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Research Article

On the Riesz Almost Convergent Sequences Space

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The purpose of this paper is to introduce new spaces $\widehat{f}$ and $\widehat{f}_0$ that consist of all sequences whose Riesz transforms of order one are in the spaces f and f_0, respectively. We also show that $\widehat{f}$ and $\widehat{f}_0$ are linearly isomorphic to the spaces f and f_0, respectively. The β- and γ-duals of the spaces $\widehat{f}$ and $\widehat{f}_0$ are computed. Furthermore, the classes $(\widehat{f}:\mu)$ and $(\mu:\widehat{f})$ of infinite matrices are characterized for any given sequence space μ and determine the necessary and sufficient conditions on a matrix A to satisfy $B_R-\text{core}(Ax)\subseteq K-\text{core}(x)$, $B_R-\text{core}(Ax)\subseteq st-\text{core}(x)$ for all $x\in\ell_\infty$.

1. Introduction and Preliminaries

Let w be the space of all real or complex valued sequences. Then, each linear subspace of w is called a sequence space. For example, the notations ℓ_∞, c, c_0, ℓ_1, cs, and bs are used for the sequence spaces of all bounded, convergent, and null sequences, absolutely convergent series, convergent series, and bounded series, respectively. Let λ and μ be two sequence spaces and $A=(a_{nk})$ an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}=\{0,1,2,\ldots\}$. Then, A defines a matrix mapping from λ to μ and is denoted by $A:\lambda\to\mu$ if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in μ where

$$(Ax)_n=\sum_k a_{nk}x_k,\quad (n\in\mathbb{N}).\tag{1.1}$$

By $(\lambda:\mu)$, we denote the class of matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if the series on the right side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\lambda$. The *matrix domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A=\{x=(x_k)\in w:Ax\in\lambda\}.\tag{1.2}$$

If we take $\lambda = c$, then c_A is called, *convergence domain* of A, and we write the limit of Ax as $\lim_A x = \lim_{n\to\infty}\sum_k a_{nk}x_k$. Further A is called regular if $\lim_A x = \lim x$ for each convergent sequence x.

Let λ be a sequence space. Then λ is called solid if and only if $\ell_\infty\lambda \subset \lambda$, [1]. The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Altay and Başar [2], Başarir [3], Aydın and Başar [4], Kirişçi and Başar [5], Şengönül and Başar [6], Polat and Başar [7], and Malkowsky et al. [8]. Finally, the new technique for deducing certain topological properties, such as AB-, KB-, and AD-properties, and solidity and monotonicity, and determining the α-, β- and γ-duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [9].

Furthermore, quite recently, Kirişçi and Başar [10] introduced the new sequence space $\widehat{f}$ derived from the space f of almost convergent sequences by means of the domain of the generalized difference matrix $B(r,s)$.

Define the sets f and f_0 by

$$\begin{gathered} f = \left\{ x = (x_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x_{k+p}}{n+1} = \alpha \text{ uniformly in } p \right\}, \\ f_0 = \left\{ x = (x_k) \in w : \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x_{k+p}}{n+1} = 0 \text{ uniformly in } p \right\}. \end{gathered} \tag{1.3}$$

If $x \in f$, then x is said to be almost convergent to the generalized limit α. When $x \in f$, we write $f - \lim x = \alpha$.

Lorentz [11] introduced this concept and obtained the necessary and sufficient conditions for an infinite matrix to contain f in its convergence domain. These conditions on an infinite matrix $A = (a_{nk})$ consist of the standard Silverman Toeplitz conditions for regularity plus the condition $\lim_{n\to\infty}\sum_k |a_{nk} - a_{n,k+1}| = 0$. Such matrices are called *strongly regular*. One of the best known strongly regular matrices is C, the Cesàro matrix of order one which is a lower triangular matrix defined by

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{1.4}$$

for all $n, k \in \mathbb{N}$.

A matrix U is called the *generalized Cesàro matrix* if it is obtained from C by shifting rows. Let $p : \mathbb{N} \to \mathbb{N}$. Then, $U = (u_{nk})$ is defined by

$$u_{nk} = \begin{cases} \dfrac{1}{n+1}, & p(n) \le k \le p(n) + n, \\ 0, & \text{otherwise}, \end{cases} \tag{1.5}$$

for all $n, k \in \mathbb{N}$.

Let us suppose that G is the set of all such matrices obtained by using all possible functions p. Now, right here, let us give a new definition for the set of almost convergent sequences that was introduced by Butković et al. [12]:

Lemma 1.1. *The set f of all almost convergent sequences is equal to the set $\bigcap_{U\in G} c_U$.*

Other one of the best known regular matrices is $R = (r_{nk})$, the Riesz matrix which is a lower triangular matrix defined by

$$r_{nk} = \begin{cases} \dfrac{r_k}{R_n}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{1.6}$$

for all $n, k \in \mathbb{N}$, where (r_k) is real sequence with $r_k \ge 0$ $(k \in \mathbb{N})$ and $R_n = r_0 + r_1 + \cdots + r_n$.

Let K be a subset of $\mathbb{N}$. The natural density δ of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|, \tag{1.7}$$

where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to the number l if, for every ε, $\delta(\{k : |x_k - l| \ge \varepsilon\}) = 0$ (see [13]). In this case, we write $st - \lim x = l$. We will also write $\mathcal{S}$ and $\mathcal{S}_0$ to denote the sets of all statistically convergent sequences and statistically null sequences. The statistically convergent sequences were studied by several authors (see [13, 14] and others).

