Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 683136, 14 pages
doi:10.1155/2012/683136

Research Article

Some New Inequalities of Opial's Type on Time Scales

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Received 28 February 2012; Accepted 16 April 2012

Academic Editor: Allan C. Peterson

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We will prove some new dynamic inequalities of Opial's type on time scales. The results not only extend some results in the literature but also improve some of them. Some continuous and discrete inequalities are derived from the main results as special cases. The results can be applied on the study of distribution of generalized zeros of half-linear dynamic equations on time scales.

1. Introduction

In 1960 Opial [1] proved that if x is absolutely continuous on $[a,b]$ with $x(a) = x(b) = 0$, then

$$\int_a^b |x(t)||x'(t)|dt \leq \frac{(b-a)}{4}\int_a^b |x'(t)|^2 dt. \tag{1.1}$$

Since the discovery of Opial's inequality much work has been done and many papers which deal with new proofs, various generalizations, and extensions have appeared in the literature. In further simplifying the proof of the Opial inequality which had already been simplified by Olech [2], Beesack [3], Levinson [4], Mallows [5], and Pederson [6], it is proved that if x is real absolutely continuous on $(0,b)$ and with $x(0) = 0$, then

$$\int_0^b |x(t)||x'(t)|dt \leq \frac{b}{2}\int_0^b |x'(t)|^2 dt. \tag{1.2}$$

These inequalities and their extensions and generalizations are the most important and fundamental inequalities in the analysis of qualitative properties of solutions of different types of differential equations.

In recent decades the asymptotic behavior of difference equations and inequalities and their applications have been and still are receiving intensive attention. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. So it is expected to see the discrete versions of the above inequalities. In fact, the discrete version of (1.1) which has been proved by Lasota [7] is given by

$$\sum_{i=1}^{h-1}|x_i \Delta x_i| \leq \frac{1}{2}\left[\frac{h+1}{2}\right]\sum_{i=1}^{h-1}|\Delta x_i|^2, \tag{1.3}$$

where $\{x_i\}_{0\leq i\leq h}$ is a sequence of real numbers with $x_0 = x_h = 0$ and $[x]$ is the greatest integer function. The discrete version of (1.2) is proved in [8, Theorem 5.2.2] and states that for a real sequence $\{x_i\}_{0\leq i\leq h}$ with $x_0 = 0$, we have

$$\sum_{i=1}^{h-1}|x_i \Delta x_i| \leq \frac{h-1}{2}\sum_{i=0}^{h-1}|\Delta x_i|^2. \tag{1.4}$$

These difference inequalities and their generalizations are also important and fundamental in the analysis of qualitative properties of solutions of difference equations.

Since the continuous and discrete inequalities are important in the analysis of qualitative properties of solutions of differential and difference equations, we also believe that the unification of these inequalities on time scales, which leads to dynamic inequalities on time scales, will play the same effective act in the analysis of qualitative properties of solutions of dynamic equations. The study of dynamic inequalities on time scales helps avoid proving results twice—once for differential inequality and once again for difference inequality. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. A cover story article in New Scientist [9] discusses several possible applications.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [10]), that is, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [11, 12] which summarize and organize much of the time scale calculus.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}, \tag{1.5}$$

where $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left-dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, is right-dense if $\sigma(t) = t$, is left-scattered if $\rho(t) < t$, and is right-scattered if $\sigma(t) > t$.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We will assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$.

Definition 1.1. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$\left|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{1.6}$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t.

We will frequently use the results in the following theorem which is due to Hilger [13].

Theorem 1.2. *Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*

(ii) *If g is continuous at t and t is right-scattered, then g is differentiable at t with*

$$g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}. \tag{1.7}$$

(iii) *If g is differentiable and t is right-dense, then*

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}. \tag{1.8}$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t) g^{\Delta}(t)$.*

In this paper we will refer to the (delta) integral which we can define as follows.

