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Research Article

A Note on Nonlocal Boundary Value Problems for Hyperbolic Schrödinger Equations

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The nonlocal boundary value problem $d^2u(t)/dt^2 + Au(t) = f(t)$ $(0 \leq t \leq 1)$, $i(du(t)/dt) + Au(t) = g(t)$ $(-1 \leq t \leq 0)$, $u(0^+) = u(0^-)$, $u_t(0^+) = u_t(0^-)$, $Au(-1) = \alpha u(\mu) + \varphi$, $0 < \mu \leq 1$, for hyperbolic Schrödinger equations in a Hilbert space H with the self-adjoint positive definite operator A is considered. The stability estimates for the solution of this problem are established. In applications, the stability estimates for solutions of the mixed-type boundary value problems for hyperbolic Schrödinger equations are obtained.

1. Introduction

Methods of solutions of nonlocal boundary value problems for partial differential equations and partial differential equations of mixed type have been studied extensively by many researches (see, e.g., [1–12] and the references given therein).

In the present paper, the nonlocal boundary value problem

$$\begin{gathered}
\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\
i\frac{du(t)}{dt} + Au(t) = g(t) \quad (-1 \leq t \leq 0), \\
u(0^+) = u(0^-), \qquad u_t(0^+) = u_t(0^-), \\
Au(-1) = \alpha u(\mu) + \varphi, \quad 0 < \mu \leq 1
\end{gathered} \tag{1.1}$$

for differential equations of hyperbolic Schrödinger type in a Hilbert space H with self-adjoint positive definite operator A is considered.

It is known that various nonlocal boundary value problems for the hyperbolic Schrödinger equations can be reduced to problem (1.1).

A function $u(t)$ is called a solution of the problem (1.1) if the following conditions are satisfied.

(i) $u(t)$ is twice continuously differentiable on the interval (0,1] and continuously differentiable on the segment $[-1,1]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on the segment $[-1,1]$.

(iii) $u(t)$ satisfies the equations and nonlocal boundary condition (1.1).

In the present paper, the stability estimates for the solution of the problem (1.1) for the hyperbolic Schrödinger equation are established. In applications, the stability estimates for the solutions of the mixed-type boundary value problems for hyperbolic Schrödinger equations are obtained.

Finally note that hyperbolic Schrödinger equations play important role in physics and engineering (see, e.g., [13–16] and the references given therein).

Furthermore, the investigation of the numerical solution of initial value problems and Schrödinger equations is the subject of extensive research activity during the last decade (indicatively [17–25] and the references given therein).

2. The Main Theorem

Let H be a Hilbert space, and let A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > \delta_0 > 0$. Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator function defined by

$$c(t) = \frac{e^{itA^{1/2}} + e^{-itA^{1/2}}}{2}. \tag{2.1}$$

Then, from the definition of the sine operator function $s(t)$

$$s(t)u = \int_0^t c(s)u\,ds, \tag{2.2}$$

it follows that

$$s(t) = A^{-1/2}\frac{e^{itA^{1/2}} - e^{-itA^{1/2}}}{2i}. \tag{2.3}$$

For the theory of cosine operator function, we refer to Fattorini [26] and Piskarev and Shaw [27].

We begin with two lemmas that will be needed as follows.

Lemma 2.1. *The following estimates hold:*

$$\|c(t)\|_{H\to H} \le 1, \qquad \left\|A^{1/2}s(t)\right\|_{H\to H} \le 1, \quad t \ge 0,$$
$$\left\|e^{\pm itA}\right\|_{H\to H} \le 1, \quad t \ge 0. \tag{2.4}$$

Lemma 2.2. *Let*

$$|\alpha| < \frac{\delta}{\sqrt{1+\delta}}. \tag{2.5}$$

Then, the operator

$$I - \alpha\left[A^{-1}c(\mu) + is(\mu)\right]e^{iA} \tag{2.6}$$

has an inverse

$$T = \left(I - \alpha\left[A^{-1}c(\mu) + is(\mu)\right]e^{iA}\right)^{-1}, \tag{2.7}$$

and the estimate

$$\|T\|_{H\to H} \le M \tag{2.8}$$

holds, where M does not depend on α and μ.