Let us consider the following functionals defined on ℓ_∞:

$$\begin{gathered} l(x) = \liminf_{k\to\infty} x_k, \qquad L(x) = \limsup_{k\to\infty} x_k, \\ q_\sigma(x) = \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{p+1} \sum_{i=0}^{p} x_{\sigma^i(n)}, \\ L^*(x) = \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{p+1} \sum_{i=0}^{p} x_{n+i}. \end{gathered} \tag{1.8}$$

In [15], the σ-core of a real bounded sequence x is defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$ and also the inequalities $q_\sigma(Ax) \le L(x)$ (σ-core of $Ax \subseteq K$-core of x) $q_\sigma(Ax) \le q_\sigma(x)$ (σ-core of $Ax \subseteq \sigma$-core of x), for all $x \in \ell_\infty$, have been studied. Here the Knopp core, in short K-core, of x is the interval $[l(x), L(x)]$. In particular, when $\sigma(n) = n+1$, since $q_\sigma(x) = L^*(x)$, σ-core of x is reduced to the Banach core, in short B-core, of x defined by the interval $[-L^*(-x), L^*(x)]$.

The concepts of B-core and σ-core have been studied by many authors [16, 17].

Recently, Fridy and Orhan [13] have introduced the notions of statistical boundedness, statistical limit superior (or briefly $st - \limsup$), and statistical limit inferior (or briefly $st - \liminf$), defined the statistical core (or briefly st-core) of a statistically bounded sequence as the closed interval $[st - \liminf x, st - \limsup x]$, and also determined the necessary and sufficient conditions for a matrix A to yield K-core$(Ax) \subseteq st$-core(x) for all $x \in \ell_\infty$.

Let us write

$$T^*(x) = \limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{j=0}^{k} \frac{x_{j+p}}{k+1}. \tag{1.9}$$

Quite recently, B_C-core of a sequence has been introduced by the closed intervals $[-T^*(-x), T^*(x)]$ and also the inequalities

$$T^*(Ax) \leq L(x), L(Ax) \leq T^*(x), T^*(Ax) \leq T^*(x), T^*(Ax) \leq st - \limsup x \tag{1.10}$$

have been studied for all $x \in \ell_\infty$ in [18].

Definition 1.2. Let $x \in \ell_\infty$. Then, B_R-core of x is defined by the closed interval $[-\tau^*(-x), \tau^*(x)]$, where

$$\begin{aligned} \tau^*(x) &= \limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j}, \\ -\tau^*(-x) &= \liminf_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j}. \end{aligned} \tag{1.11}$$

Therefore, it is easy to see that B_R-core of x is ℓ if and only if $\widehat{f} - \lim x = \ell$.

As known, the method to obtain a new sequence space by using the convergence field of an infinite matrix is an old method in the theory of sequence spaces. However, the study of the convergence field of an infinite matrix in the space of almost convergent sequences is new.

2. The Sequence Spaces $\widehat{f}$ and $\widehat{f}_0$

In this section we introduce the new spaces $\widehat{f}$ and $\widehat{f}_0$ as the sets of all sequences such that their R-transforms are in the spaces f and f_0, respectively, that is

$$\begin{aligned} \widehat{f} &= \left\{ x = (x_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} = \alpha \text{ uniformly in } p \right\}, \\ \widehat{f}_0 &= \left\{ x = (x_k) \in w : \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} = \alpha \text{ uniformly in } p \right\}. \end{aligned} \tag{2.1}$$

With the notation of (1.2), we can write $\widehat{f} = f_R$ and $\widehat{f}_0 = (f_0)_R$. Define the sequence $y = (y_k)$, which will be frequently used, as the R-transform of a sequence $x = (x_k)$, that is,

$$y_n = \sum_{k=0}^{n} \frac{r_k x_k}{R_n} \quad \forall n \in \mathbb{N}. \tag{2.2}$$

If $R = C$, which is Cesàro matrix, order 1, then the space $\widehat{f}$ and $\widehat{f}_0$ correspond to the spaces $\widetilde{f}$ and $\widetilde{f}_0$ (see [18]).

Suppose that $G' = \{\mathcal{G} : \mathcal{G} = U \circ R, U \in G$ and R is Riesz matrix$\}$. Then we have the following proposition.

Proposition 2.1. $\widehat{f} = \bigcap_{T\in G'} c_T$ *dir.*

Proof. The proof is similar to the proof of Lemma 1.1 so we omit the details, (see [12]). □

Consider the function $\|\cdot\|_{\widehat{f}} : \widehat{f} \to \mathbb{R}$, and define

$$\|x\|_{\widehat{f}} = \sup_n \left| \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} \right|. \tag{2.3}$$

The function $\|\cdot\|_{\widehat{f}}$ is a norm and $(\widehat{f}, \|\cdot\|_{\widehat{f}})$ is BK-space. The proof of this is as follows.

Theorem 2.2. *The sets $\widehat{f}$ and $\widehat{f}_0$ are linear spaces with the coordinate wise addition and scalar multiplication that is the BK- space with the $\|x\|_{\widehat{f}} = \|Rx\|_f$.*

Proof. The first part of the theorem can be easily proved. We prove the second part of the theorem. Since (1.2) holds and f and $\widehat{f}_0$ are the BK-spaces [1] with respect to their natural norm, also the matrix R is normal and gives the fact that the spaces $\widehat{f}$ and $\widehat{f}_0$ are BK-spaces. □

Theorem 2.3. *The sequence spaces $\widehat{f}$ and $\widehat{f}_0$ are linearly isomorphic to the spaces f and f_0, respectively.*

Proof. Since the fact "the spaces $\widehat{f}_0$ and f_0 are linearly isomorphic" can also be proved in a similar way, we consider only the spaces $\widehat{f}$ and f. In order to prove the fact that $\widehat{f} \cong f$, we should show the existence of a linear bijection between the spaces $\widehat{f}$ and f. Consider the transformation T defined, with the notation of (2.2), from $\widehat{f}$ to f by $x \mapsto y = Tx$. The linearity of T is clear. Further, it is trivial that $x = \theta = (0, 0, \ldots)$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in \widehat{f}$, and define the sequence $x = (x_k)$ by

$$x_k = \frac{1}{r_k}\left(R_k y_k - R_{k-1} y_{k-1}\right), \quad (k \in \mathbb{N}). \tag{2.4}$$