Definition 1.3. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a). \tag{1.9}$$

It can be shown (see [11]) that if $g \in C_{\mathrm{rd}}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t, \tag{1.10}$$

and the integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t). \tag{1.11}$$

However, the study of dynamic inequalities of the Opial types on time scales has been started by Bohner and Kaymakçalan [14] in 2001, only recently received a lot of attention and few papers have been written, see [14–17] and the references cited therein. For contributions of different types of inequalities on time scales, we refer also the reader to the papers [18–22] and the references cited therein. In the following, we recall some of the related results that have been established for dynamic inequalities on time scales that serve and motivate the contents of this paper.

In [14] the authors extended the inequality (1.1) on time scales and proved that if $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(0) = 0$, then

$$\int_0^h |x(t) + x^\sigma(t)|\left|x^\Delta(t)\right|\Delta t \le h\int_0^h \left|x^\Delta(t)\right|^2\Delta t. \tag{1.12}$$

Also in [14] the authors proved that if r and q are positive rd-continuous functions on $[0,b]$, $\int_a^b(\Delta t/r(t)) < \infty$, q nonincreasing, and $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(0) = 0$, then

$$\int_0^b q^\sigma(t)\left|(x(t) + x^\sigma(t))x^\Delta(t)\right|\Delta t \le \int_0^b \frac{\Delta t}{r(t)}\int_0^b r(t)q(t)\left|x^\Delta(t)\right|^2\Delta t. \tag{1.13}$$

Karpuz et al. [15] proved an inequality similar to inequality (1.13) replaced $q^\sigma(t)$ by $q(t)$ of the form

$$\int_0^b q(t)\left|(x(t) + x^\sigma(t))x^\Delta(t)\right|\Delta t \le K_q(a,b)\int_0^b \left|x^\Delta(t)\right|^2\Delta t, \tag{1.14}$$

where q is a positive rd-continuous function on $[0,b]$, and $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(0) = 0$ and

$$K_q(a,b) = \left(2\int_a^b q^2(u)(\sigma(u) - a)\Delta u\right)^{1/2}. \tag{1.15}$$

Wong et al. [16] and Sirvastava et al. [17] proved that if r is a positive rd-continuous function on $[a,b]$, we have

$$\int_a^b r(t)|x(t)|^p\left|x^\Delta(t)\right|^q\Delta t \le \frac{q}{p+q}(b-a)^p\int_a^b r(t)\left|x^\Delta(t)\right|^{p+q}\Delta t, \tag{1.16}$$

where $x : [0,b]\cap\mathbb{T} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$.

Following this trend, to develop the qualitative theory of dynamic inequalities on time scales, we will prove some new inequalities of Opial′s type. Some special cases on continuous and discrete spaces are derived and compared by previous results. The main results in this paper can be considered as the continuation of the paper [23] that has been published by the author and can be applied on the study of distribution of the generalized zeros of the half-linear dynamic equation:

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q(t)(x^{\sigma}(t))^{\gamma} = 0, \quad \text{on } [a,b]_{\mathbb{T}}, \tag{1.17}$$

and according to the limited space the applications of these inequalities will be discussed in a different paper.

2. Main Results

In this section, we will prove the main results and this will be done by making use of the Hölder inequality (see [11, Theorem 6.13]):

$$\int_a^h |f(t)g(t)|\Delta t \le \left[\int_a^h |f(t)|^{\gamma}\Delta t\right]^{1/\gamma}\left[\int_a^h |g(t)|^{\nu}\Delta t\right]^{1/\nu}, \tag{2.1}$$

where $a, h \in \mathbb{T}$ and $f,\ g \in C_{\mathrm{rd}}(\mathbb{I},\mathbb{R})$, $\gamma > 1$ and $1/\nu + 1/\gamma = 1$, and inequality (see [24, page 500])

$$|a+b|^r \le 2^{r-1}\left(|a|^r + |b|^r\right), \quad \text{for } r \ge 1, \tag{2.2}$$

where a, b are positive real numbers. We also need the formula

$$(x^{\gamma}(t))^{\Delta} = \int_0^1 \gamma[hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \tag{2.3}$$

