second-order singular semipositone boundary value problem (BVP):

$$\begin{gathered}
-u''(t) = \lambda\left[f(t,u(t)) - q(t)\right], \quad 0 < t < 1, \\
\alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\
\gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s),
\end{gathered} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\alpha, \gamma \geq 0$, $\beta, \delta > 0$ are constants such that $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$, and the integrals in (1.1) are given by Stieltjes integral with a signed measure, that is, ξ, η are suitable functions of bounded variation, $q : (0,1) \to [0,+\infty)$ is a Lebesgue integral and may have finitely many singularities in $[0,1]$, $f : (0,1) \times (0,\infty) \to [0,\infty)$ is continuous, $f(t,x)$ may be singular at $t = 0$, $t = 1$, and $x = 0$.

Semipositone BVPs occur in models for steady-state diffusion with reactions [1], and interest in obtaining conditions for the existence of positive solutions of such problems has been ongoing for many years. For a small sample of such work, we refer the reader to the papers of Agarwal et al. [2, 3], Kosmatov [4], Lan [5–7], Liu [8], Ma et al. [9, 10], and Yao [11]. In [12], the second-order m-point BVP,

$$-u''(t) = \lambda f(t,u(t)), \quad t \in (0,1),$$
$$u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

is studied, where $a_i, b_i > 0$ ($i = 1,2,\ldots,m-2$), $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, λ is a positive parameter. By using the Krasnosel'skii fixed point theorem in cones, the authors established the conditions for the existence of at least one positive solution to (1.2), assuming that $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \sum_{i=1}^{m-2} b_i < 1$, $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous, and there exists $A > 0$ such that $f(t,u) \geq -A$ for $(t,u) \in [0,1] \times [0,+\infty)$. If the constant A is replaced by any continuous function $A(t)$ on $[0,1]$, f also has a lower bound and the existence results are still true.

Recently, Webb and Infante [13] studied arbitrary-order semipositone boundary value problems. The existence of multiple positive solutions is established via a Hammerstein integral equation of the form:

$$u(t) = \int_0^1 k(t,s) g(s) f(s,u(s)) ds, \tag{1.3}$$

where k is the corresponding Green function, $g \in L^1[0,1]$ is nonnegative and may have pointwise singularities, $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty)$ satisfies the Carathéodory conditions and $f(t,u) \geq -A$ for some $A > 0$. Although A is a constant, because of the term g, [13] includes nonlinearities that are bounded below by an integral function. It is worth mentioning that the boundary conditions cover both local and nonlocal types. Nonlocal boundary conditions are quite general, involving positive linear functionals on the space $C[0,1]$, given by Stieltjes integrals.

For the cases where the nonlinear term takes only nonnegative values, the existence of positive solutions of nonlinear boundary value problems with nonlocal boundary conditions, including multipoint and integral boundary conditions, has been extensively studied by many researchers in recent years [14–25]. Kong [17] studied the second-order singular BVP:

$$u''(t) + \lambda f(t,u(t)) = 0, \quad t \in (0,1),$$
$$u(0) = \int_0^1 u(s) d\xi(s), \tag{1.4}$$
$$u(1) = \int_0^1 u(s) d\eta(s),$$

where λ is a positive parameter, $f : (0,1)\times(0,+\infty) \to [0,+\infty)$ is continuous, $\xi(s)$ and $\eta(s)$ are nondecreasing, and the integrals in (1.4) are Riemann-Stieltjes integrals. Sufficient conditions are obtained for the existence and uniqueness of a positive solution by using the mixed monotone operator theory.

Inspired by the above work, the purpose of this paper is to establish the existence of positive solutions to BVP (1.1). By using the fixed point theorem on a cone, some new existence results are obtained for the case where the nonlinearity is allowed to be sign changing. We will address here that the problem tackled has several new features. Firstly, as $q \in L^1[0,1]$, the perturbed effect of q on f may be so large that the nonlinearity may tend to negative infinity at some singular points. Secondly, the BVP (1.1) possesses singularity, that is, the perturbed term q may has finitely many singularities in $[0,1]$, and $f(t,x)$ is allowed to be singular at $t = 0$, $t = 1$, and $x = 0$. Obviously, the problem in question is different from those in [2–13]. Thirdly, $\int_0^1 u(s)d\xi(s)$ and $\int_0^1 u(s)d\eta(s)$ denote the Stieltjes integrals where ξ,η are of bounded variation, that is, $d\xi$ and $d\eta$ can change sign. This includes the multipoint problems and integral problems as special cases.

The rest of this paper is organized as follows. In Section 2, we present some lemmas and preliminaries, and we transform the singularly perturbed problem (1.1) to an equivalent approximate problem by constructing a modified function. Section 3 gives the main results and their proofs. In Section 4, two examples are given to demonstrate the validity of our main results.

Let K be a cone in a Banach space E. For $0 < r < R < +\infty$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K : r \le \|x\| \le R\}$. The proof of the main theorem of this paper is based on the fixed point theory in cone. We list here one lemma [26, 27] which is needed in our following argument.

Lemma 1.1. *Let K be a positive cone in real Banach space E, $T : \overline{K}_{r,R} \to K$ is a completely continuous operator. If the following conditions hold:*

(i) $\|Tx\| \le \|x\|$ *for* $x \in \partial K_R$,

(ii) *there exists* $e \in \partial K_1$ *such that* $x \ne Tx + me$ *for any* $x \in \partial K_r$ *and* $m > 0$,

then, T has a fixed point in $\overline{K}_{r,R}$.

Remark 1.2. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively, then Lemma 1.1 is still true.

2. Preliminaries and Lemmas

Denote

$$\begin{gathered}
\phi_1(t) = \frac{1}{\rho}\big(\delta + \gamma(1-t)\big), \quad \phi_2(t) = \frac{1}{\rho}\big(\beta + \alpha t\big), \quad e(t) = G(t,t), \quad t \in [0,1], \\
k_1 = 1 - \int_0^1 \phi_1(t)d\xi(t), \quad k_2 = \int_0^1 \phi_2(t)d\xi(t), \quad k_3 = \int_0^1 \phi_1(t)d\eta(t), \\
k_4 = 1 - \int_0^1 \phi_2(t)d\eta(t), \quad k = k_1k_4 - k_2k_3, \quad \sigma = \frac{\rho}{(\alpha+\beta)(\gamma+\delta)},
\end{gathered} \tag{2.1}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}(\beta+\alpha s)(\delta+\gamma(1-t)), & 0\le s\le t\le 1,\\ (\beta+\alpha t)(\delta+\gamma(1-s)), & 0\le t\le s\le 1.\end{cases} \tag{2.2}$$

Obviously,

$$e(t)=\rho\phi_1(t)\phi_2(t)=\frac{1}{\rho}(\beta+\alpha t)(\delta+\gamma(1-t)),\quad t\in[0,1], \tag{2.3}$$

$$\sigma e(t)e(s)\le G(t,s)\le e(s)\quad (\text{or } e(t))\le\sigma^{-1},\quad t,s\in[0,1]. \tag{2.4}$$

Throughout this paper, we adopt the following assumptions.