Then, we have

$$\begin{aligned} f_R - \lim x &= \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} \text{ uniformly in } p \\ &= \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i\left((1/r_i)\ \left(R_i y_{i+p} - R_{i-1} y_{i+p-1}\right)\right)}{R_j} \text{ uniformly in } p \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} y_{j+p} \text{ uniformly in } p \\ &= f - \lim y \text{ uniformly in } p, \end{aligned} \tag{2.5}$$

which shows that $x \in \hat{f}$. Consequently, we see that T is surjective. Hence, T is a linear bijection that therefore shows that the spaces $\hat{f}$ and f are linearly isomorphic, as desired. This completes the proof. □

Theorem 2.4. *The spaces $\hat{f}$ and $\hat{f}_0$ are not solid sequence spaces.*

Proof. If we take $u = (u_k) = (R_0/r_0, -(R_0/r_0 + R_1/r_1), (R_1/r_1 + R_2/r_2), \ldots, (-1)^k(R_k/r_k + R_{k+1}/r_{k+1}), \ldots)$ and $v = (v_k) = (1, -1, 1, \ldots, (-1)^k, \ldots)$, then we see that $u \in \hat{f}$ and $v \in \ell_\infty$. Let uv be t, that is, $(uv)_k = t_k$. Then, since $(t_k) = (R_0/r_0, (R_0/r_0 + R_1/r_1), (R_1/r_1 + R_2/r_2), \ldots, (R_k/r_k + R_{k+1}/r_{k+1}), \ldots)$ and it is not hard to see by taking into account the definition Riesz matrix that $f_R - \lim t = \lim_{n\to\infty} \sum_{j=0}^{n} 1/(n+1) \sum_{i=0}^{j} (r_i\ t_{i+p})/R_j = \infty$. This shows that the multiplication $\ell_\infty \hat{f}$ of the spaces ℓ_∞ and $\hat{f}$ is not a subset of $\hat{f}$ and therefore the space $\hat{f}$ is not solid. The proof for the space $\hat{f}_0$ is similar to the proof of the space $\hat{f}$, so we omit it. □

Theorem 2.5. *Let the spaces $\hat{f}$ and $\hat{f}_0$ be given. Then,*

(1) *the inclusion $\hat{f}_0 \subset \hat{f}$ holds and the space $\hat{f}$ is not a subset of the space ℓ_∞,*

(2) *if $(1/R_n) \in c$ and $(r_k) \in \ell_1$, then $\ell_\infty \subset \hat{f}$ strictly holds.*

Proof. (1) Clearly, the inclusion $\hat{f}_0 \subset \hat{f}$ holds. Let us consider the sequence given by

$$u_k = \begin{cases} -\left(\dfrac{R_k}{r_k} + \dfrac{R_{k+1}}{r_{k+1}}\right), & k \text{ is odd}, \\ \dfrac{R_k}{r_k} + \dfrac{R_{k+1}}{r_{k+1}}, & k \text{ is even}, \end{cases} \tag{2.6}$$

Since $(R_k) \to \infty$ $(k \to \infty)$, the sequence (u_k) is not a bounded sequence. But clearly $u \in \hat{f}$. This shows that to us, the space $\hat{f}$ is not a subset of the space ℓ_∞.

(2) If $(1/R_n) \in c$ and $(r_k) \in \ell_1$, then for all $x \in \ell_\infty$ we have $Rx \in c$. Therefore, since $\lim(Rx) = f - \lim(Rx)$, we see that $x \in \hat{f}$. □

In Theorem 2.6, we will use some similar techniques that are due to Móricz and Rhoades [19].

Theorem 2.6. *Define the sequences* (α_n) *and* (β_n) *by*

$$\alpha_n = \inf_{p>0} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j}, \qquad \beta_n = \sup_{p>0} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} \tag{2.7}$$

for all $n \in \mathbb{N}$. *Then,* $\alpha_n \le \beta_n$ *for each* $n \in \mathbb{N}$ *and*

(i) *the sequence* (α_{2^n}) *is nondecreasing,*

(ii) *the sequence* (β_{2^n}) *is nonincreasing.*

Proof. It is trivial that

$$\alpha_n = \inf_{p>0} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} \le \sup_{p>0} \sum_{j=0}^{n} \frac{1}{n+1} \sum_{i=0}^{j} \frac{r_i x_{i+p}}{R_j} = \beta_n \tag{2.8}$$

for each $n \in \mathbb{N}$.

Since the part (ii) can be proved in a similar way, we prove only part (i)

$$\begin{aligned}
\alpha_{2^{n+1}} &= \inf_{p>0} \sum_{j=p}^{p+2^{n+1}} \frac{1}{2^{n+1}+1} \sum_{i=0}^{j} \frac{r_i x_i}{R_j} \\
&= \inf_{p>0} \frac{1}{2^{n+1}+1} \left[\frac{2^n+1}{2^n+1} \left(\sum_{j=p}^{p+2^{n+1}} \sum_{i=0}^{j} \frac{r_i x_i}{R_j} \right) \right] \\
&= \inf_{p>0} \frac{1}{2^{n+1}+1} \left[\frac{1}{2^n+1} \left(\sum_{j=p}^{p+2^{n+1}} \sum_{i=0}^{j} \frac{r_i x_i}{R_j} \right) (2^n+1) \right] \\
&= \inf_{p>0} \frac{1}{2^{n+1}+1} \left[\frac{1}{2^n+1} \left(\sum_{j=p}^{p+2^{n}} \sum_{i=0}^{j} \frac{r_i x_i}{R_j} \right) (2^n+1) \right. \\
&\qquad\qquad \left. + \frac{1}{2^n+1} \left(\sum_{j=p+2^n+1}^{p+2^{n+1}} \sum_{i=0}^{j} \frac{r_i x_i}{R_j} \right) (2^n+1) \right] \\
&\ge \frac{1}{2^{n+1}+1} \left[(2^n+1)\alpha_{2^n} + (2^n+1)\alpha_{2^n} \right] = \alpha_{2^n}.
\end{aligned} \tag{2.9}$$

This step completes the proof. $\square$

Theorem 2.7. $\lim_{p\to\infty}(\beta_{2^p}-\alpha_{2^p})=0$ *if and only if* $x\in\widehat{f}$.