which is a simple consequence of Keller's chain rule [11, Theorem 1.90]. Now, we are ready to state and prove the main results.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$ and p, q be positive real numbers such that $p \ge 1$, and let r, s be nonnegative rd-continuous functions on $(X,b)_{\mathbb{T}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [a,X] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, (and y^{Δ} does not change sign in $(a,X)_{\mathbb{T}}$), then one has*

$$\int_a^X s(x)\left|y(x) + y^{\sigma}(x)\right|^p\left|y^{\Delta}(x)\right|^q\Delta x \le K_1(a,X,p,q)\int_a^X r(x)\left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{2.4}$$

where

$$K_1(a,X,p,q) = 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{p+q-1}\Delta x\right)^{p/(p+q)} + 2^{p-1}\sup_{a\le x\le X}\left(\mu^p(x)\frac{s(x)}{r(x)}\right). \tag{2.5}$$

Proof. Since $y^\Delta(t)$ does not change sign in $(a,X)_{\mathbb{T}}$, we have

$$|y(x)| = \int_a^x \left|y^\Delta(t)\right|\Delta t, \quad \text{for } x\in[a,X]_{\mathbb{T}}. \tag{2.6}$$

This implies that

$$|y(x)| = \int_a^x \frac{1}{(r(t))^{1/(p+q)}}(r(t))^{1/(p+q)}\left|y^\Delta(t)\right|\Delta t. \tag{2.7}$$

Now, since r is nonnegative on $(a,X)_{\mathbb{T}}$, then it follows from the Hölder inequality (2.1) with

$$f(t) = \frac{1}{(r(t))^{1/(p+q)}}, \qquad g(t) = (r(t))^{1/(p+q)}\left|y^\Delta(t)\right|, \qquad \gamma = \frac{p+q}{p+q-1}, \qquad \nu = p+q, \tag{2.8}$$

that

$$\int_a^x \left|y^\Delta(t)\right|\Delta t \le \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{(p+q-1)/(p+q)}\left(\int_a^x r(t)\left|y^\Delta(t)\right|^{p+q}\Delta t\right)^{1/(p+q)}. \tag{2.9}$$

Then, for $a\le x\le X$, we get (note that $y(a)=0$) that

$$|y(x)|^p \le \left(\int_a^x \frac{1}{(r(t))^{1/(p+q-1)}}\Delta t\right)^{p((p+q-1)/(p+q))}\left(\int_a^x r(t)\left|y^\Delta(t)\right|^{p+q}\Delta t\right)^{p/(p+q)}. \tag{2.10}$$

Since $y^\sigma = y+\mu y^\Delta$, we have

$$y(x)+y^\sigma(x) = 2y(x)+\mu y^\Delta(x). \tag{2.11}$$

Applying inequality (2.2), we get (where $p\ge 1$) that

$$|y+y^\sigma|^p \le 2^{p-1}\left(2^p|y|^p+\mu^p\left|y^\Delta\right|^p\right) = 2^{2p-1}|y|^p + 2^{p-1}\mu^p\left|y^\Delta\right|^p. \tag{2.12}$$

Setting

$$z(x) := \int_a^x r(t) \left| y^\Delta(t) \right|^{p+q} \Delta t, \tag{2.13}$$

we see that $z(a) = 0$, and

$$z^\Delta(x) = r(x) \left| y^\Delta(x) \right|^{p+q} > 0. \tag{2.14}$$

From this, we get that

$$\left| y^\Delta(x) \right|^{p+q} = \frac{z^\Delta(x)}{r(x)}, \qquad \left| y^\Delta(x) \right|^{q} = \left(\frac{z^\Delta(x)}{r(x)} \right)^{q/(p+q)}. \tag{2.15}$$