Proof. Actually, the proof of estimate (2.8) is based on the following estimate:

$$\left\|-\alpha\left[A^{-1}c(\mu) + is(\mu)\right]e^{iA}\right\|_{H\to H} < 1. \tag{2.9}$$

Using the definitions of cosine and sine operator functions, $A \ge \delta I$, $\delta > 0$ (positivity), and $A = A^*$ (self-adjointness property), we obtain

$$\begin{aligned}
\left\|-\alpha\left[A^{-1}c(\mu) + is(\mu)\right]e^{iA}\right\|_{H\to H} &\le \sup_{\delta\le\rho<\infty}\left|-\alpha\left[\frac{1}{\rho}\cos\left(\sqrt{\rho}\mu\right) + \frac{i}{\sqrt{\rho}}\sin\left(\sqrt{\rho}\mu\right)\right]e^{i\rho}\right| \\
&\le \sup_{\delta\le\rho<\infty}|\alpha|\left|\frac{1}{\rho}\cos\left(\sqrt{\rho}\mu\right) + \frac{i}{\sqrt{\rho}}\sin\left(\sqrt{\rho}\mu\right)\right|\left|e^{i\rho}\right| \\
&\le |\alpha|\sup_{\delta\le\rho<\infty}\sqrt{\frac{1}{\rho^2}\cos^2\left(\sqrt{\rho}\mu\right) + \frac{1}{\rho}\sin^2\left(\sqrt{\rho}\mu\right)} \\
&\le |\alpha|\frac{\sqrt{1+\rho}}{\rho}.
\end{aligned} \tag{2.10}$$

Since

$$\frac{\sqrt{1+\rho}}{\rho} \leq \frac{\sqrt{1+\delta}}{\delta}, \tag{2.11}$$

we have that

$$\left\| -\alpha\left[A^{-1}c(\mu)+is(\mu)\right]e^{iA}\right\|_{H\to H} < \frac{\delta}{\sqrt{1+\delta}}\cdot\frac{\sqrt{1+\delta}}{\delta}=1. \tag{2.12}$$

Hence, Lemma 2.2 is proved. □

Now, we will obtain the formula for solution of problem (1.1). It is known that for smooth data of initial value problems

$$\begin{aligned} &\frac{d^2u(t)}{dt^2}+Au(t)=f(t) \quad (0\leq t\leq 1),\\ &u(0)=u_0, \qquad u'(0)=u_0',\\ &i\frac{du(t)}{dt}+Au(t)=g(t) \quad (-1\leq t\leq 0),\\ &u(-1)=u_{-1}, \end{aligned} \tag{2.13}$$

there are unique solutions of problems (2.13), and following formulas hold:

$$u(t)=c(t)u(0)+s(t)u'(0)+\int_0^t s(t-y)f(y)dy, \quad 0\leq t\leq 1, \tag{2.14}$$

$$u(t)=e^{i(t+1)A}u_{-1}-i\int_{-1}^t e^{i(t-y)A}g(y)dy, \quad -1\leq t\leq 0. \tag{2.15}$$

Using (2.14), (2.15), and (1.1), we can write

$$\begin{aligned} u(t)=[c(t)+iAs(t)]&\left\{e^{iA}u_{-1}-i\int_{-1}^0 e^{-iAy}g(y)dy\right\}\\ &-is(t)g(0)+\int_0^t s(t-y)f(y)dy. \end{aligned} \tag{2.16}$$

Now, using the nonlocal boundary condition

$$Au(-1)=\alpha u(\mu)+\varphi, \tag{2.17}$$

we obtain the operator equation:

$$\begin{aligned}&\left\{I-\alpha\left[A^{-1}c(\mu)+is(\mu)\right]e^{iA}\right\}u_{-1}\\&=\alpha\left\{-iA^{-1}c(\mu)\int_{-1}^{0}e^{-iAy}g(y)dy\right.\\&\qquad\left.-s(\mu)\left[iA^{-1}g(0)-\int_{-1}^{0}e^{-iAy}g(y)dy\right]+A^{-1}\int_{0}^{\mu}s(\mu-y)f(y)dy\right\}+A^{-1}\varphi.\end{aligned}\tag{2.18}$$