(H_1) $k_1,k_4\in(0,1]$, $k_2\ge 0$, $k_3\ge 0$, $k>0$, and

$$\mathcal{G}_\xi(s)=\int_0^1 G(t,s)d\xi(t)\ge 0,\quad \mathcal{G}_\eta(s)=\int_0^1 G(t,s)d\eta(t)\ge 0,\quad s\in[0,1]. \tag{2.5}$$

(H_2) $q:(0,1)\to[0,+\infty)$ is a Lebesgue integral and $\int_0^1 q(t)dt>0$.

(H_3) For any $(t,x)\in(0,1)\times(0,+\infty)$,

$$0\le f(t,x)\le p(t)(g(x)+h(x)), \tag{2.6}$$

where $p\in C(0,1)$ with $p>0$ on $(0,1)$ and $\int_0^1 p(t)dt<+\infty$, $g>0$ is continuous and nonincreasing on $(0,+\infty)$, $h\ge 0$ is continuous on $[0,+\infty)$, and for any constant $r>0$,

$$0<\int_0^1 p(s)g(re(s))ds<+\infty. \tag{2.7}$$

Remark 2.1. If $d\xi$ and $d\eta$ are two positive measures, then the assumption (H_1) can be replaced by a weaker assumption:

(H_1') $k_1>0$, $k_4>0$, $k>0$.

Remark 2.2. It follows from (2.4) and (H_3) that

$$\int_0^1 e(s)p(s)g(re(s))ds\le\sigma^{-1}\int_0^1 p(s)g(re(s))ds<+\infty. \tag{2.8}$$

For convenience, in the rest of this paper, we define several constants as follows:

$$L_1 = 1 + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 d\xi(\tau) + \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 d\eta(\tau),$$
$$L_2 = \sigma\left[1 + \frac{k_4(\gamma+\delta)+k_3(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\xi(\tau) + \frac{k_2(\gamma+\delta)+k_1(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\eta(\tau)\right],$$
$$L_3 = 1 + \frac{k_4\delta+k_3\beta}{k\beta\delta}\int_0^1 e(\tau)d\xi(\tau) + \frac{k_2\delta+k_1\beta}{k\beta\delta}\int_0^1 e(\tau)d\eta(\tau). \tag{2.9}$$

Remark 2.3. If $x \in C[0,1] \cap C^2(0,1)$ satisfies (1.1), and $x(t) > 0$ for any $t \in (0,1)$, then we say that x is a $C[0,1] \cap C^2(0,1)$ positive solution of BVP (1.1).

Lemma 2.4. *Assume that* (H_1) *holds. Then, for any* $y \in L^1[0,1]$*, the problem,*

$$\begin{gathered} -u''(t) = y(t), \quad t \in (0,1), \\ \alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s), \end{gathered} \tag{2.10}$$

has a unique solution

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.11}$$

where

$$H(t,s) = G(t,s) + \frac{k_4\phi_1(t)+k_3\phi_2(t)}{k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{k_2\phi_1(t)+k_1\phi_2(t)}{k}\int_0^1 G(\tau,s)d\eta(\tau). \tag{2.12}$$

Proof. The proof is similar to Lemma 2.2 of [28], so we omit it. □

Lemma 2.5. *Suppose that* (H_1) *holds, then Green's function* $H(t,s)$ *defined by* (2.12) *possesses the following properties:*

(i) $H(t,s) \le L_1 e(s)$, $t,s \in [0,1]$;

(ii) $L_2 e(t)e(s) \le H(t,s) \le L_3 e(t)$, $t,s \in [0,1]$,

where L_1, L_2, *and* L_3 *are defined by* (2.9).

Proof. (i) It follows from (2.4) that

$$\begin{aligned}
H(t,s) &\le e(s) + \frac{k_4(\gamma+\delta) + k_3(\alpha+\beta)}{\rho k}\int_0^1 e(s)d\xi(\tau) \\
&\quad + \frac{k_2(\gamma+\delta) + k_1(\alpha+\beta)}{\rho k}\int_0^1 e(s)d\eta(\tau) \\
&= L_1 e(s), \quad t,s \in [0,1].
\end{aligned} \tag{2.13}$$

(ii) By the monotonicity of ϕ_1, ϕ_2 and the definition of $G(t,s)$, we have

$$\begin{aligned}
\frac{e(t)}{\alpha+\beta} &\le \phi_1(t) = \frac{e(t)}{\rho\phi_2(t)} = \frac{e(t)}{\alpha t+\beta} \le \frac{e(t)}{\beta}, \quad t \in [0,1], \\
\frac{e(t)}{\gamma+\delta} &\le \phi_2(t) = \frac{e(t)}{\rho\phi_1(t)} = \frac{e(t)}{\gamma(1-t)+\delta} \le \frac{e(t)}{\delta}, \quad t \in [0,1].
\end{aligned} \tag{2.14}$$

By (2.4) and the left-hand side of inequalities (2.14), we have

$$\begin{aligned}
H(t,s) &\ge \sigma e(t)e(s) + \sigma e(t)e(s)\frac{k_4/(\alpha+\beta) + k_3/(\gamma+\delta)}{k}\int_0^1 e(\tau)d\xi(\tau) \\
&\quad + \sigma e(t)e(s)\frac{k_2/(\alpha+\beta) + k_1/(\gamma+\delta)}{k}\int_0^1 e(\tau)d\eta(\tau) \\
&= \sigma e(t)e(s)\left[1 + \frac{k_4(\gamma+\delta) + k_3(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\xi(\tau) \right. \\
&\quad \left. + \frac{k_2(\gamma+\delta) + k_1(\alpha+\beta)}{\rho k}\int_0^1 e(\tau)d\eta(\tau)\right] \\
&= L_2 e(t)e(s), \quad t,s \in [0,1].
\end{aligned} \tag{2.15}$$

Similarly, by (2.4) and the right-hand side of inequalities (2.14), we have

$$\begin{aligned}
H(t,s) &= G(t,s) + \frac{k_4\phi_1(t) + k_3\phi_2(t)}{k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{k_2\phi_1(t) + k_1\phi_2(t)}{k}\int_0^1 G(\tau,s)d\eta(\tau) \\
&\le e(t) + e(t)\left[\frac{k_4/\beta + k_3/\delta}{k}\int_0^1 G(\tau,s)d\xi(\tau) + \frac{k_2/\beta + k_1/\delta}{k}\int_0^1 G(\tau,s)d\eta(\tau)\right] \\
&\le e(t) + e(t)\left[\frac{k_4\delta + k_3\beta}{k\beta\delta}\int_0^1 e(\tau)d\xi(\tau) + \frac{k_2\delta + k_1\beta}{k\beta\delta}\int_0^1 e(\tau)d\eta(\tau)\right]
\end{aligned}$$

$$\begin{aligned}
&= e(t)\left[1 + \frac{k_4\delta + k_3\beta}{k\beta\delta}\int_0^1 e(\tau)d\xi(\tau) + \frac{k_2\delta + k_1\beta}{k\beta\delta}\int_0^1 e(\tau)d\eta(\tau)\right] \\
&= L_3 e(t), \quad t, s \in [0,1].
\end{aligned} \tag{2.16}$$