Proof. Suppose that $\lim_{n\to\infty}(\beta_{2^n}-\alpha_{2^n})=0$. For each n, choose r to satisfy $2^r\le n\le 2^{r+1}$. We may write n in a dyadic representation of the form $n=\sum_{i=0}^{r} n_i 2^i$, where each n_i is 0 or 1, $i=0,1,2,\dots,r-1$, and $n_r=1$. Then,

$$\begin{aligned}
\frac{1}{n+1}\sum_{j=p}^{p+n}\sum_{i=0}^{j}\frac{r_i x_i}{R_j} &= \frac{1}{n+1}\sum_{j=p}^{p+\sum_{i=0}^{r} n_i 2^i}\sum_{i=0}^{j}\frac{r_i x_i}{R_j}\\
&= \frac{1}{n+1}\left[\frac{1}{2^0+1}\left(\sum_{j=p}^{p+n_0 2^0}\sum_{i=0}^{j}\frac{r_i x_i}{R_j}\right)\left(2^0+1\right)\right.\\
&\qquad + \frac{1}{2^1+1}\left(\sum_{j=n_1 2^1}^{n_2 2^2}\sum_{i=0}^{j}\frac{r_i x_i}{R_j}\right)\left(2^1+1\right)\\
&\qquad \left. +\cdots+\frac{1}{2^r+1}\left(\sum_{j=n_{r-1}2^{r-1}}^{n_r 2^r}\sum_{i=0}^{j}\frac{r_i x_i}{R_j}\right)(2^r+1)\right]\\
&= \frac{1}{n+1}\left[(2^r+1)\alpha_{2^r}+\left(2^{r-1}+1\right)\alpha_{2^{r-1}}+\cdots+2^0\alpha_{2^0}\right]\\
&\ge \frac{1}{n+1}\sum_{j=0}^{r} n_j\left(2^j+1\right)\alpha_j
\end{aligned} \tag{2.10}$$

since $n_j\in\{0,1\}$, and hence

$$\begin{aligned}
\inf_{p>0}\sum_{j=p}^{p+n}\frac{1}{n+1}\sum_{i=0}^{j}\frac{r_i x_i}{R_j} &= \alpha_n \ge \frac{1}{n+1}\sum_{j=0}^{r} n_j\left(2^j+1\right)\alpha_{2^j},\\
\sup_{p>0}\sum_{j=p}^{p+n}\frac{1}{n+1}\sum_{i=0}^{j}\frac{r_i x_i}{R_j} &= \beta_n \le \frac{1}{n+1}\sum_{j=0}^{r} n_j\left(2^j+1\right)\alpha_{2^j}.
\end{aligned} \tag{2.11}$$

Thus,

$$0\le\beta_n-\alpha_n\le\frac{1}{n+1}\sum_{j=0}^{r} n_j\left(2^j+1\right)\left(\beta_{2^j}-\alpha_{2^j}\right). \tag{2.12}$$

If T is the lower triangular matrix with nonzero entries $t_{nk}=n_k(2^k+1)/(n+1)$, then, T is a regular matrix so that $\lim_{r\to\infty}(\beta_{2^r}-\alpha_{2^r})=0$. From the equality (2.12), we see that $\lim_{n\to\infty}(\beta_n-\alpha_n)=0$.

Conversely, assume that $x\in\widehat{f}$. Then, since

$$\lim_{n\to\infty}\sum_{j=p}^{p+n}\frac{1}{n+1}\sum_{i=0}^{j}\frac{r_i x_i}{R_j}=\alpha \tag{2.13}$$

implies

$$\lim_{n\to\infty}\alpha_n = \inf_{p>0}\lim_{n\to\infty}\sum_{j=p}^{p+n}\frac{1}{n+1}\sum_{i=0}^{j}\frac{r_i x_i}{R_j} = \alpha,$$
$$\lim_{n\to\infty}\beta_n = \sup_{p>0}\lim_{n\to\infty}\sum_{j=p}^{p+n}\frac{1}{n+1}\sum_{i=0}^{j}\frac{r_i x_i}{R_j} = \alpha, \tag{2.14}$$

we have

$$\lim_{n\to\infty}\alpha_n - \lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\left(\alpha_n - \beta_n\right) = 0. \tag{2.15}$$

If we take $n = 2^p$, then the proof of sufficiency is obtained. This step completes the proof. □

3. Some Duals of the Spaces $\widehat{f}$ and $\widehat{f}_0$

In this section, by using techniques in [9], we have stated and proved the theorems determining the β- and γ-duals of the spaces $\widehat{f}_0$ and $\widehat{f}$. For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu) = \left\{z = (z_k) \in w : xz = (x_k z_k) \in \mu \ \forall x = (x_k) \in \lambda\right\}. \tag{3.1}$$

With the notation of (3.1), the α-, β-, and γ-duals of a sequence space λ, which are, respectively, denoted by λ^α, λ^β, and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \qquad \lambda^\beta = S(\lambda, cs), \qquad \lambda^\gamma = S(\lambda, bs). \tag{3.2}$$

The following two lemmas are introduced in [20] which we need in proving Theorems 3.3 and 3.4.