Since the operator

$$I-\alpha\left[A^{-1}c(\mu)+is(\mu)\right]e^{iA}\tag{2.19}$$

has an inverse

$$T=(I-\alpha[A^{-1}c(\mu)+is(\mu)]e^{iA})^{-1},\tag{2.20}$$

for the solution of the operator equation (2.18), we have the formula

$$\begin{aligned}u_{-1}=T\Bigg(&\alpha\left\{-iA^{-1}c(\mu)\int_{-1}^{0}e^{-iAy}g(y)dy\right.\\&\left.-s(\mu)\left[iA^{-1}g(0)-\int_{-1}^{0}e^{-iAy}g(y)dy\right]+A^{-1}\int_{0}^{\mu}s(\mu-y)f(y)dy\right\}+A^{-1}\varphi\Bigg)\end{aligned}\tag{2.21}$$

Thus, for the solution of the nonlocal boundary value problem (1.1) we obtain (2.15), (2.16), and (2.21).

Theorem 2.3. *Suppose that $\varphi\in D(A^{1/2}), f(0)\in D(A^{1/2})$, and $g(0)\in D(A^{1/2})$. Let $f(t)$ be continuously differentiable on $[0,1]$ and let $g(t)$ be twice continuously differentiable on $[-1,0]$ functions. Then, there is a unique solution of the problem (1.1) and the following stability inequalities*

$$\begin{aligned}&\max_{-1\le t\le 1}\|u(t)\|_H\\&\le M\left[\left\|A^{-1/2}\varphi\right\|_H+\left\|A^{-1/2}g(0)\right\|_H+\max_{-1\le t\le 0}\left\|A^{-1}g'(t)\right\|_H+\max_{0\le t\le 1}\left\|A^{-1/2}f(t)\right\|_H\right],\end{aligned}\tag{2.22}$$

$$\begin{aligned}&\max_{-1\le t\le 1}\left\|\frac{du(t)}{dt}\right\|_H+\max_{-1\le t\le 1}\left\|A^{1/2}u(t)\right\|_H\\&\quad\le M\left[\|\varphi\|_H+\|g(0)\|_H+\max_{-1\le t\le 0}\left\|A^{-1/2}g'(t)\right\|_H+\max_{0\le t\le 1}\|f(t)\|_H\right],\end{aligned}\tag{2.23}$$

$$\begin{aligned}&\max_{-1\le t\le 0}\left\|\frac{du(t)}{dt}\right\|_H+\max_{0\le t\le 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H+\max_{-1\le t\le 1}\|Au(t)\|_H\\&\quad\le M\left[\left\|A^{1/2}f'(t)\right\|\left\|A^{1/2}\varphi\right\|_H+\left\|A^{1/2}g(0)\right\|_H+\|g'(0)\|_H\right.\\&\qquad\left.+\max_{-1\le t\le 0}\|g''(t)\|_H+\left\|A^{1/2}f(0)\right\|_H+\max_{0\le t\le 1}\left\|A^{1/2}f'(t)\right\|\right]_H\end{aligned}\tag{2.24}$$

hold, where M is independent of $f(t),\ t\in[0,1], g(t),\ t\in[-1,0]$, and φ.

Note that there are three inequalities in Theorem 2.3 on the stability of solution, stability of first derivative of solution and stability of second derivative of solution. That means the solution of problem (1.1) $u(t)$ and its first and second derivatives are continuously dependent on $f(t), g(t)$ and φ.