The proof of Lemma 2.5 is completed. □

Lemma 2.6. *Suppose that* (H_1) *and* (H_2) *hold. Then, the boundary value problem,*

$$\begin{aligned}
-w''(t) &= 2\lambda p(t), \quad t \in (0,1), \\
\alpha w(0) - \beta w'(0) &= \int_0^1 w(s)d\xi(s), \\
\gamma w(1) + \delta w'(1) &= \int_0^1 w(s)d\eta(s),
\end{aligned} \tag{2.17}$$

has unique solution

$$w(t) = 2\lambda \int_0^1 H(t,s)p(s)ds, \tag{2.18}$$

which satisfies

$$w(t) \le 2\lambda L_3 e(t) \int_0^1 p(s)ds, \quad t \in [0,1]. \tag{2.19}$$

Proof. It follows from (2.11), Lemma 2.5, (H_1) and (H_2) that (2.18) and (2.19) hold. □

Let $X = C[0,1]$ be a real Banach space with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$ for $x \in X$. We let

$$K = \{x \in X : x \text{ is concave on } [0,1],\ x(t) \ge \Lambda e(t)\|x\| \text{ for } t \in [0,1]\}, \tag{2.20}$$

where $\Lambda = L_2/L_1$. Clearly, K is a cone of X.

For any $u \in X$, let us define a function $[\cdot]^+$:

$$[u(t)]^+ = \begin{cases} u(t), & u(t) \ge 0, \\ 0, & u(t) < 0. \end{cases} \tag{2.21}$$

Next, we consider the following approximate problem of (1.1):

$$\begin{gathered}
-x''(t) = \lambda\left[f\left(t, [x(t) - w(t)]^{+}\right) + q(t)\right], \quad t \in (0,1), \\
\alpha x(0) - \beta x'(0) = \int_0^1 x(s)d\xi(s), \\
\gamma x(1) + \delta x'(1) = \int_0^1 x(s)d\eta(s).
\end{gathered} \tag{2.22}$$

Lemma 2.7. *If $x \in C[0,1] \cap C^2(0,1)$ is a positive solution of problem* (2.22) *with $x(t) \geq w(t)$ for any $t \in [0,1]$, then $x - w$ is a positive solution of the singular semipositone differential equation* (1.1)*.*

Proof. If x is a positive solution of (2.22) such that $x(t) \geq w(t)$ for any $t \in [0,1]$, then from (2.22) and the definition of $[u(t)]^{+}$, we have

$$\begin{gathered}
-x''(t) = \lambda\left[f(t, x(t) - w(t)) + q(t)\right], \quad t \in (0,1), \\
\alpha x(0) - \beta x'(0) = \int_0^1 x(s)d\xi(s), \\
\gamma x(1) + \delta x'(1) = \int_0^1 x(s)d\eta(s).
\end{gathered} \tag{2.23}$$

Let $u = x - w$, then $u'' = x'' - w''$, which implies that

$$-x'' = -u'' - w'' = -u'' + 2\lambda q(t). \tag{2.24}$$

Thus, (2.23) becomes

$$\begin{gathered}
-u''(t) = \lambda\left[f(t, u(t)) - q(t)\right], \quad t \in (0,1), \\
\alpha u(0) - \beta u'(0) = \int_0^1 u(s)d\xi(s), \\
\gamma u(1) + \delta u'(1) = \int_0^1 u(s)d\eta(s),
\end{gathered} \tag{2.25}$$

that is, $x - w$ is a positive solution of (1.1). The proof is complete. □

To overcome singularity, we consider the following approximate problem of (2.22):

$$\begin{gathered}
-x''(t) = \lambda\left[f\left(t, [x(t) - w(t)]^{+} + n^{-1}\right) + q(t)\right], \quad t \in (0,1), \\
\alpha x(0) - \beta x'(0) = \int_0^1 x(s)d\xi(s), \\
\gamma x(1) + \delta x'(1) = \int_0^1 x(s)d\eta(s),
\end{gathered} \tag{2.26}$$

where n is a positive integer. For any $n \in \mathbb{N}$, let us define a nonlinear integral operator $T_n^\lambda : K \to X$ as follows:

$$T_n^\lambda x(t) = \lambda \int_0^1 H(t,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds. \tag{2.27}$$

It is obvious that solving (2.26) in $C[0,1] \cap C^2(0,1)$ is equivalent to solving the fixed point equation $T_n^\lambda x = x$ in the Banach space $C[0,1]$.

Lemma 2.8. *Assume that (H_1)–(H_3) hold, then for each $n \in \mathbb{N}$, $\lambda > 0$, $R > r \geq 4\lambda L_3 \Lambda^{-1} \int_0^1 q(s)ds$, $T_n^\lambda : \overline{K}_{r,R} \to K$ is a completely continuous operator.*

Proof. Let $n \in \mathbb{N}$ be fixed. For any $x \in K$, by (2.27) we have

$$\begin{aligned} \left(T_n^\lambda x\right)''(t) &= -\lambda\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] \leq 0, \\ T_n^\lambda x(0) &= \lambda \int_0^1 H(0,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \geq 0, \\ T_n^\lambda x(1) &= \lambda \int_0^1 H(1,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \geq 0, \end{aligned} \tag{2.28}$$

which implies that T_n^λ is nonnegative and concave on $[0,1]$. For any $x \in K$ and $t \in [0,1]$, it follows from Lemma 2.5 that

$$\begin{aligned} T_n^\lambda x(t) &= \lambda \int_0^1 H(t,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\ &\leq \lambda L_1 \int_0^1 e(s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds. \end{aligned} \tag{2.29}$$

Thus,

$$\left\|T_n^\lambda x\right\| \leq \lambda L_1 \int_0^1 e(s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds. \tag{2.30}$$

On the other hand, from Lemma 2.5, we also obtain

$$\begin{aligned} T_n^\lambda x(t) &= \lambda \int_0^1 H(t,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\ &\geq \lambda L_2 e(t) \int_0^1 e(s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds. \end{aligned} \tag{2.31}$$

So,

$$T_n^\lambda x(t) \geq \Lambda e(t) \left\|T_n^\lambda x\right\|, \quad t \in [0,1]. \tag{2.32}$$

This yields that $T_n^\lambda(K) \subset K$.