Also since s is nonnegative on $(a, X)_{\mathbb{T}}$, we have from (2.12) and (2.15) that

$$\begin{aligned} s(x)\left|y(x)+y^\sigma(x)\right|^p \left|y^\Delta(x)\right|^q &\le 2^{2p-1} s(x) |y(x)|^p \left|y^\Delta(x)\right|^q + 2^{p-1}\mu^p(x) s(x) \left|y^\Delta\right|^{p+q} \\ &\le 2^{2p-1} s(x) \left(\frac{1}{r(x)}\right)^{q/(p+q)} \times \left(\int_a^x \frac{1}{r^{1/(p+q-1)}(t)} \Delta t \right)^{p((p+q-1)/(p+q))} \\ &\quad \times (z(x))^{p/(p+q)} \left(z^\Delta(x)\right)^{q/(p+q)} + 2^{p-1}\mu^p(x) s(x) \left(\frac{z^\Delta(x)}{r(x)} \right). \end{aligned} \tag{2.16}$$

This implies that

$$\begin{aligned} &\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p \left|y^\Delta(x)\right|^q \Delta x \\ &\quad\le 2^{2p-1} \int_a^X s(x) \left(\frac{1}{r(x)}\right)^{q/(p+q)} \times \left(\int_a^x \frac{1}{r^{1/(p+q-1)}(t)} \Delta t \right)^{p((p+q-1)/(p+q))} \\ &\qquad \times (z(x))^{p/(p+q)} \left(z^\Delta(x)\right)^{q/(p+q)} \Delta x + 2^{p-1} \int_a^X \left(\mu^p \frac{s(x)}{r(x)} \right) z^\Delta(x) \Delta x \\ &\quad\le 2^{2p-1} \int_a^X s(x) \left(\frac{1}{r(x)}\right)^{q/(p+q)} \times \left(\int_a^x \frac{1}{r^{1/(p+q-1)}(t)} \Delta t \right)^{p((p+q-1)/(p+q))} \\ &\qquad \times (z(x))^{p/(p+q)} \left(z^\Delta(x)\right)^{q/(p+q)} \Delta x + 2^{p-1} \max_{a\le x\le X} \left(\mu^p \frac{s(x)}{r(x)} \right) \int_a^X z^\Delta(x) \Delta x. \end{aligned} \tag{2.17}$$

Supposing that the integrals in (2.17) exist and again applying the Hölder inequality (2.1) with indices $p+q/p$ and $p+q/q$ on the first integral on the right hand side, we have

$$\begin{aligned}
&\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\\
&\quad\le 2^{2p-1}\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}\\
&\qquad\times\left(\int_a^X z^{p/q}(x)z^\Delta(x)\Delta x\right)^{q/(p+q)}+2^{p-1}\sup_{a\le x\le X}\left(\mu^p\frac{s(x)}{r(x)}\right)\int_a^X z^\Delta(x)\Delta x.
\end{aligned}\tag{2.18}$$

From (2.14), and the chain rule (2.3), we obtain

$$z^{p/q}(x)z^\Delta(x)\le\frac{q}{p+q}\left(z^{(p+q)/q}(x)\right)^\Delta.\tag{2.19}$$

Substituting (2.19) into (2.18) and using the fact that $z(a)=0$, we have that

$$\begin{aligned}
&\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\\
&\quad\le 2^{2p-1}\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}\\
&\qquad\times\left(\frac{p}{p+q}\right)^{q/(p+q)}\left(\int_a^X\left(z^{(p+q)/q}(t)\right)^\Delta\Delta t\right)^{q/(p+q)}+2^{p-1}\sup_{a\le x\le X}\left(\mu^p\frac{s(x)}{r(x)}\right)\int_a^X z^\Delta(x)\Delta x\\
&\quad=\left(\int_a^X s^{(p+q)/p}(x)\left(\frac{1}{r(x)}\right)^{q/p}\left(\int_a^x\frac{1}{r^{1/(p+q-1)}(t)}\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)}\\
&\qquad\times 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)}z(\mathrm{X})+2^{p-1}\sup_{a\le x\le X}\left(\mu^p\frac{s(x)}{r(x)}\right)z(\mathrm{X}).
\end{aligned}\tag{2.20}$$

Using (2.13), we have from the last inequality that

$$\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_1(a,b,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x,\tag{2.21}$$

which is the desired inequality (2.4). The proof is complete. □

Here, we only state the following theorem, since its proof is the same as that of Theorem 2.1, with $[a,X]$ replaced by $[b,X]$ and $|y(x)|=\int_x^b|y^\Delta(t)|\Delta t$.