Proof. First, estimate (2.22) will be obtained. Using formula (2.21) and integration by parts, we obtain

$$\begin{aligned} u_{-1} = T\Bigg(\alpha \Bigg\{ & -A^{-2} c(\mu) \left[g(0) - e^{iA} g(-1) - \int_{-1}^{0} e^{-iAy} g'(y) dy \right] \\ & + iA^{-1} s(\mu) \left(e^{iA} g(-1) + \int_{-1}^{0} e^{-iAy} g'(y) dy \right) \\ & + A^{-1} \int_{0}^{\mu} s(\mu - y) f(y) dy \Bigg\} + A^{-1}\varphi \Bigg). \end{aligned} \tag{2.25}$$

Using estimates (2.4), and (2.8), we get

$$\|u_{-1}\|_H \le M\left[\left\| A^{-1/2}\varphi \right\|_H + \left\| A^{-1} g(0) \right\|_H + \max_{-1\le t\le 0} \left\| A^{-1} g'(t) \right\|_H + \max_{0\le t\le 1} \left\| A^{-1} f(t) \right\|_H \right]. \tag{2.26}$$

Applying $A^{1/2}$ to the formula (2.25) and using estimates (2.4) and (2.8), we can write

$$\left\| A^{1/2} u_{-1} \right\|_H \le M\left[\|\varphi\|_H + \left\| A^{-1/2} g(0) \right\|_H + \left\| A^{-1/2} g'(t) \right\|_H + \max_{0\le t\le 1} \left\| A^{-1/2} f(t) \right\|_H \right]. \tag{2.27}$$

Using formulas (2.15) and (2.16) and integration by parts, we obtain

$$\begin{aligned} u(t) &= e^{i(t+1)A} u_{-1} + A^{-1}\left[g(t) - e^{i(t+1)A} g(-1) - \int_{-1}^{t} e^{i(t-y)A} g'(y) dy \right], \quad -1 \le t \le 0, \\ u(t) &= [c(t) + iAs(t)]\left\{ e^{iA} u_{-1} + A^{-1}\left(g(0) - e^{iA} g(-1) - \int_{-1}^{0} e^{-iAy} g'(y) dy \right) \right\} \\ &\quad - is(t) g(0) + \int_{0}^{t} s(t-y) f(y) dy, \quad 0 \le t \le 1. \end{aligned} \tag{2.28}$$

Using estimates (2.4) we get

$$\begin{aligned} \|u(t)\|_H &\le M\left[\|u_{-1}\|_H + \left\| A^{-1} g(0) \right\|_H + \max_{-1\le t\le 0} \left\| A^{-1} g'(t) \right\|_H \right], \quad -1 \le t \le 0, \\ \|u(t)\|_H &\le M\left[\left\| A^{1/2} u_{-1} \right\|_H + \left\| A^{-1/2} g(0) \right\|_H + \max_{-1\le t\le 0} \left\| A^{-1} g'(t) \right\|_H + \max_{0\le t\le 1} \left\| A^{-1/2} f(t) \right\|_H \right], \\ &\qquad 1 \le t \le 1. \end{aligned} \tag{2.29}$$

Then, from estimates (2.26), (2.27), and (2.29) it follows (2.22).

Second, (2.23) will be obtained. Applying $A^{1/2}$ to the formula (2.25) and using estimates (2.4),and (2.8), we obtain

$$\left\|A^{1/2}u_{-1}\right\|_{H} \leq M\left[\|\varphi\|_{H} + \left\|A^{-1/2}g(0)\right\|_{H} + \max_{-1\leq t\leq 0}\left\|A^{-1/2}g'(t)\right\|_{H} + \max_{0\leq t\leq 1}\left\|A^{-1/2}f(t)\right\|_{H}\right]. \tag{2.30}$$

Applying A to the formula (2.25) and using estimates (2.4), (2.8), we get

$$\|Au_{-1}\|_{H} \leq M\left[\left\|A^{1/2}\varphi\right\|_{H} + \|g(0)\|_{H} + \max_{-1\leq t\leq 0}\|g'(t)\|_{H} + \max_{0\leq t\leq 1}\|f(t)\|_{H}\right]. \tag{2.31}$$

