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Research Article

Approximate Solutions of Delay Parabolic Equations with the Dirichlet Condition

Deniz Agirseven

Department of Mathematics, Trakya University, 22030 Edirne, Turkey

Correspondence should be addressed to Deniz Agirseven, denizagirseven@yahoo.com

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Finite difference and homotopy analysis methods are used for the approximate solution of the initial-boundary value problem for the delay parabolic partial differential equation with the Dirichlet condition. The convergence estimates for the solution of first and second orders of difference schemes in Hölder norms are obtained. A procedure of modified Gauss elimination method is used for the solution of these difference schemes. Homotopy analysis method is applied. Comparison of finite difference and homotopy analysis methods is given on the problem.

1. Introduction

Increase in interest in the theoretical aspects of numerical methods for delay differential equations points out that delay differential equations are capable of generating extensive and conceivable models for phenomena in many branches of sciences. Numerical solutions of the delay ordinary differential equations have been studied mostly for ordinary differential equations (cf., e.g., [1–14] and the references therein). Nevertheless, delay partial differential equations are less in demand than delay ordinary differential equations. Different kinds of problems for delay partial differential equations are solved by using operator approach (see, e.g., [15–17]).

In recent years, Ashyralyev and Sobolevskii considered the initial-value problem for linear delay partial differential equations of parabolic type in the spaces $C(E_\alpha)$ of functions defined on the segment $[0,\infty)$ with values in a Banach space E_α and the stability inequalities were established under stronger assumption than the necessary condition of the stability of the differential problem. The stability estimates for the solutions of difference schemes of the first- and second-order accuracy difference schemes for approximately solving this initial-value problem for delay differential equations of parabolic type were presented. They obtained the stability estimates in Hölder norms for solutions of the initial-value problem of the delay differential and difference equations of the parabolic type [15, 16]. Gabriella used extrapolation spaces to solve Banach spaces valued delay differential equations with unbounded delay operators. The author proved regularity properties of various types of solutions and investigated the existence of strong and weak solutions for a class of abstract semilinear delay equations [17].

In this paper, finite difference (see, e.g., [18–28]) and homotopy analysis methods (HAM) (see, e.g., [29–37]) for the approximate solutions of the delay differential equation of the parabolic type

$$\begin{aligned}
&u_t(t,x) + (-a(x)u_{xx}(t,x) + b(x)u_x(t,x) + c(x)u(t,x)) \\
&\quad = d(t)(-a(x)u_{xx}(t-\omega,x) + b(x)u_x(t-\omega,x) \\
&\qquad + c(x)u(t-\omega,x)), \quad 0 < t < \infty, \ x \in (0,l), \\
&u(t,x) = g(t,x), \quad -\omega \le t \le 0, \ x \in [0,l], \\
&u(t,0) = u(t,l) = 0, \quad t \ge 0,
\end{aligned} \tag{1.1}$$

are studied. Here $g(t,x)$ $(t \in (-\infty,0),\ x \in [0,l])$, $a(x), b(x), c(x)$ $(x \in (0,\infty))$ are given smooth bounded functions and $a(x) \ge a > 0$.

Difference schemes which are accurate to first and second orders for the approximate solution of problem (1.1) are presented. The convergence estimates for the solution of these difference schemes are obtained. For the numerical study, procedure of modified Gauss elimination method is used to solve these difference schemes. Homotopy analysis method is applied to find the solution of problem (1.1). The numerical results are obtained at the same points for each method. Comparison of finite difference and homotopy analysis methods is given on the problem.

2. The Finite Difference Method

In this section, the first and second orders of accuracy in t for the approximate solution of problem (1.1) are considered. The convergence estimates for the solution of these difference schemes are established. A procedure of modified Gauss elimination method is used to solve these difference schemes.

2.1. The Difference Scheme, Convergence Estimates

The discretization of problem (1.1) is carried out in two steps. In the first step, we define the grid space

$$[0,L]_h = \{x = x_n : x_n = nh,\ 0 \le n \le M,\ Mh = L\}. \tag{2.1}$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}_h^{\alpha} = \overset{\circ}{C}{}^{\alpha}[0,L]_h$, $\alpha \in [0,1)$, of all grid functions $\varphi^h = \{\varphi_n\}_{n=1}^{M-1}$ defined on $[0,L]_h$ with $\varphi_0 = \varphi_M = 0$ equipped with the norm

$$\left\|\varphi^h\right\|_{\overset{\circ}{C}{}_h^{\alpha}} = \left\|\varphi^h\right\|_{C_h} + \sup_{1\le n<n+r\le M-1} \left|\varphi_{n+r} - \varphi_n\right| (rh)^{-\alpha},$$
$$\left\|\varphi^h\right\|_{C_h} = \max_{1\le n\le M-1} \left|\varphi_n\right|. \tag{2.2}$$