Theorem 2.2. *Let $\mathbb{T}$ be a time scale with X, $b \in \mathbb{T}$ and p, q be positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(X, b)_{\mathbb{T}}$ such that $\int_X^b r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [X, b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(b) = 0$, (and y^Δ does not change sign in $(X, b)_{\mathbb{T}}$), then one has*

$$\int_X^b s(x)\left|y(x) + y^\sigma(x)\right|^p \left|y^\Delta(x)\right|^q \Delta x \leq K_2(X, b, p, q) \int_X^b r(x)\left|y^\Delta(x)\right|^{p+q} \Delta x, \tag{2.22}$$

where

$$\begin{aligned} K_2(X, b, p, q) = {} & 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \\ & \times \left(\int_X^b (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_x^b r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)} \Delta x\right)^{p/(p+q)} \\ & + 2^{p-1} \sup_{X \leq x \leq b}\left(\mu^p(x)\frac{s(x)}{r(x)}\right). \end{aligned} \tag{2.23}$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $y^\sigma = y$ and $\mu(x) = 0$. Then from Theorems 2.1 and 2.2 we have the following integral inequalities.

Corollary 2.3. *Assume that p, q be positive real numbers such that $p \geq 1$, and let r, s be nonnegative continuous functions on $(a, X)_{\mathbb{R}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)dt < \infty$. If $y : [a, X] \cap \mathbb{R} \to \mathbb{R}^+$ is differentiable with $y(a) = 0$, (and y^Δ does not change sign in $(a, X)_{\mathbb{R}}$), then one has*

$$\int_a^X s(x)|y(x)|^p |y'(x)|^q dx \leq C_1(a, X, p, q) \int_a^X r(x)|y'(x)|^{p+q} dx, \tag{2.24}$$

where

$$\begin{aligned} C_1(a, X, p, q) = {} & 2^{p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \\ & \times \left(\int_a^X (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_a^x r^{-1/(p+q-1)}(t)dt\right)^{(p+q-1)} dx\right)^{p/(p+q)}. \end{aligned} \tag{2.25}$$

Corollary 2.4. *Assume that p, q be positive real numbers such that $p \geq 1$, and let r, s be nonnegative continuous functions on $(X, b)_{\mathbb{R}}$ such that $\int_X^b r^{-1/(p+q-1)}(t)dt < \infty$. If $y : [X, b] \cap \mathbb{R} \to \mathbb{R}^+$ is delta differentiable with $y(b) = 0$, (and y' does not change sign in $(X, b)_{\mathbb{R}}$), then one has*

$$\int_X^b s(x)|y(x)|^p |y'(x)|^q dx \leq C_2(X, b, p, q) \int_X^b r(x)|y'(x)|^{p+q} dx, \tag{2.26}$$

where

$$C_2(X,b,p,q) = 2^{p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_X^b (s(x))^{(p+q)/p}(r(x))^{-q/p}\left(\int_x^b r^{-1/(p+q-1)}(t)dt\right)^{(p+q-1)}dx\right)^{p/(p+q)}. \tag{2.27}$$

In the following, we assume that there exists $h \in (a,b)$ which is the unique solution of the equation:

$$K(p,q) = K_1(a,h,p,q) = K_2(h,b,p,q) < \infty, \tag{2.28}$$

where $K_1(a,h,p,q)$ and $K_2(h,b,p,q)$ are defined as in Theorems 2.1 and 2.2. Note that since

$$\begin{aligned}\int_a^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x = \int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \\ + \int_X^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x,\end{aligned} \tag{2.29}$$

then the proof will be a combination of Theorems 2.1 and 2.2.

Theorem 2.5. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$ and p, q be positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(a,b)_{\mathbb{T}}$ such that $\int_a^b r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [a,b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0 = y(b)$, (and y^Δ does not change sign in $(a,b)_{\mathbb{T}}$), then one has*

$$\int_a^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \leq K(p,q)\int_a^b r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x. \tag{2.30}$$

For $r = s$ in Theorem 2.1, we obtain the following result.