Applying $A^{1/2}$ to the formulas (2.28), and using estimates (2.4) we can write

$$\begin{aligned}
\left\|A^{1/2}u(t)\right\|_{H} &\leq M\left[\left\|A^{1/2}u_{-1}\right\|_{H} + \left\|A^{-1/2}g(0)\right\|_{H} + \max_{-1\leq t\leq 0}\left\|A^{-1/2}g'(t)\right\|_{H}\right], \quad -1 \leq t \leq 0,\\
\left\|A^{1/2}u(t)\right\|_{H} &\leq M\left[\|Au_{-1}\|_{H} + \|g(0)\|_{H} + \max_{-1\leq t\leq 0}\left\|A^{-1/2}g'(t)\right\|_{H} + \max_{0\leq t\leq 1}\|f(t)\|_{H}\right], \quad 0 \leq t \leq 1.
\end{aligned} \tag{2.32}$$

Combining estimates (2.30), (2.31),and (2.32), we get estimate (2.23).

Third, estimate (2.24) will be obtained. Using formula (2.25) and integration by parts, we obtain

$$\begin{aligned}
u_{-1} = \Bigg(\alpha\Bigg\{&-A^{-2}c(\mu)\Bigg(g(0) - e^{-iA}g(-1) - iA^{-1}\\
&\times\left[g'(0) - e^{iA}g'(-1) - \int_{-1}^{0} e^{-iAy}g''(y)dy\right]\Bigg) + iA^{-1}s(\mu)\\
&\times\left[e^{iA}g(-1) + iA^{-1}\left(g'(0) - e^{iA}g'(-1) - \int_{-1}^{0} e^{-iAy}g''(y)dy\right)\right]\\
&-A^{-2}\left[f(\mu) + c(\mu)f(0) - \int_{0}^{\mu} c(\mu-y)f'(y)dy\right]\Bigg\} + A^{-1}\varphi\Bigg)
\end{aligned} \tag{2.33}$$

Applying A to formula (2.33) and using estimates (2.4) and (2.8), we get

$$\begin{aligned}
\|Au_{-1}\|_{H} \leq M\Bigg[&\left\|A^{1/2}\varphi\right\|_{H} + \|g(0)\|_{H} + \left\|A^{-1/2}g'(0)\right\|_{H}\\
&+\max_{-1\leq t\leq 0}\left\|A^{-1/2}g''(t)\right\|_{H} + \|f(0)\|_{H} + \max_{0\leq t\leq 1}\left\|A^{-1/2}f'(t)\right\|_{H}\Bigg].
\end{aligned} \tag{2.34}$$

Applying $A^{3/2}$ to formula (2.33) and using estimates (2.4), and (2.8) we can write

$$\begin{aligned} \left\|A^{3/2}u_{-1}\right\|_H \leq M\Big[&\|A\varphi\|_H + \left\|A^{1/2}g(0)\right\|_H + \|g'(0)\|_H \\ &+ \max_{-1\leq t\leq 0}\|g''(t)\|_H + \left\|A^{1/2}f(0)\right\|_H + \max_{0\leq t\leq 1}\|f'(t)\|_H\Big]. \end{aligned} \tag{2.35}$$

Using formulas (2.28), and integration by parts, we obtain

$$\begin{aligned} u(t) &= e^{i(t+1)A}u_{-1} + A^{-1}\Big[g(t) - e^{i(t+1)A}g(-1) \\ &\qquad - iA^{-1}\Big(g'(t) - e^{i(t+1)A}g'(-1) - \int_{-1}^{t} e^{i(t-y)A}g''(y)\,dy\Big)\Big], \quad -1 \leq t \leq 0, \\ u(t) &= \big[c(t) + iAs(t)\big]\Big\{e^{iA}u_{-1} + A^{-1}\Big(g(0) - e^{iA}g(-1) \\ &\qquad - iA^{-1}\Big[g'(0) - e^{iA}g'(-1) - \int_{-1}^{0} e^{-iAy}g''(y)\,dy\Big]\Big)\Big\} \\ &\quad - is(t)g(0) - A^{-1}\Big[f(t) - c(t)f(0) - \int_0^t c(t-y)f'(y)\,dy\Big], \quad 0 \leq t \leq 1. \end{aligned} \tag{2.36}$$