Moreover, $C_\tau(E) = C([0,\infty)_\tau, E)$ is the Banach space of all grid functions $f^\tau = \{f_k\}_{k=1}^{\infty}$ defined on

$$[0,\infty)_\tau = \{t_k = k\tau,\ k = 0,1,\dots\} \tag{2.3}$$

with values in E equipped with the norm

$$\|f^\tau\|_{C_\tau(E)} = \sup_{1\le k<\infty} \|f_k\|_E. \tag{2.4}$$

To the differential operator A generated by problem (1.1), we assign the difference operators A_h^x, B_h^x by the formulas

$$A_h^x \varphi^h(x) = \left\{ -a(x_n) \frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{h^2} + b(x_n) \frac{\varphi_{n+1} - \varphi_{n-1}}{2h} + c(x_n)\varphi_n \right\}_1^{M-1},$$
$$B_h^x(t)\varphi^h(x) = d(t) A_h^x \varphi^h, \tag{2.5}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_1^{M-1}$ satisfying the conditions $\varphi_0 = \varphi_M = 0$. It is well known that A_h^x is a strongly positive operator in C_h. With the help of A_h^x and $d(t)A_h^x$, we arrive at the initial value problem

$$\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = d(t) A_h^x u^h(t-w,x), \quad 0 < t < \infty,\ 0 < x < L,$$
$$u^h(t,x) = g^h(t,x), \quad -\omega \le t \le 0,\ 0 \le x \le L. \tag{2.6}$$

In the second step, we consider difference schemes of first and second orders of accuracy

$$\frac{1}{\tau}\left(u_k^h(x) - u_{k-1}^h(x)\right) + A_h^x u_k^h(x) = d(t_k) A_h^x u_{k-N}^h(x), \quad t_k = k\tau,\ 1 \le k,\ N\tau = w,$$
$$u_k^h(x) = g^h(t_k, x), \quad t_k = k\tau,\ -N \le k \le 0, \tag{2.7}$$

$$\frac{1}{\tau}\left(u_k^h(x) - u_{k-1}^h(x)\right) + \left(A_h^x + \frac{1}{2}\tau (A_h^x)^2\right) u_k^h(x)$$
$$= \frac{1}{2}\left(I + \frac{\tau}{2} A_h^x\right) d\left(t_k - \frac{\tau}{2}\right) A_h^x \left(u_{k-N}^h(x) + u_{k-N-1}^h(x)\right), \quad t_k = k\tau,\ 1 \le k,$$
$$u_k^h = g^h(t_k, x), \quad t_k = k\tau,\ -N \le k \le 0. \tag{2.8}$$

Theorem 2.1. *Assume that*

$$\sup_{0\le t<\infty}|d(t)| \le \frac{1-\alpha}{M2^{2-\alpha}}. \tag{2.9}$$

Suppose that problem (1.1) *has a smooth solution* $u(t,x)$ *and*

$$\int_0^\infty \left[\max_{0\le x\le L}|u_{ss}(s,x)| + \sup_{0<x<x+y<L}\frac{|u_{ss}(s,x+y)-u_{ss}(s,x)|}{y^{2\alpha}}\right]ds<\infty,$$
$$\int_0^\infty \left[\max_{0\le x\le L}|u_{xxxx}(s,x)| + \sup_{0<x<x+y<L}\frac{|u_{xxxx}(s,x+y)-u_{xxxx}(s,x)|}{y^{2\alpha}}\right]ds<\infty. \tag{2.10}$$

Then, for the solution of difference scheme (2.7)*, the following convergence estimate holds:*

$$\sup_k \left\| u_k^h - u^h(t_k,\cdot)\right\|_{\overset{\circ}{C}{}_h^{2\alpha}} \le M_1\left(\tau+h^2\right) \tag{2.11}$$

with M_1 *being a real number independent of* τ*,* α*, and* h*.*

Proof. Using notations of A_h^x and B_h^x, we can obtain the following formula for the solution:

$$u_k^h(x) = R^k g^h(0,x) + \sum_{j=1}^{k} R^{k-j+1}B_j^x g^h(t_{j-N},x)\tau, \quad 1\le k\le N,$$
$$u_k^h(x) = R^{k-nN}u_{nN}^h(x) + \sum_{j=n\,N+1}^{k} R^{k-j+1}B_j^x u_{j-N}^h(x)\tau,$$
$$nN\le k\le (n+1)N, \tag{2.12}$$

where $R=(I+\tau A_h^x)^{-1}$. The proof of Theorem 2.1 is based on the formulas (2.12), on the convergence theorem, on the difference schemes in $C_\tau(E_\alpha^h)$ (see [38]), on the estimate

$$\|\exp\{-t_k A_h^x\}\|_{C_h\to C_h}\le M, \quad k\ge 0, \tag{2.13}$$

and on the fact that in $E_\alpha^h = E_\alpha(A_h^x, C_h)$ the norms are equivalent to the norms in $\overset{\circ}{C}{}_h^{2\alpha}$ uniformly in h for $0<\alpha<1/2$ (see, [18]). □