Next, we prove that $T_n^\lambda : \overline{K}_{r,R} \to K$ is completely continuous. Suppose $x_m \in \overline{K}_{r,R}$ and $x_0 \in \overline{K}_{r,R}$ with $\|x_m - x_0\| \to 0\,(m \to \infty)$. Notice that

$$\begin{aligned}
&\left|[x_m(s)-w(s)]^+ - [x_0(s)-w(s)]^+\right| \\
&\quad= \left|\frac{|x_m(s)-w(s)|+x_m(s)-w(s)}{2} - \frac{|x_0(s)-w(s)|+x_0(s)-w(s)}{2}\right| \\
&\quad= \left|\frac{|x_m(s)-w(s)|-|x_0(s)-w(s)|}{2} + \frac{x_m(s)-x_0(s)}{2}\right| \\
&\quad\le |x_m(s)-x_0(s)|.
\end{aligned} \tag{2.33}$$

This, together with the continuity of f, implies

$$\begin{aligned}
&\left|f\left(s,[x_m(s)-w(s)]^+ + n^{-1}\right) + q(s) - \left[f\left(s,[x_0(s)-w(s)]^+ + n^{-1}\right) + q(s)\right]\right| \\
&\quad= \left|f\left(s,[x_m(s)-w(s)]^+ + n^{-1}\right) - f\left(s,[x_0(s)-w(s)]^+ + n^{-1}\right)\right| \longrightarrow 0, \quad m \longrightarrow \infty.
\end{aligned} \tag{2.34}$$

Using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\left\|T_n^\lambda x_m - T_n^\lambda x_0\right\| \\
&\le \lambda L_1 \int_0^1 e(s)\left|f\left(s,[x_m(s)-w(s)]^+ + n^{-1}\right) - f\left(s,[x_0(s)-w(s)]^+ + n^{-1}\right)\right| \quad ds \longrightarrow 0,\ m \longrightarrow \infty.
\end{aligned} \tag{2.35}$$

So, $T_n^\lambda : \overline{K}_{r,R} \to K$ is continuous.

Let $B \subset \overline{K}_{r,R}$ be any bounded set, then for any $x \in B$, we have $x \in K$, $r \le \|x\| \le R$. Therefore, we have

$$\begin{aligned}
&[x(t)-w(t)]^+ \le x(t) \le \|x\| \le R, \quad t \in [0,1], \\
&x(t)-w(t) \ge x(t) - 2\lambda L_3 e(t)\int_0^1 q(s)ds \ge x(t) - 2\lambda L_3 \frac{x(t)}{\Lambda r}\int_0^1 q(s)ds \\
&\qquad \ge \frac{1}{2}x(t) \ge \frac{1}{2}r\Lambda e(t) > 0, \quad t \in [0,1].
\end{aligned} \tag{2.36}$$

By (H_3), we have

$$L_r := \int_0^1 p(s) g\left(\frac{1}{2}\Lambda r e(s)\right) ds < +\infty. \tag{2.37}$$

It is easy to show that $T_n^\lambda(B)$ is uniformly bounded. In order to show that T_n^λ is a compact operator, we only need to show that $T_n^\lambda(B)$ is equicontinuous. By the continuity of $H(t,s)$ on

$[0,1]\times[0,1]$, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for any $t_1, t_2, s \in [0,1]$ and $|t_1 - t_2| < \delta_1$, we have

$$|H(t_1, s) - H(t_2, s)| < \varepsilon. \tag{2.38}$$

By (2.36)–(2.37), and (2.27), we have

$$\begin{aligned}
\left|T_n^\lambda x(t_1) - T_n^\lambda x(t_2)\right| &\leq \lambda \int_0^1 |H(t_1, s) - H(t_2, s)| \left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\
&< \varepsilon\lambda \int_0^1 \left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\
&\leq \varepsilon\lambda \int_0^1 \Big[p(s)\Big(g\Big([x(s) - w(s)]^+ + n^{-1}\Big) + h\Big([x(s) - w(s)]^+ + n^{-1}\Big)\Big) \\
&\qquad + q(s)\Big] ds \\
&\leq \varepsilon\lambda \left[L_r + (1 + h^*(R)) \int_0^1 [p(s) + q(s)] ds\right],
\end{aligned} \tag{2.39}$$

where

$$h^*(r) = \max_{y \in [0, 1+r]} h(y). \tag{2.40}$$

This means that $T_n^\lambda(B)$ is equicontinuous. By the Arzela-Ascoli theorem, $T_n^\lambda(B)$ is a relatively compact set. Now since λ and n are given arbitrarily, the conclusion of this lemma is valid. □

3. Main Results

Theorem 3.1. *Assume that conditions (H_1)–(H_3) are satisfied. Further assume that the following condition holds.*

(H_4) *There exists an interval $[a,b] \subset (0,1)$ such that*

$$\liminf_{u \to +\infty} \min_{t \in [a,b]} \frac{f(t,u)}{u} = +\infty. \tag{3.1}$$

Then, there exists $\lambda^* > 0$ such that the BVP (1.1) has at least one positive solution $u(t) \in C[0,1] \cap C^2(0,1)$ provided $\lambda \in (0, \lambda^*)$. Furthermore, the solution also satisfies $u(t) \geq \tilde{l}e(t)$ for some positive constant $\tilde{l}$.

Proof. Take $r > 4L_3\Lambda^{-1}\int_0^1 q(s)ds$. Let

$$\lambda^* = \min\left\{1, \frac{r}{2L_1\int_0^1 e(s)p(s)g((1/2)r\Lambda e(s))ds}, \frac{r}{2L_1(h^*(r)+1)\int_0^1 e(s)\left[p(s)+q(s)\right]ds}\right\}, \tag{3.2}$$

where h^* is defined by (2.40). For any $\lambda \in (0, \lambda^*)$, $x \in \partial K_r$, noticing that $\lambda^* \leq 1$, we have

$$\begin{aligned} &[x(t)-w(t)]^+ \leq x(t) \leq \|x\| \leq r, \quad t \in [0,1], \\ &x(t)-w(t) \geq x(t) - 2\lambda L_3 e(t)\int_0^1 q(s)ds \geq x(t) - 2L_3 e(t)\int_0^1 q(s)ds \\ &\qquad \geq x(t) - 2L_3\frac{x(t)}{\Lambda r}\int_0^1 q(s)ds \geq \frac{1}{2}x(t) \geq \frac{1}{2}r\Lambda e(t) > 0, \quad t \in [0,1]. \end{aligned} \tag{3.3}$$