Lemma 3.1. *$A \in (f : \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}| < \infty. \tag{3.3}$$

Lemma 3.2. *$A \in (f : c)$ if and only if*

$$\lim_{n\to\infty}\sum_k a_{nk} = a,$$
$$\lim_{n\to\infty} a_{nk} = a_k; \quad k \in \mathbb{N}, \tag{3.4}$$
$$\lim_{n\to\infty}\sum_k |\Delta(a_{nk} - a_k)| = 0.$$

Theorem 3.3. *The γ-duals of the spaces $\widehat{f}$ and $\widehat{f}_0$ are the set $d_1 \cap d_2$, where*

$$d_1 = \left\{ a = (a_k) \in w : \sum_k \left| R_k \Delta\left(\frac{a_k}{r_k}\right) \right| < \infty \right\},$$
$$d_2 = \left\{ a = (a_k) \in w : \left\{ \frac{a_n}{r_n} R_n \right\} \in \ell_\infty \right\}. \quad (3.5)$$

Proof. Define the matrix $T = (t_{nk})$ via the sequence $a = (a_k) \in w$ by

$$t_{nk} = \begin{cases} R_k \Delta\left(\dfrac{a_k}{r_k}\right), & 0 \le k \le n-1, \\ \dfrac{a_n}{r_n} R_n, & n = k, \\ 0, & \text{otherwise}, \end{cases} \quad (3.6)$$

for all $n, k \in \mathbb{N}$. Here, $\Delta(a_k/r_k) = (a_k/r_k - a_{k+1}/r_{k+1})$. By using (2.2), we derive that

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n-1} R_k \Delta\left(\frac{a_k}{r_k}\right) y_k + \frac{a_n}{r_n} R_n y_n \\ &= (Ty)_n, \quad (n \in \mathbb{N}). \end{aligned} \quad (3.7)$$

From (3.7), we see that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in \widehat{f}$ if and only if $Ty \in \ell_\infty$ whenever $y = (y_k) \in f$. Then, we derive by Lemma 3.1 that

$$\sum_{k=0}^{n} \left| R_k \Delta\left(\frac{a_k}{r_k}\right) \right| < \infty, \qquad \left\{ \frac{a_n}{r_n} R_n \right\} \in \ell_\infty, \quad (3.8)$$

which yields the desired result $\widehat{f}^\gamma = \widehat{f}_0^\gamma = d_1 \cap d_2$. □

Theorem 3.4. *Define the set d_3 by*

$$d_3 = \left\{ a = (a_k) \in w : \sum_k \left| \Delta\left[R_k \Delta\left(\frac{a_k}{r_k}\right) - a_k \right] \right| < \infty \right\}. \quad (3.9)$$

Then, $\widehat{f}^\beta = d_3 \cap cs$.

Proof. Consider equality (3.7), again. Thus, we deduce that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \widehat{f}$ if and only if $Ty \in c$ whenever $y = (y_k) \in f$. It is obvious that the columns of that matrix T, defined by (3.6), are in the space c. Therefore, we derive the consequence from Lemma 3.2 that $\widehat{f}^\beta = d_3 \cap cs$. □

4. Some Matrix Mappings Related to the Spaces $\widehat{f}$ and $\widehat{f}_0$

In this section, we characterize the matrix mappings from $\widehat{f}$ into any given sequence space via the concept of the dual summability methods of the new type introduced by Başar [21].

Note that some researchers, such as, Başar [21], Başar and Çolak [22], Kuttner [23], and Lorentz and Zeller [24], worked on the dual summability methods. Now, following Başar [21], we give a short survey about dual summability methods of the new type.

Let us suppose that the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$, which are connected by the relation (2.2) to the sequences $z = (z_n)$ and $t = (t_n)$, respectively, that is,

$$z_n = (Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}),$$
$$t_n = (By)_n = \sum_k b_{nk}y_k, \quad (n \in \mathbb{N}). \tag{4.1}$$

It is clear here that the method B is applied to the R-transform of the sequence $x = (x_k)$ while the method A is directly applied to the entries of the sequence $x = (x_k)$. So, the methods A and B are essentially different.

Let us assume that the matrix product BR exists, which is a much weaker assumption than the conditions on the matrix B belonging to any matrix class, in general. The methods A and B in (4.1), (4.2) are called *dual summability methods of the new type* if z_n reduces to t_n (or t_n reduces to z_n) under the application of formal summation by parts. This leads us to the fact that BR exists and is equal to A and $(BR)x = B(Rx)$ formally holds if one side exists. This statement is equivalent to the following relation between the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$:

$$a_{nk} := \sum_{j=k}^{\infty} \frac{r_j}{R_j} b_{nj} \quad \text{or} \quad b_{nk} := R_k\left(\frac{a_{nk}}{r_k} - \frac{a_{n,k+1}}{r_{k+1}}\right) = R_k\Delta\left(\frac{a_{nk}}{r_k}\right) \tag{4.2}$$

for all $n, k \in \mathbb{N}$.

Now, we give the following theorem concerning the dual matrices of the new type.

Theorem 4.1. *Let $A = (a_{nk})$ and $B = (b_{nk})$ be the dual matrices of the new type and μ any given sequence space. Then, $A \in (\widehat{f} : \mu)$ if and only if $B \in (f : \mu)$ and*

$$\left\{\left(\frac{R_n}{r_n}\right)a_{nk}\right\}_{n\in\mathbb{N}} \in c_0 \tag{4.3}$$

for every fixed $k \in \mathbb{N}$.