Applying A to the formulas (2.36), and using estimates (2.4), we get

$$\begin{aligned} \|Au(t)\|_H &\leq M\Big[\|Au_{-1}\|_H + \left\|A^{1/2}g(0)\right\|_H + \|g'(0)\|_H + \max_{-1\leq t\leq 0}\|g''(t)\|_H\Big], \quad -1 \leq t \leq 0, \\ \|Au(t)\|_H &\leq M\Big[\left\|A^{3/2}u_{-1}\right\|_H + \left\|A^{1/2}g(0)\right\|_H + \|g'(0)\|_H \\ &\qquad + \max_{-1\leq t\leq 0}\|g''(t)\|_H + \left\|A^{1/2}f(0)\right\|_H + \max_{0\leq t\leq 1}\left\|A^{1/2}f'(t)\right\|_H\Big], \quad 0 \leq t \leq 1. \end{aligned} \tag{2.37}$$

From (2.34) and (2.35) and estimates (2.37) it follows (2.24). This completes the proof of Theorem 2.3. □

Remark 2.4. We can obtain the same stability results for the solution of the following multi-point nonlocal boundary value problem:

$$\begin{aligned} &\frac{d^2u(t)}{dt^2} + Au(t) = f(t) \quad (0 \leq t \leq 1), \\ &i\frac{d(t)}{dt} + Au(t) = g(t) \quad (-1 \leq t \leq 0), \\ &Au(-1) = \sum_{j=1}^{N} \alpha_j u(\mu_j) + \varphi, \\ &0 < \mu_j \leq 1, \ 1 \leq j \leq N, \end{aligned} \tag{2.38}$$

for differential equations of mixed type in a Hilbert space H with self-adjoint positive definite operator A.

3. Applications

Initially, the mixed problem for the hyperbolic Schrödinger equation

$$\begin{gathered}
v_{yy} - (a(x)v_x)_x + \delta v = f(y,x), \quad 0 < y < 1,\ 0 < x < 1, \\
iv_y - (a(x)v_x)_x + \delta v = g(y,x), \quad -1 < y < 0,\ 0 < x < 1, \\
-(a(x)v_x(-1,x))_x + \delta v(-1,x) = \alpha v(1,x) + \varphi(x), \quad 0 \le x \le 1, \\
v(y,0) = v(y,1), \qquad v_x(y,0) = v_x(y,1), \quad -1 \le y \le 1, \\
v(0^+,x) = v(0^-,x), \qquad v_y(0^+,x) = v_y(0^-,x), \quad 0 \le x \le 1, \\
|\alpha| < \frac{\delta}{\sqrt{1+\delta}}
\end{gathered} \tag{3.1}$$

is considered, where $\delta = \text{const} > 0$. The problem (3.1) has a unique smooth solution $v(y,x)$ for smooth $a(x) \ge a > 0$ $(x \in (0,1))$, $\varphi(x)$ $(x \in [0,1])$, $f(y,x)$ $(y \in [0,1],\ x \in [0,1])$, and $g(y,x)$ $(y \in [-1,0], x \in [0,1])$ functions.

We introduce the Hilbert space $L_2[0,1]$ of all the square integrable functions defined on $[0,1]$ and Hilbert spaces $W_2^1[0,1]$ and $W_2^2[0,1]$ equipped with norms

$$\begin{aligned}
\|\varphi\|_{W_2^1[0,1]} &= \left(\int_0^1 |\varphi(x)|^2 dx\right)^{1/2} + \left(\int_0^1 |\varphi_x(x)|^2 dx\right)^{1/2}, \\
\|\varphi\|_{W_2^2[0,1]} &= \left(\int_0^1 |\varphi(x)|^2 dx\right)^{1/2} + \left(\int_0^1 |\varphi_x(x)|^2 dx\right)^{1/2} + \left(\int_0^1 |\varphi_{xx}(x)|^2 dx\right)^{1/2},
\end{aligned} \tag{3.2}$$

respectively. This allows us to reduce the mixed problem (3.1) to the nonlocal boundary value problem (1.1) in Hilbert space H with a self-adjoint positive definite operator A defined by problem (3.1).