For any $\lambda \in (0, \lambda^*)$, by (3.3), we have

$$\begin{aligned} \left|T_n^\lambda x(t)\right| &= \lambda\int_0^1 H(t,s)\left[f\left(s, [x(s)-w(s)]^+ + n^{-1}\right) + q(s)\right]ds \\ &\leq \lambda L_1\int_0^1 e(s)\left[p(s)\left(g\left([x(s)-w(s)]^+ + n^{-1}\right) + h\left([x(s)-w(s)]^+ + n^{-1}\right)\right) + q(s)\right]ds \\ &\leq \lambda L_1\int_0^1 e(s)\left[p(s)\left(g\left(\frac{1}{2}r\Lambda e(s)\right) + h^*(r)\right) + q(s)\right]ds \\ &\leq \lambda L_1\int_0^1 e(s)p(s)g\left(\frac{1}{2}r\Lambda e(s)\right)ds + \lambda L_1(h^*(r)+1)\int_0^1 e(s)\left[p(s)+q(s)\right]ds \\ &\leq \frac{r}{2} + \frac{r}{2} = r, \end{aligned} \tag{3.4}$$

which means that

$$\left\|T_n^\lambda x\right\| \leq \|x\|, \quad x \in \partial K_r. \tag{3.5}$$

On the other hand, choose a real number $M > 0$ such that $\lambda M L_2 l_1^2 \Lambda\int_a^b e(s)ds > 2$, where $l_1 = \min_{0\leq t\leq 1} e(t) > 0$, L_2 is defined by (2.9). By (H_4), there exists $N > 0$ such that for any $t \in [a,b]$, we have

$$f(t,u) \geq Mu, \quad u \geq N. \tag{3.6}$$

Take $R > \max\{r, 2N/l_1\Lambda\}$. Next, we take $\varphi_1 \equiv 1 \in \partial K_1 = \{x \in K : \|x\| = 1\}$, and for any $x \in \partial K_R$, $m > 0$, $n \in \mathbb{N}$, we will show

$$x \neq T_n^{\lambda} x + m\varphi_1. \tag{3.7}$$

Otherwise, there exist $x_0 \in \partial K_R$ and $m_0 > 0$ such that

$$x_0 = T_n^{\lambda} x_0 + m_0\varphi_1. \tag{3.8}$$

From $x_0 \in \partial K_R$, we know that $\|x_0\| = R$. Then, for $t \in [a, b]$, we have

$$\begin{aligned} x_0(t) - w(t) &\geq x_0(t) - 2\lambda L_3 e(t) \int_0^1 q(s)ds \geq x_0(t) - 2L_3 \frac{x_0(t)}{\Lambda R} \int_0^1 q(s)ds \\ &\geq \frac{1}{2} x_0(t) \geq \frac{1}{2} R\Lambda e(t) \geq \frac{l_1 R\Lambda}{2} \geq N > 0. \end{aligned} \tag{3.9}$$

So, by (3.6), (3.9), we have

$$\begin{aligned} x_0(t) &= \lambda \int_0^1 H(t,s)\left[f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right) + q(s)\right]ds + m_0 \\ &\geq \lambda L_2 e(t) \int_0^1 e(s) f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right)ds + m_0 \\ &\geq \lambda L_2 e(t) \int_a^b e(s) f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right)ds + m_0 \\ &\geq \lambda M L_2 e(t) \int_a^b e(s) \left([x_0(s) - w(s)]^+ + n^{-1}\right)ds + m_0 \\ &\geq \frac{1}{2} \lambda M L_2 l_1^2 \Lambda R \int_a^b e(s)ds + m_0 \\ &\geq R + m_0 > R. \end{aligned} \tag{3.10}$$

This implies that $R > R$, which is a contradiction. This yields that (3.7) holds. By (3.5), (3.7), and Lemma 1.1, for any $n \in \mathbb{N}$ and $\lambda \in (0, \lambda^*)$, we obtain that T_n^{λ} has a fixed point x_n in $\overline{K}_{r,R}$.

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence of solutions of the boundary value problems (2.26). It is easy to see that they are uniformly bounded. Next, we show that $\{x_n\}_{n=1}^{\infty}$ are equicontinuous on $[0, 1]$. From $x_n \in \overline{K}_{r,R}$, we know that

$$\begin{aligned} &[x_n(t) - w(t)]^+ \leq x_n(t) \leq \|x_n\| \leq R, \quad t \in [0,1], \\ &x_n(t) - w(t) \geq x_n(t) - 2\lambda L_3 e(t) \int_0^1 q(s)ds \geq x_n(t) - 2\lambda L_3 \frac{x_n(t)}{\Lambda \|x_n\|} \int_0^1 q(s)ds \\ &\qquad \geq \frac{1}{2} x_n(t) \geq \frac{1}{2} \Lambda e(t) \|x_n\| \geq \frac{1}{2} \Lambda r e(t) > 0, \quad t \in [0,1]. \end{aligned} \tag{3.11}$$

For any $\varepsilon > 0$, by the continuity of $H(t,s)$ in $[0,1] \times [0,1]$, there exists $\delta_2 > 0$ such that for any $t_1, t_2, s \in [0,1]$ and $|t_1 - t_2| < \delta_2$, we have

$$|H(t_1,s) - H(t_2,s)| < \varepsilon. \tag{3.12}$$

This, combined with (2.11) and (2.37), implies that for any $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta_2$, we have

$$\begin{aligned}
|x_n(t_1) - x_n(t_2)| &\leq \int_0^1 |H(t_1,s) - H(t_2,s)| \left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\
&< \varepsilon \int_0^1 \left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right] ds \\
&\leq \varepsilon \int_0^1 \left[p(s)\left(g\left([x(s) - w(s)]^+ + n^{-1}\right) + h\left([x(s) - w(s)]^+ + n^{-1}\right)\right) + q(s)\right] ds \\
&\leq \varepsilon \left[L_r + (1 + h^*(R)) \int_0^1 (p(s) + q(s)) ds\right].
\end{aligned} \tag{3.13}$$

By the Ascoli-Arzela theorem, the sequence $\{x_n\}_{n=1}^{\infty}$ has a subsequence being uniformly convergent on $[0,1]$. Without loss of generality, we still assume that $\{x_n\}_{n=1}^{\infty}$ itself uniformly converges to x on $[0,1]$. Since $\{x_n\}_{n=1}^{\infty} \in \overline{K}_{r,R} \subset K$, we have $x_n \geq 0$. By (2.26), we have

$$\begin{aligned}
x_n(t) = x_n\left(\frac{1}{2}\right) &+ \left(t - \frac{1}{2}\right) x_n'\left(\frac{1}{2}\right) \\
&- \lambda \int_{1/2}^t ds \int_{1/2}^s \left[f\left(\tau, x_n(\tau) - w(\tau) + n^{-1}\right) + q(\tau)\right] d\tau, \quad t \in (0,1).
\end{aligned} \tag{3.14}$$