Proof. Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ are dual matrices of the new type, that is to say (4.2) holds and μ is an any given sequence space. Since the spaces $\widehat{f}$ and f are linearly isomorphic, now let $A \in (\widehat{f} : \mu)$ and $y = (y_k) \in f$. Then, BR exists and $(a_{nk})_{k\in\mathbb{N}} \in d_2 \cap cs$, which yields that $(b_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, By exists for each $y \in f$, and thus letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} b_{nk} y_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{r_j}{R_j} b_{nj} x_k \tag{4.4}$$

for all $m, n \in \mathbb{N}$, we have by (4.2) that $By = Ax$, which gives the result $B \in (f : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \widehat{f}^{\beta}$ for each $n \in \mathbb{N}$ and $B \in (f : \mu)$ hold, and take any $x = (x_k) \in \widehat{f}$. Then, Ax exists. Therefore, we obtain from the equality

$$\begin{aligned} \sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m-1} R_k \Delta \frac{a_{nk}}{r_k} y_k + \frac{a_{nm}}{r_n} y_m \\ &= \sum_{k=0}^{m} b_{nk} y_k; \quad (n \in \mathbb{N}), \end{aligned} \tag{4.5}$$

as $n \to \infty$ that $Ax = By$, and this shows that $A \in (\widehat{f} : \mu)$. This completes the proof. □

Theorem 4.2. *Suppose that the entries of the infinite matrices $D = (d_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} = \sum_{j=0}^{n} \frac{r_j d_{jk}}{R_j}, \quad (n, k \in \mathbb{N}) \tag{4.6}$$

and μ is any given sequence space. Then, $D \in (\mu : \widehat{f})$ if and if only $E \in (\mu : f)$.

Proof. Let $x = (x_k) \in \mu$, and consider the following equality with (4.6):

$$\sum_{j=0}^{n} \frac{r_j}{R_j} \sum_{k=0}^{m} d_{jk} x_k = \sum_{k=0}^{m} e_{nk} x_k; \quad (m, n, k \in \mathbb{N}), \tag{4.7}$$

which yields as $m \to \infty$ that $Dx \in \widehat{f}$ whenever $x \in \mu$ if and if only $Ex \in f$ whenever $x \in \mu$. This step completes the proof. □

Now, right here, we give the following propositions that are obtained from Lemmas 3.2 and 3.1 and Theorems 4.1 and 4.2.

Proposition 4.3. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then,*

$$A = (a_{nk}) \in \left(\widehat{f} : \ell_\infty\right) \Longleftrightarrow \begin{cases} (1) \quad \lim_{n\to\infty} \sum_k \left| \Delta \left(\sum_{j=k}^{\infty} \frac{r_j a_{nj}}{R_j} - a_k \right) \right| = 0, \\ (2) \quad \{a_{nk}\} \in \widehat{f}^{\beta} \ \forall n \in \mathbb{N}. \end{cases} \tag{4.8}$$

Proposition 4.4. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then,*

$$A = (a_{nk}) \in \left(\widehat{f} : c\right) \Longleftrightarrow \begin{cases} (3) \quad \lim_{n\to\infty} \sum_k \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} = a, \\ (4) \quad \lim_{n\to\infty} \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} = a_k \text{ for each } k \in \mathbb{N}, \\ (5) \quad \lim_{n\to\infty} \sum_k \left| \Delta\left(\sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} - a_k \right) \right| = 0, \\ (6) \quad \{a_{nk}\}_{k\in\mathbb{N}} \in \widehat{f}^{\beta} \ \forall n \in \mathbb{N}. \end{cases} \tag{4.9}$$

Proposition 4.5. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then,*

$$A = (a_{nk}) \in \left(\ell_\infty : \widehat{f}\right) \Longleftrightarrow \begin{cases} (7) \quad \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} \right| < \infty, \\ (8) \quad f - \lim_{n\to\infty} \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} = \alpha_k \text{ exists for each fixed } k \in \mathbb{N}, \\ (9) \quad \lim_{m\to\infty} \sum_k \left| \sum_{i=0}^{m} \dfrac{1}{m+1} \sum_{j=k}^{\infty} \dfrac{r_j a_{n+i,j}}{R_j} - \alpha_k \right| = 0 \text{ uniformly in } n. \end{cases} \tag{4.10}$$

Proposition 4.6. *Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. Then,*

$$A = (a_{nk}) \in \left(c : \widehat{f}\right) \Longleftrightarrow \begin{cases} (10) \quad \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} \right| < \infty, \\ (11) \quad f - \lim_{n\to\infty} \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} = \alpha_k \text{ exists for each fixed } k \in \mathbb{N}, \\ (12) \quad f - \lim_{n\to\infty} \sum_k \sum_{j=k}^{\infty} \dfrac{r_j a_{nj}}{R_j} = \alpha. \end{cases} \tag{4.11}$$

5. Core Theorems

In this section, we give some core theorems related to the space $\widehat{f}$. We need the following lemma due to Das [25] for the proof of next theorem.

Lemma 5.1. *Let $\|c\| = \|c_{nj}(p)\| < \infty$ and $\lim_{n\to\infty} \sup_{p\in\mathbb{N}} |c_{nj}(p)| = 0$. Then, there is a $y = (y_j) \in \ell_\infty$ such that $\|y\| \le 1$ and*

$$\limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_j c_{nj}(p) y_j = \limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_j \left| c_{nj}(p) \right|. \tag{5.1}$$

Theorem 5.2. *B_R–core$(Ax) \subseteq K$–core(x) for all $x \in \ell_\infty$ if and only if $A \in (c : \widehat{f})_{\text{reg}}$ and*

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_k \frac{1}{n+1} \left| \sum_{j=0}^{n} \frac{1}{R_j} \sum_{i=0}^{j} r_i a_{i+p,k} \right| = 1. \tag{5.2}$$

Proof. Necessity: Suppose first that $B_R - \text{core}(Ax) \subseteq K - \text{core}(x)$ for all $x \in \ell_\infty$. If $x \in \widehat{f}$, then we have $\tau^*(Ax) = -\tau^*(-Ax)$. By this hypothesis, we get

$$-L(-x) \le -\tau^*(-Ax) \le \tau^*(Ax) \le L(x). \tag{5.3}$$

If $x \in c$, then $L(x) = -L(-x) = \lim x$. So, we have $\widehat{f} - \lim Ax = \tau^*(Ax) = -\tau^*(-Ax) = \lim x$, which implies that $A \in (c, \widehat{f})_{\text{reg}}$.