Corollary 2.6. *Let $\mathbb{T}$ be a time scale with a, $X \in \mathbb{T}$ and p, q be positive real numbers such that $p \geq 1$, and let r be a nonnegative rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1/(p+q-1)}(t)\Delta t < \infty$. If $y : [a,X] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, (and y^Δ does not change sign in $(a,X)_{\mathbb{T}}$) then one has*

$$\int_a^X r(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \leq K_1^*(a,X,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{2.31}$$

where

$$K_1^*(a, X, p, q) = 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X r(x)\left(\int_a^x r^{-1/(p+q-1)}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{p/(p+q)} + 2^{p-1}\sup_{a\le x\le X}\left(\mu^p(x)\right). \tag{2.32}$$

From Theorems 2.2 and 2.5 one can derive similar results by setting $r = s$. The details are left to the reader.

On a time scale $\mathbb{T}$, we note from the chain rule (2.3) that

$$\begin{aligned}\left((t-a)^{p+q}\right)^{\Delta} &= (p+q)\int_0^1 \left[h(\sigma(t)-a)+(1-h)(t-a)\right]^{p+q-1}dh \\ &\ge (p+q)\int_0^1 \left[h(t-a)+(1-h)(t-a)\right]^{p+q-1}dh \\ &= (p+q)(t-a)^{p+q-1}.\end{aligned} \tag{2.33}$$

This implies that

$$\int_a^X (x-a)^{(p+q-1)}\Delta x \le \int_a^X \frac{1}{(p+q)}\left((x-a)^{p+q}\right)^{\Delta}\Delta x = \frac{(X-a)^{p+q}}{(p+q)}. \tag{2.34}$$

From this and (2.32) (by putting $r(t) = 1$), we get that that

$$\begin{aligned}K_1^*(a, X, p, q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)} \times \left(\int_a^X (x-a)^{(p+q-1)}\Delta x\right)^{p/(p+q)} \\ &\le 2^{2p-1}\left(\frac{q}{p+q}\right)^{q/(p+q)}\left(\frac{(X-a)^{p+q}}{(p+q)}\right)^{p/(p+q)} + 2^{p-1}\max_{a\le x\le X}\left(\mu^p(x)\right) \\ &= 2^{p-1}\max_{a\le x\le X}\left(\mu^p(x)\right) + 2^{2p-1}\frac{q^{q/(p+q)}}{p+q}(X-a)^p.\end{aligned} \tag{2.35}$$

So setting $r = 1$ in (2.31) and using (2.35), we have the following result.

Corollary 2.7. *Let $\mathbb{T}$ be a time scale with $a, X \in \mathbb{T}$ and p, q be positive real numbers such that $p \ge 1$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, (and y^{Δ} does not change sign in $(a, X)_{\mathbb{T}}$), then one has*

$$\int_a^X \left|y(x)+y^{\sigma}(x)\right|^p\left|y^{\Delta}(x)\right|^q\Delta x \le L(a, b, p, q)\int_a^X \left|y^{\Delta}(x)\right|^{p+q}\Delta x, \tag{2.36}$$

where

$$L(a,b,p,q) := \left(2^{2p-1} \frac{q^{q/(p+q)}}{p+q} \times (X-a)^p + 2^{p-1} \sup_{a \le x \le X} \mu^p(x) \right). \tag{2.37}$$

Remark 2.8. Note that when $\mathbb{T} = \mathbb{R}$, we have $y^\sigma = y$, $\mu(x) = 0$ and then the inequality (2.36) becomes

$$\int_a^X |y(x)|^p |y'(x)|^q dx \le 2^{p-1} \frac{q^{q/(p+q)}}{(p+q)} \times (X-a)^p \int_a^X |y'(x)|^{p+q} dx. \tag{2.38}$$

Note also that when $p = 1$ and $q = 1$, then the inequality (2.38) becomes

$$\int_a^X |y(x)||y'(x)| dx \le \frac{(X-a)}{2} \int_a^X |y'(x)|^2 dx, \tag{2.39}$$

which is the Opial inequality (1.2).