From (3.14), we know that $\{x_n'(1/2)\}_{n=1}^{\infty}$ is bounded sets. Without loss of generality, we may assume $x_n'(1/2) \to c_1$ as $n \to \infty$. Then, by (3.14) and the Lebesgue dominated convergence theorem, we have

$$x(t) = x\left(\frac{1}{2}\right) + c_1\left(t - \frac{1}{2}\right) - \lambda \int_{1/2}^t ds \int_{1/2}^s [f(\tau, x(\tau) - w(\tau)) + q(\tau)] d\tau, \quad t \in (0,1). \tag{3.15}$$

By (3.15), direct computation shows that

$$-x''(t) = \lambda [f(t, x(t) - w(t)) + q(t)], \quad 0 < t < 1. \tag{3.16}$$

On the other hand, letting $n \to \infty$ in the following boundary conditions:

$$\alpha x_n(0) - \beta x_n'(0) = \int_0^1 x_n(s)d\xi(s),$$
$$\gamma x_n(1) + \delta x_n'(1) = \int_0^1 x_n(s)d\eta(s), \tag{3.17}$$

we deduce that x is a positive solution of BVP (2.22).

Let $u(t) = x(t) - w(t)$ and $\tilde{l} = (1/2)\Lambda r$. By (3.11) and the convergence of sequence $\{x_n\}_{n=1}^{\infty}$, we have $u(t) \geq \tilde{l}e(t) > 0$, $t \in [0,1]$. It then follows from Lemma 2.7 that BVP (1.1) has at least one positive solution u satisfying $u \geq \tilde{l}e(t)$ for any $t \in [0,1]$. The proof is completed. □

Theorem 3.2. *Assume that conditions* (H_1)–(H_3) *are satisfied. In addition, assume that the following condition holds.*

(H_5) *There exists an interval* $[c,d] \subset (0,1)$ *such that*

$$\liminf_{u\to+\infty} \min_{t\in[c,d]} f(t,u) > \frac{4L_3\int_0^1 q(s)ds}{\Lambda L_2 l_2 \int_c^d e(s)ds}, \tag{3.18}$$

where $l_2 = \min_{c\leq t\leq d} e(t)$ *and*

$$\lim_{u\to+\infty} \frac{h(u)}{u} = 0. \tag{3.19}$$

Then there exists $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one positive solution* $u(t) \in C[0,1] \cap C^2(0,1)$ *provided* $\lambda \in (\lambda^*, +\infty)$. *Furthermore, the solution also satisfies* $u(t) \geq \tilde{l}e(t)$ *for some positive constant* $\tilde{l}$.

Proof. By (3.18), there exists $N_0 > 0$ such that, for any $t \in [c,d]$, $u \geq N_0$, we have

$$f(t,u) \geq \frac{4L_3\int_0^1 q(s)ds}{\Lambda L_2 l_2 \int_c^d e(s)ds}. \tag{3.20}$$

Choose $\lambda^* = N_0/2l_2L_3\int_0^1 q(s)ds$. Let $r = 4\lambda L_3\Lambda^{-1}\int_0^1 q(s)ds$ as $\lambda > \lambda^*$. Next, we take $\varphi_1 \equiv 1 \in \partial K_1 = \{x \in K : \|x\| = 1\}$, and for any $x \in \partial K_r$, $m > 0$, $n \in \mathbb{N}$, we will show that

$$x \neq T_n^\lambda x + m\varphi_1. \tag{3.21}$$

Otherwise, there exist $x_0 \in \partial K_r$ and $m_0 > 0$ such that

$$x_0 = T_n^\lambda x_0 + m_0\varphi_1. \tag{3.22}$$

From $x_0 \in \partial K_r$, we know that $\|x_0\| = r$, and

$$x_0(t) - w(t) \geq \Lambda r e(t) - 2\lambda L_3 e(t) \int_0^1 q(s)ds = 2\lambda L_3 e(t) \int_0^1 q(s)ds \geq \frac{N_0}{l_2} e(t). \tag{3.23}$$

So, we have $x_0(t) - w(t) \geq N_0$ on $t \in [c, d]$, $x_0 \in \partial K_r$. Then, by (3.20) we have

$$\begin{aligned} x_0(t) &= \lambda \int_0^1 H(t,s) \left[f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right) + q(s) \right] ds + m_0 \\ &\geq \lambda L_2 e(t) \int_0^1 e(s) f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right) ds + m_0 \\ &\geq \lambda L_2 e(t) \int_c^d e(s) f\left(s, [x_0(s) - w(s)]^+ + n^{-1}\right) ds + m_0 \\ &\geq \lambda L_2 e(t) \frac{4L_3 \int_0^1 q(s)ds}{\Lambda L_2 l_2 \int_c^d e(s)ds} \int_c^d e(s)ds + m_0 \\ &\geq 4\lambda \Lambda^{-1} L_3 \int_0^1 q(s)ds + m_0 \\ &= r + m_0 > r. \end{aligned} \tag{3.24}$$

This implies that $r > r$, which is a contradiction. This yields that (3.21) holds.

On the other hand, by (3.19) and the continuity of $h(u)$ on $[0, +\infty)$, we have

$$\lim_{u \to +\infty} \frac{h^*(u)}{u} = 0, \tag{3.25}$$

where $h^*(u)$ is defined by (2.40). For

$$\epsilon = \left[4\lambda L_1 \int_0^1 e(s)[p(s) + q(s)]ds \right]^{-1}, \tag{3.26}$$

there exists $M_0 > 0$ such that when $x > M_0$, for any $0 \leq y \leq x$, we have $h(y) \leq \epsilon x$. Take

$$R > \max\left\{ 2, r, M_0, 2\lambda L_1 \int_0^1 e(s)p(s)g(\Lambda e(s))ds \right\}. \tag{3.27}$$

Then, for any $x \in \partial K_R, t \in [0,1]$, we have

$$\begin{aligned}
&[x(t) - w(t)]^+ \leq x(t) \leq \|x\| \leq R, \\
&x(t) - w(t) \geq x(t) - 2\lambda L_3 e(t) \int_0^1 q(s)ds \geq x(t) - 2\lambda L_3 \frac{x(t)}{\Lambda R} \int_0^1 q(s)ds \\
&\geq \frac{1}{2}x(t) \geq \frac{1}{2}R\Lambda e(t) > \Lambda e(t) > 0, \quad t \in [0,1].
\end{aligned} \tag{3.28}$$