Theorem 3.1. *The solutions of the nonlocal boundary value problem* (3.1) *satisfy the following stability estimates:*

$$\begin{aligned}
&\max_{-1\le y\le 1} \|v_y(y,\cdot)\|_{L_2[0,1]} + \max_{-1\le y\le 1} \|v(y,\cdot)\|_{W_2^1[0,1]} \\
&\quad \le M\left[\|\varphi\|_{L_2[0,1]} + \|g(0,\cdot)\|_{L_2[0,1]} + \max_{-1\le y\le 0}\|g_y(y,\cdot)\|_{L_2[0,1]} + \max_{0\le y\le 1}\|f(y,\cdot)\|_{L_2[0,1]}\right], \\
&\max_{-1\le y\le 1} \|v(y,\cdot)\|_{W_2^2[0,1]} + \max_{-1\le y\le 0} \|v_y(y,\cdot)\|_{L_2[0,1]} + \max_{0\le y\le 1}\|v_{yy}(y,\cdot)\|_{L_2[0,1]} \\
&\quad \le M\Bigg[\|\varphi\|_{W_2^1[0,1]} + \|g(0,\cdot)\|_{L_2[0,1]} + \|g_y(0,\cdot)\|_{L_2[0,1]} \\
&\qquad + \max_{-1\le y\le 0}\|g_{yy}(y,\cdot)\|_{L_2[0,1]} + \|f(0,\cdot)\|_{W_2^1[0,1]} + \max_{0\le y\le 1}\|f_y(y,\cdot)\|_{W_2^1[0,1]}\Bigg],
\end{aligned} \tag{3.3}$$

where M does not depend on not only $f(y,x)$ $(y \in [0,1], x \in [0,1])$ and $g(y,x)(y \in [-1,0], x \in [0,1])$ but also $\varphi(x)(x \in [0,1])$.

The proof of Theorem 3.1 is based on the abstract Theorem 2.3 and symmetry properties of the space operator defined by problem (3.1).

Next, we consider the mixed nonlocal boundary value problem for the multidimensional hyperbolic Schrödinger equation:

$$\begin{aligned}
&v_{yy} - \sum_{r=1}^{m} \left(a_r(x) v_{x_r}\right)_{x_r} = f(y,x), \quad 0 \le y \le 1, \\
&x = (x_1, \ldots, x_m) \in \Omega, \\
&i v_y - \sum_{r=1}^{m} \left(a_r(x) v_{x_r}\right)_{x_r} = g(y,x), \quad -1 \le y \le 0, \\
&x = (x_1, \ldots, x_m) \in \Omega, \\
&-\sum_{r=1}^{n} \left(a_r(x) v_{x_r}(-1,x)\right)_{x_r} = v(1,x) + \varphi(x), \quad x \in \overline{\Omega}, \\
&u(y,x) = 0, \quad x \in S, \quad -1 \le y \le 1,
\end{aligned} \tag{3.4}$$

where Ω is the unit open cube in the m-dimensional Euclidean space $\mathbb{R}^m$:

$$(x : x = (x_1, \ldots, x_m),\ 0 < x_k < 1,\ 1 \le k \le m) \tag{3.5}$$

with boundary S and $\overline{\Omega} = \Omega \cup S$. Here, $a_r(x)$ $(x \in \Omega), \varphi(x)$ $(x \in \overline{\Omega})$, and $f(y,x)$ $(y \in (0,1), x \in \Omega), g(y,x)$ $(y \in (-1,0), x \in \Omega)$ are given smooth functions in $[0,1] \times \Omega$ and $a_r(x) \ge a > 0$.