Now, let us consider the sequence $C = (c_{nj}(p))$ of infinite matrices defined by

$$c_{nj}(p) = \frac{1}{n+1} \sum_{j=0}^{n} \frac{1}{R_j} \sum_{i=0}^{j} r_i a_{i+p,k} \quad \forall n, i, p \in \mathbb{N}. \tag{5.4}$$

Then, it is easy to see that the conditions of Lemma 5.1 are satisfied for the matrix sequence C. Thus, by using the hypothesis, we can write

$$\begin{aligned} 1 \le \liminf_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_j \left| c_{nj}(p) \right| &\le \limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_j \left| c_{nj}(p) \right| \\ &= \limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_j c_{nj}(p) y_j \\ &= \tau^*(Ay) \le L(y) \le \|y\| \le 1. \end{aligned} \tag{5.5}$$

This gives the necessity of (5.2).

Sufficiency: Conversely, let $A \in (c : \widehat{f})_{\text{reg}}$ and (5.2) hold for all $x \in \ell_\infty$. For any real number $\tilde{\lambda}$ we write $\tilde{\lambda}^+ = \max\{\tilde{\lambda}, 0\}$ and $\tilde{\lambda}^- = \max\{-\tilde{\lambda}, 0\}$; then $|\tilde{\lambda}| = \tilde{\lambda}^+ + \tilde{\lambda}^-$ and $\tilde{\lambda} = \tilde{\lambda}^+ - \tilde{\lambda}^-$. Therefore, for any given $\varepsilon > 0$, there is a $j_0 \in \mathbb{N}$ such that $x_j < L(x) + \varepsilon$ for all $j > j_0$. Now, we can write

$$\begin{aligned} \sum_j c_{nj}(p) x_j &= \sum_{j<j_0} c_{nj}(p) x_j + \sum_{j\ge j_0} (c_{nj}(p))^+ x_j - \sum_{j\ge j_0} (c_{nj}(p))^- x_j \\ &\le \|x\| \sum_{j<j_0} \left| c_{nj}(p) \right| + [L(x) + \varepsilon] \sum_j \left| c_{nj}(p) \right| \\ &\quad + \|x\| \sum_j \left[\left| c_{nj}(p) \right| - c_{nj}(p) \right]. \end{aligned} \tag{5.6}$$

Thus, by applying $\limsup_{n\to\infty} \sup_{p\in\mathbb{N}}$ and using the hypothesis, we have $\tau^*(Ax) \le L(x) + \varepsilon$. This completes the proof since ε is arbitrary and $x \in \ell_\infty$. □

In particular $r_i = 1$ for all i since R is reduced to Cesàro matrix, see [18].

Theorem 5.3. *B_C–core$(Ax) \subseteq K$–core(x) for all $x \in \ell_\infty$ if and only if $A \in (c : \widetilde{f})_{\text{reg}}$ and*

$$\lim_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_i \frac{1}{n+1} \left| \sum_{k=0}^{n} \frac{1}{k+1} \sum_{j=0}^{k} a_{j+p,i} \right| = 1. \tag{5.7}$$

Theorem 5.4. *$A \in (\mathcal{S} \cap \ell_\infty : \widehat{f})_{\text{reg}}$ if and only if $A \in (c : \widehat{f})_{\text{reg}}$ and*

$$\lim_{n\to\infty} \sum_{k\in E} \frac{1}{n+1} \left| \sum_{j=0}^{n} \frac{1}{R_j} \sum_{i=0}^{j} r_i a_{i+p,k} \right| = 0 \text{ uniformly in } p \tag{5.8}$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

Proof. Necessity: Let $A \in (S \cap \ell_\infty, \widehat{f})_{\text{reg}}$. Then, $A \in (c, \widehat{f})_{\text{reg}}$ immediately follows from the fact that $c \subset S \cap \ell_\infty$. Now, define a sequence $t = (t_k)$ for $x \in \ell_\infty$ as

$$t_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases} \tag{5.9}$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. Then, $st - \lim t_k = 0$ and $t \in S_0$, and so we have $At \in \widehat{f}_0$. On the other hand, since $At = \sum_{k\in E} a_{nk} t_k$, the matrix $B = (b_{nk})$ defined by

$$b_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E, \end{cases} \tag{5.10}$$

for all n, must belong to the class $(\ell_\infty, \widehat{f}_0)$. Hence, the necessity of (5.8) follows from Proposition 4.5.

Sufficiency: Conversely, suppose that $A \in (c, \widehat{f})_{\text{reg}}$ and (5.8) holds. Let $x \in S \cap \ell_\infty$ and $st - \lim x = \ell$. Write $E = \{k : |x_k - \ell| \geq \varepsilon\}$ for any given $\varepsilon > 0$ so that $\delta(E) = 0$. Since $A \in (c, \widehat{f})_{\text{reg}}$ and $\widehat{f} - \lim \sum_k a_{nk} = 1$, we have

$$\begin{aligned} \widehat{f} - \lim(Ax) &= \widehat{f} - \lim\left(\sum_k a_{nk}(x_k - \ell) + \ell \sum_k a_{nk} \right) \\ &= \widehat{f} - \lim\left(\sum_k a_{nk}(x_k - \ell) + \ell \right) \\ &= \lim_{n\to\infty} \sup_{p\in\mathbb{N}} \sum_k \frac{1}{n+1} \sum_{j=0}^{n} \frac{1}{R_j} \sum_{i=0}^{j} r_i a_{i+p,k}(x_k - \ell) + \ell. \end{aligned} \tag{5.11}$$

On the other hand, since

$$\left|\sum_k \frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{R_j}\sum_{i=0}^{j} r_i a_{i+p,k}(x_k-\ell)\right| \le \|x\|\sum_{k\in E}\frac{1}{n+1}\left|\sum_{j=0}^{n}\frac{1}{R_j}\sum_{i=0}^{j} r_i a_{i+p,k}\right| + \varepsilon\|A\|, \tag{5.12}$$

condition (5.8) implies that

$$\lim_{n\to\infty}\sum_k \frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{R_j}\sum_{i=0}^{j} r_i a_{i+p,k}(x_k-\ell) = 0 \text{ uniformly in } p. \tag{5.13}$$

Hence, $\widehat{f}-\lim(Ax) = st-\lim x$; that is, $A \in (\mathrm{S}\cap m, \widehat{f})_{\mathrm{reg}}$, which completes the proof. □

Similarly, $r_i = 1$ for all i since R is reduced to Cesàro matrix, see [18].