It follows from (3.19) and (3.28) that

$$\begin{aligned}
\left|T_n^\lambda x(t)\right| &= \lambda \int_0^1 H(t,s)\left[f\left(s, [x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right]ds \\
&\leq \lambda L_1 \int_0^1 e(s)p(s)g\left([x(s) - w(s)]^+ + n^{-1}\right)ds \\
&+ \lambda L_1 \int_0^1 e(s)\left[p(s)h\left([x(s) - w(s)]^+ + n^{-1}\right) + q(s)\right]ds \\
&\leq \lambda L_1 \int_0^1 e(s)p(s)g(\Lambda e(s))ds + \lambda L_1 \int_0^1 e(s)\left[p(s)\epsilon(R+1) + q(s)\right]ds \\
&\leq \lambda L_1 \int_0^1 e(s)p(s)g(\Lambda e(s))ds + \epsilon\lambda L_1(R+2) \int_0^1 e(s)\left[p(s) + q(s)\right]ds \\
&\leq \frac{R}{2} + \frac{R}{2} = R,
\end{aligned} \tag{3.29}$$

which means that

$$\left\|T_n^\lambda x\right\| \leq \|x\|, \quad x \in \partial K_R. \tag{3.30}$$

By (3.21), (3.30), and Lemma 1.1, for any $n \in \mathbb{N}$ and $\lambda > \lambda^*$, we obtain that T_n^λ has a fixed point x_n in $\overline{K}_{r,R}$ satisfying $r \leq \|x_n\| \leq R$. The rest of proof is similar to Theorem 3.1. The proof is complete. □

Remark 3.3. From the proof of Theorem 3.2, we can see that if (H_5) is replaced by the following condition.

(H_5') There exists an interval $[c,d] \subset (0,1)$ such that

$$\liminf_{u \to +\infty} \min_{t \in [c,d]} f(t,u) = +\infty, \qquad \lim_{u \to +\infty} \frac{h(u)}{u} = 0, \tag{3.31}$$

then, the conclusion of Theorem 3.2 is still true.

4. Applications

In this section, we construct two examples to demonstrate the application of our main results.

Example 4.1. Consider the following 4-point boundary value problem:

$$\begin{gathered}
-u''(t) = \lambda\left[\frac{1}{\sqrt{t(1-t)}}\left(\frac{1}{u^2}+u^2+1\right)-q(t)\right], \quad 0<t<1,\\
u(0)-u'(0)=\frac{1}{4}u\left(\frac{1}{3}\right)+\frac{1}{9}u\left(\frac{2}{3}\right),\\
u(1)+u'(1)=\frac{3}{8}u\left(\frac{1}{3}\right)+u\left(\frac{2}{3}\right),
\end{gathered} \tag{4.1}$$

where $\lambda > 0$ is a parameter and

$$q(t) = \frac{1}{4+3\sqrt[3]{4}}\left[\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{1-t}}+\frac{1}{\sqrt[3]{(t-1/2)^2}}\right]. \tag{4.2}$$

The BVP (4.1) can be regarded as a boundary value problem of the form of (1.1). In this situation, $\alpha=\beta=\gamma=\delta=1$ and

$$\xi(s)=\begin{cases}0, & s\in\left[0,\dfrac{1}{3}\right),\\ \dfrac{1}{4}, & s\in\left[\dfrac{1}{3},\dfrac{2}{3}\right),\\ \dfrac{13}{36}, & s\in\left[\dfrac{2}{3},1\right],\end{cases}\qquad \eta(s)=\begin{cases}0, & s\in\left[0,\dfrac{1}{3}\right),\\ \dfrac{3}{8}, & s\in\left[\dfrac{1}{3},\dfrac{2}{3}\right),\\ \dfrac{11}{8}, & s\in\left[\dfrac{2}{3},1\right].\end{cases} \tag{4.3}$$

Let

$$f(t,x)=\frac{1}{\sqrt{t(1-t)}}\left(\frac{1}{x^2}+x^2+1\right), \quad \text{for } (t,x)\in(0,1)\times(0,+\infty) \tag{4.4}$$

and let $p(t)=1/\sqrt{t(1-t)}$, $g(x)=1/x^2$, $h(x)=x^2+1$. By direct calculation, we have $\int_0^1 p(t)dt = \pi$, $\int_0^1 q(t)dt=1$, and

$$\begin{gathered}
\rho=3,\quad \sigma=\frac{3}{4},\quad \phi_1(t)=\frac{2-t}{3},\quad \phi_2(t)=\frac{1+t}{3},\quad e(t)=\frac{1}{3}(2-t)(1+t),\\
k_1=\frac{263}{324},\quad k_2=\frac{14}{81},\quad k_3=\frac{47}{72},\quad k_4=\frac{5}{18},\quad k=\frac{73}{648},\\
L_1=11,\quad L_2=\frac{227}{36},\quad L_3=\frac{109}{9}.
\end{gathered} \tag{4.5}$$

Clearly, the conditions (H_1)–(H_3) hold. Taking $t \in [1/4, 3/4] \subset [0,1]$, we have

$$\liminf_{x\to+\infty} \min_{t\in[1/4,3/4]} \frac{f(t,x)}{x} = \liminf_{x\to+\infty} \min_{t\in[1/4,3/4]} \frac{\left(1/\sqrt{t(1-t)}\right)(1/x^2 + x^2 + 1)}{x} = +\infty. \tag{4.6}$$

Thus (H_4) also holds. Consequently, by Theorem 3.1, we infer that the singular BVP (4.1) has at least one positive solution provided λ is small enough.

Example 4.2. Consider the following problem:

$$\begin{gathered} -u''(t) = \lambda\left[\frac{1}{\sqrt{t(1-t)}}\left(\frac{1}{u^2} + \sqrt{u} + 1\right) - \frac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}}\right], \quad 0 < t < 1, \\ u(0) - u'(0) = -\int_0^1 u(t)\cos 2\pi t dt, \\ u(1) + u'(1) = \frac{1}{5}u\left(\frac{1}{4}\right) + u\left(\frac{3}{4}\right), \end{gathered} \tag{4.7}$$

where $\lambda > 0$ is a parameter. Let

$$\begin{gathered} f(t,x) = \frac{1}{\sqrt{t(1-t)}}\left(\frac{1}{x^2} + \sqrt{x} + 1\right), \quad (t,x) \in (0,1) \times (0,+\infty), \\ q(t) = \frac{\sqrt{2}}{\sqrt[4]{t^3(1-t)}}, \quad p(t) = \frac{1}{\sqrt{t(1-t)}}, \quad g(x) = \frac{1}{x^2}, \quad h(x) = \sqrt{x} + 1. \end{gathered} \tag{4.8}$$