We introduce the Hilbert space $L_2(\overline{\Omega})$ of all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2} \tag{3.6}$$

and Hilbert spaces $W_2^1(\overline{\Omega})$ and $W_2^2(\overline{\Omega})$ defined on $\overline{\Omega}$, equipped with norms

$$\begin{aligned}
\|\varphi\|_{W_2^1(\overline{\Omega})} &= \|\varphi\|_{L_2(\overline{\Omega})} + \left\{ \int \cdots \int_{x \in \overline{\Omega}} \sum_{r=1}^{n} |\varphi_{x_r}|^2 dx_1 \cdots dx_n \right\}^{1/2}, \\
\|\varphi\|_{W_2^2(\overline{\Omega})} &= \|\varphi\|_{L_2(\overline{\Omega})} + \left\{ \int \cdots \int_{x \in \overline{\Omega}} \sum_{r=1}^{n} |\varphi_{x_r}|^2 dx_1 \cdots dx_n \right\}^{1/2} \\
&\quad + \left\{ \int \cdots \int_{x \in \overline{\Omega}} \sum_{r=1}^{n} |\varphi_{x_r x_r}|^2 dx_1 \cdots dx_n \right\}^{1/2},
\end{aligned} \tag{3.7}$$

respectively. The problem (3.4) has a unique smooth solution $v(y,x)$ for smooth $a_r(x), f(y,x)$, and $g(y,x)$ functions. This allows us to reduce the mixed problem (3.4) to the nonlocal boundary value problem (1.1) in Hilbert space H with a self-adjoint positive definite operator A defined by problem (3.4).

Theorem 3.2. *The following stability inequalities for solutions of the nonlocal boundary value problem* (3.4)

$$\begin{aligned}
&\max_{-1\le y\le 1}\|v_y(y,\cdot)\|_{L_2(\overline{\Omega})}+\max_{-1\le y\le 1}\|v(y,\cdot)\|_{W_2^1(\overline{\Omega})}\\
&\quad\le M\left[\|g(0,\cdot)\|_{L_2(\overline{\Omega})}+\max_{-1\le y\le 0}\|g_y(y,\cdot)\|_{L_2(\overline{\Omega})}+\max_{0\le y\le 1}\|f(y,\cdot)\|_{L_2(\overline{\Omega})}+\|\varphi\|_{L_2(\overline{\Omega})}\right],\\
&\max_{-1\le y\le 1}\|v(y,\cdot)\|_{W_2^2(\overline{\Omega})}+\max_{-1\le y\le 0}\|v_y(y,\cdot)\|_{L_2(\overline{\Omega})}+\max_{0\le y\le 1}\|v_{yy}(y,\cdot)\|_{L_2(\overline{\Omega})}\\
&\quad\le M\Bigg[\|\varphi\|_{W_2^1(\overline{\Omega})}+\|g(0,\cdot)\|_{L_2(\overline{\Omega})}+\|g_y(0,\cdot)\|_{L_2(\overline{\Omega})}\\
&\qquad+\max_{-1\le y\le 0}\|g_{yy}(y,\cdot)\|_{L_2(\overline{\Omega})}+\|f(0,\cdot)\|_{W_2^1(\overline{\Omega})}+\max_{0\le y\le 1}\|f_y(y,\cdot)\|_{W_2^1(\overline{\Omega})}\Bigg]
\end{aligned} \tag{3.8}$$

hold. Here, M is independent of $f(y,x)$ $(y\in[0,1],\ x\in[0,1])$, $g(y,x)$ $(y\in[-1,0],\ x\in[0,1])$, and $\varphi(x)$ $(x\in[0,1])$.

The proof of Theorem 3.2 is based on the abstract Theorem 2.3, symmetry properties of the space operator defined by problem (3.4), and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$ in Sobolevskii [28].

Theorem 3.3. *For the solutions of the elliptic differential problem*

$$\begin{gathered}
-\sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r}=\omega(x),\quad x\in\Omega,\\
u(x)=0,\quad x\in S,
\end{gathered} \tag{3.9}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{m}\|u_{x_r x_r}\|_{L_2(\overline{\Omega})}\le M\|\omega\|_{L_2(\overline{\Omega})}. \tag{3.10}$$

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