Theorem 5.5. *$A \in (S\cap \ell_\infty : \widetilde{f})_{\mathrm{reg}}$ if and only if $A \in (c : \widetilde{f})_{\mathrm{reg}}$ and*

$$\lim_{n\to\infty}\sum_{i\in E}\frac{1}{n+1}\left|\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k} a_{j+p,i}\right| = 0 \text{ uniformly in } p \tag{5.14}$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

Theorem 5.6. *B_R–core$(Ax) \subseteq st$–core(x) for all $x \in \ell_\infty$ if and only if $A \in (\mathcal{S}\cap \ell_\infty : \widehat{f})_{\mathrm{reg}}$ and* (5.2) *holds.*

Proof. Necessity: Let B_R − core$(Ax) \subseteq st$ − core(x) for all $x \in \ell_\infty$. Then, $\tau^*(Ax) \le \beta(x)$ for all $x \in \ell_\infty$, where $\beta(x) = st - \limsup x$. Hence, since $\beta(x) = st - \limsup x \le L(x)$ for all $x \in \ell_\infty$ (see [13]), we have (5.2) from Theorem 5.2. Furthermore, one can also easily see that $-\beta(-x) \le -\tau^*(-Ax) \le \tau^*(Ax) \le \beta(x)$, that is,

$$st - \liminf x \le -\tau^*(-Ax) \le \tau^*(Ax) \le st - \limsup x. \tag{5.15}$$

If $x \in \mathcal{S}\cap \ell_\infty$, then $st - \liminf x = st - \limsup x = st - \lim x$. Thus, the last inequality implies that $st - \lim x = -\tau^*(-Ax) = \tau^*(Ax) = \widehat{f} - \lim\ Ax$, that is, $A \in (\mathcal{S}\cap \ell_\infty : \widehat{f})_{\mathrm{reg}}$.

Sufficiency: Conversely, assume that $A \in (\mathcal{S}\cap \ell_\infty : \widehat{f})_{\mathrm{reg}}$ and (5.2) hold. If $x \in \ell_\infty$, then $\beta(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E = \{l : x_i > \beta(x) + \varepsilon\}$ for a given $\varepsilon > 0$. Then it is obvious that $\delta(E) = 0$ and $x_i \le \beta(x) + \varepsilon$ if $l \notin E$.

For any real number $\widetilde{\lambda}$ we write $\widetilde{\lambda}^{+} = \max\{\widetilde{\lambda}, 0\}$ and $\widetilde{\lambda}^{-} = \max\{-\widetilde{\lambda}, 0\}$ whence $|\widetilde{\lambda}| = \widetilde{\lambda}^{+} + \widetilde{\lambda}^{-}$, $\widetilde{\lambda} = \widetilde{\lambda}^{+} - \widetilde{\lambda}^{-}$ and $|\widetilde{\lambda}| - \widetilde{\lambda} = 2\widetilde{\lambda}^{-}$. Now, we can write

$$\begin{aligned}
\sum_{j} c_{nj}(p)x_j &= \sum_{j<j_0} c_{nj}(p)x_j + \sum_{j\geq j_0} c_{nj}(p)x_j \\
&= \sum_{j<j_0} c_{nj}(p)x_j + \sum_{j\geq j_0} c^{+}_{nj}(p)x_j - \sum_{j\geq j_0} c^{-}_{nj}(p)x_j \\
&\leq \|x\| \sum_{j<j_0} \left|c_{nj}(p)\right| + \sum_{\substack{j\geq j_0 \\ j\notin E}} c^{+}_{nj}(p)x_j + \sum_{\substack{j\geq j_0 \\ j\in E}} c^{+}_{nj}(p)x_j \\
&\quad + \|x\| \sum_{j\geq j_0} \left[\left|c_{nj}(p)\right| - c_{nj}(p)\right] \leq \|x\| \sum_{j<j_0} \left|c_{nj}(p)\right| \\
&\quad + \left[\beta(x)+\varepsilon\right] \sum_{\substack{j\geq j_0 \\ j\notin E}} \left|c_{nj}(p)\right| + \|x\| \sum_{\substack{j\geq j_0 \\ j\in E}} \left|c_{nj}(p)\right| \\
&\quad + \|x\| \sum_{j\geq j_0} \left[\left|c_{nj}(p)\right| - c_{nj}(p)\right].
\end{aligned} \tag{5.16}$$

By applying the operator $\limsup_{n\to\infty}\sup_{p\in\mathbb{N}}$ and using the hypothesis, we obtained that $\tau^{*}(Ax) \leq \beta(x)+\varepsilon$. Since ε is arbitrary, we conclude that $\tau^{*}(Ax) \leq \beta(x)$ for all $x \in \ell_\infty$, that is, $B_R - \operatorname{core}(Ax) \subseteq st - \operatorname{core}(x)$ for all $x \in \ell_\infty$ and the proof is complete. Now if $r_i = 1$ for all i, then R is reduced to Cesàro matrix and we have

$$B_C - \operatorname{core}(Ax) \subseteq st - \operatorname{core}(x), \quad \forall x \in \ell_\infty \text{ if and only if } A \in \left(S \cap \ell_\infty : \widetilde{f}\right)_{\text{reg}} \tag{5.17}$$

and (5.7) holds, see [18]. □

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