Then, $\int_0^1 p(t)dt = \pi$, $\int_0^1 q(t)dt = 2\pi$. Here, $d\xi(t) = -\cos 2\pi t\, dt$, so the measure $d\xi$ changes sign on $[0,1]$. By direct calculation, we have

$$\begin{gathered} k_1 = 1, \quad k_2 = 0, \quad k_3 = \frac{8}{15}, \quad k_4 = \frac{1}{3}, \quad k = \frac{1}{3}, \\ \mathcal{G}_\xi(s) = \frac{1}{4\pi^2}(1 - \cos 2\pi s) \ge 0, \quad \mathcal{G}_\eta(s) \ge 0, \end{gathered} \tag{4.9}$$

where

$$\begin{gathered} \mathcal{G}_\eta(s) = \begin{cases} \dfrac{1}{5}G_2\left(\dfrac{1}{4}, s\right) + G_2\left(\dfrac{3}{4}, s\right), & 0 \le s < \dfrac{1}{4}, \\ \dfrac{1}{5}G_1\left(\dfrac{1}{4}, s\right) + G_2\left(\dfrac{3}{4}, s\right), & \dfrac{1}{4} \le s \le \dfrac{3}{4}, \\ \dfrac{1}{5}G_1\left(\dfrac{1}{4}, s\right) + G_1\left(\dfrac{3}{4}, s\right), & \dfrac{3}{4} < s \le 1, \end{cases} \\ G(t,s) = \begin{cases} G_1(t,s) = \dfrac{(2-s)(1+t)}{3}, & 0 \le t \le s \le 1, \\ G_2(t,s) = \dfrac{(2-t)(1+s)}{3}, & 0 \le s \le t \le 1. \end{cases} \end{gathered} \tag{4.10}$$

Taking $t \in [1/4, 3/4] \subset [0,1]$, we have

$$\liminf_{x \to +\infty} \min_{t\in[1/4,3/4]} f(t,x) = \liminf_{x \to +\infty} \min_{t\in[1/4,3/4]} \frac{1}{\sqrt{t(1-t)}}\left(\frac{1}{x^2} + \sqrt{x} + 1\right) = +\infty,$$
$$\lim_{x \to +\infty} \frac{h(x)}{x} = \lim_{x \to +\infty} \frac{\sqrt{x}+1}{x} = 0. \tag{4.11}$$

So all assumptions of Theorem 3.2 are satisfied. By Theorem 3.2, we know that BVP (4.7) has at least one positive solution provided λ is large enough.

Acknowledgments

J. Jiang and L. Liu were supported financially by the National Natural Science Foundation of China (11071141, 11126231), the Natural Science Foundation of Shandong Province of China (ZR2011AQ008), and a Project of Shandong Province Higher Educational Science and Technology Program (J11LA06). Y. Wu was supported financially by the Australia Research Council through an ARC Discovery Project Grant.

References

[1] R. Aris, *Introduction to the Analysis of Chemical Reactors*, Prentice Hall, New Jersey, NJ, USA, 1965.

[2] R. P. Agarwal, S. R. Grace, and D. O'Regan, "Existence of positive solutions to semipositone Fredholm integral equations," *Funkcialaj Ekvacioj*, vol. 45, no. 2, pp. 223–235, 2002.

[3] R. P. Agarwal and D. O'Regan, "A note on existence of nonnegative solutions to singular semipositone problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 36, pp. 615–622, 1999.

[4] N. Kosmatov, "Semipositone m-point boundary-value problems," *Electronic Journal of Differential Equations*, vol. 2004, no. 119, pp. 1–7, 2004.

[5] K. Q. Lan, "Multiple positive solutions of semi-positone Sturm-Liouville boundary value problems," *The Bulletin of the London Mathematical Society*, vol. 38, no. 2, pp. 283–293, 2006.

[6] K. Q. Lan, "Positive solutions of semi-positone Hammerstein integral equations and applications," *Communications on Pure and Applied Analysis*, vol. 6, no. 2, pp. 441–451, 2007.

[7] K. Q. Lan, "Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. 5979–5993, 2009.

[8] Y. Liu, "Twin solutions to singular semipositone problems," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 248–260, 2003.

[9] R. Ma, "Existence of positive solutions for superlinear semipositone m-point boundary-value problems," *Proceedings of the Edinburgh Mathematical Society*, vol. 46, no. 2, pp. 279–292, 2003.

[10] R. Y. Ma and Q. Z. Ma, "Positive solutions for semipositone m-point boundary-value problems," *Acta Mathematica Sinica (English Series)*, vol. 20, no. 2, pp. 273–282, 2004.

[11] Q. Yao, "An existence theorem of a positive solution to a semipositone Sturm-Liouville boundary value problem," *Applied Mathematics Letters*, vol. 23, no. 12, pp. 1401–1406, 2010.

[12] H. Feng and D. Bai, "Existence of positive solutions for semipositone multi-point boundary value problems," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2287–2292, 2011.

[13] J. R. L. Webb and G. Infante, "Semi-positone nonlocal boundary value problems of arbitrary order," *Communications on Pure and Applied Analysis*, vol. 9, no. 2, pp. 563–581, 2010.

[14] A. Boucherif, "Second-order boundary value problems with integral boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 1, pp. 364–371, 2009.

[15] J. R. Graef and L. Kong, "Solutions of second order multi-point boundary value problems," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 145, no. 2, pp. 489–510, 2008.

[16] Z. Yang, "Positive solutions of a second-order integral boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 321, no. 2, pp. 751–765, 2006.

[17] L. Kong, "Second order singular boundary value problems with integral boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 5, pp. 2628–2638, 2010.
[18] J. Jiang, L. Liu, and Y. Wu, "Second-order nonlinear singular Sturm-Liouville problems with integral boundary conditions," *Applied Mathematics and Computation*, vol. 215, no. 4, pp. 1573–1582, 2009.
[19] B. Liu, L. Liu, and Y. Wu, "Positive solutions for singular second order three-point boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 66, no. 12, pp. 2756–2766, 2007.
[20] R. Ma and Y. An, "Global structure of positive solutions for nonlocal boundary value problems involving integral conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 10, pp. 4364–4376, 2009.
[21] X. Zhang and W. Ge, "Positive solutions for a class of boundary-value problems with integral boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 2, pp. 203–215, 2009.
[22] J. R. L. Webb, "Remarks on positive solutions of some three point boundary value problems, Dynamical systems and differential equations (Wilmington, NC, 2002)," *Discrete and Continuous Dynamical Systems A*, pp. 905–915, 2003.
[23] J. R. L. Webb and G. Infante, "Positive solutions of nonlocal boundary value problems involving integral conditions," *Nonlinear Differential Equations and Applications*, vol. 15, no. 1-2, pp. 45–67, 2008.
[24] J. R. L. Webb and G. Infante, "Non-local boundary value problems of arbitrary order," *Journal of the London Mathematical Society*, vol. 79, no. 1, pp. 238–258, 2009.
[25] J. R. L. Webb and M. Zima, "Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 1369–1378, 2009.
[26] H. Amann, "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM Review*, vol. 18, no. 4, pp. 620–709, 1976.
[27] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.
[28] Z. Yang, "Existence of nontrivial solutions for a nonlinear Sturm-Liouville problem with integral boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 1, pp. 216–225, 2008.