When $\mathbb{T} = \mathbb{N}$, we have form (2.36) the following discrete Opial's type inequality.

Corollary 2.9. *Assume that p, q be positive real numbers such that $p \ge 1$ and $\{r_i\}_{0 \le i \le N}$ be a nonnegative real sequence. If $\{y_i\}_{0 \le i \le N}$ is a sequence of positive real numbers with $y(0) = 0$, then*

$$\begin{aligned} &\sum_{n=1}^{N-1} r(n) |y(n) + y(n+1)|^p |\Delta y(n)|^q \\ &\quad \le \left(2^{2p-1} \frac{q^{q/(p+q)} (N-a)^p}{(p+q)} + 2^{p-1} \right) \sum_{n=0}^{N-1} r(n) |\Delta y(n)|^{p+q}. \end{aligned} \tag{2.40}$$

The inequality (2.36) has immediate application to the case where $y(a) = y(b) = 0$. Choose $X = (a+b)/2$ and apply (2.32) to $[a, c]$ and $[c, b]$ and then add we obtain the following inequality.

Corollary 2.10. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$ and p, q be positive real numbers such that $p \ge 1$. If $y : [a, b] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b |y(x) + y^\sigma(x)|^p \left| y^\Delta(x) \right|^q \Delta x \le F(a,b,p,q) \int_a^b \left| y^\Delta(x) \right|^{p+q} \Delta x, \tag{2.41}$$

where

$$F(a,b,p,q) := 2^{2p-1} \frac{q^{q/(p+q)}}{p+q} \left(\frac{b-a}{2} \right)^p + 2^{p-1} \sup_{a \le x \le b} (\mu^p(x)). \tag{2.42}$$

From this inequality, we have the following discrete Opial type inequality.

Corollary 2.11. *Assume that p, q be positive real numbers such that $p \geq 1$. If $\{y_i\}_{0\leq i\leq N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1} r(n)\left|y(n)+y(n+1)\right|^{p}\left|\Delta y(n)\right|^{q} \leq \left(2^{2p-1}\frac{q^{q/(p+q)}}{p+q}\left(\frac{N-a}{2}\right)^{p}+2^{p-1}\right)\sum_{n=0}^{N-1} r(n)\left|\Delta y(n)\right|^{p+q}. \tag{2.43}$$

By setting $p = q = 1$ in (2.41) we have the following Opial type inequality on a time scale.

Corollary 2.12. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $y : [a, X] \cap \mathbb{T} \to \mathbb{R}^{+}$ is delta differentiable with $y(a) = 0 = y(b)$, then one has*

$$\int_a^b \left|y(x)+y^{\sigma}(x)\right|\left|y^{\Delta}(x)\right|\Delta x \leq \left(\frac{(b-a)}{2}+\sup_{a\leq x\leq b}(\mu(x))\right)\int_a^b \left|y^{\Delta}(x)\right|^{2}\Delta x. \tag{2.44}$$

As special cases from (2.44) on the continuous and discrete spaces, that is, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, we have the following inequalities.

Corollary 2.13. *If $y : [a, b] \cap \mathbb{T} \to \mathbb{R}$ is differentiable with $y(a) = 0 = y(b)$, then one has the Opial inequality*

$$\int_a^b |y(x)||y'(x)|dx \leq \frac{(b-a)}{4}\int_a^b |y'(x)|^{2}dx. \tag{2.45}$$

Corollary 2.14. *If $\{y_i\}_{0\leq i\leq N}$ is a sequence of real numbers with $y(0) = 0 = y(N)$, then*

$$\sum_{n=1}^{N-1}|y(n)+y(n+1)||\Delta y(n)| \leq \left(\frac{N}{2}+1\right)\sum_{n=0}^{N-1}|\Delta y(n)|^{2}. \tag{2.46}$$

Acknowledgment

This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Centre.

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