Theorem 2.2. *Assume that assumption* (2.9) *of Theorem 2.1 and the following conditions hold:*

$$\int_0^\infty \left[\max_{0\le x\le L}|u_{sss}(s,x)| + \sup_{0<x<x+y<L}\frac{|u_{sss}(s,x+y)-u_{sss}(s,x)|}{y^{2\alpha}}\right]ds<\infty,$$
$$\int_0^\infty \left[\max_{0\le x\le L}|u_{xxss}(s,x)| + \sup_{0<x<x+y<L}\frac{|u_{xxss}(s,x+y)-u_{xxss}(s,x)|}{y^{2\alpha}}\right]ds<\infty, \tag{2.14}$$
$$\int_0^\infty \left[\max_{0\le x\le L}|u_{xxxxs}(s,x)| + \sup_{0<x<x+y<L}\frac{|u_{xxxxs}(s,x+y)-u_{xxxxs}(s,x)|}{y^{2\alpha}}\right]ds<\infty.$$

Then for the solution of difference scheme (2.8), *the following convergence estimate is satisfied:*

$$\sup_k\left\|u_k^h-u^h(t_k,\cdot)\right\|_{\overset{\circ}{C}{}_h^{2\alpha}}\le M_2\left(\tau^2+h^2\right) \tag{2.15}$$

with M_2 being a real number independent of τ, α, and h.

Proof. Using notations of A_h^x and B_h^x again, we can obtain the following formula for the solution:

$$u_k^h(x)=R^k g^h(0,x)+\sum_{j=1}^{k}R^{k-j+1}\left(I+\frac{\tau A_h^x}{2}\right)\left(g^h(t_{j-N},x)-g^h(t_{j-N-1},x)\right)\tau,\quad 1\le k\le N,$$
$$u_k^h(x)=R^{k-nN}u_{nN}^h(x)+\sum_{j=nN+1}^{k}R^{k-j+1}\left(I+\frac{\tau A_h^x}{2}\right)B_j^x\frac{1}{2}\left(u_{j-N}^h(x)+u_{j-N-1}^h(x)\right)\tau,$$
$$nN\le k\le(n+1)N, \tag{2.16}$$

where $R=(I+\tau A_h^x+(\tau A_h^x)^2/2)^{-1}$. The proof of Theorem 2.2 is based on the formulas (2.16), on the convergence theorem, on the difference schemes in $C_\tau(E_\alpha^h)$ (see, [38]), on the estimate (2.13), and on the equivalence of the norms as in Theorem 2.1.

Finally, the numerical methods are given in the following section for the solution of delay parabolic differential equation with the Dirichlet condition. The method is illustrated by numerical examples. □

2.2. Numerical Results

We consider the initial-boundary-value problem

$$\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}+(0.1)\frac{\partial^2 u(t-1,x)}{\partial x^2}=0,\quad t>0,\ 0<x<\pi,$$
$$u(t,x)=e^{-t}\sin x,\quad -1\le t\le 0,\ 0\le x\le\pi, \tag{2.17}$$
$$u(t,0)=u(t,\pi)=0,\quad t\ge 0,$$

for the delay parabolic differential equation.

The exact solution of this problem for $t \in [n-1,n]$, $n = 0,1,2,\ldots, x \in [0,\pi]$ is

$$u(t,x) = \begin{cases} e^{-t}\sin x, & -1 \le t \le 0, \\ e^{-t}\{1+(0.1)\ et\}\sin x, & 0 \le t \le 1, \\ e^{-t}\left\{1+(0.1)\ et + \dfrac{[(0.1)e(t-1)]^2}{2!}\right\}\sin x, & 1 \le t \le 2, \\ \quad\vdots & \\ e^{-t}\left\{1+(0.1)\ et + \dfrac{[(0.1)e(t-1)]^2}{2!} + \cdots + \dfrac{[(0.1)e(t-n)]^{(n+1)}}{(n+1)!}\right\}\sin x, & n \le t \le n+1, \\ \quad\vdots & \end{cases} \tag{2.18}$$

For the approximate solution of delay parabolic equation (2.17), consider the set of grid points

$$[-1,\infty]_\tau \times [0,\pi]_h = \{(t_k,x_n) : t_k = k\tau,\ -N \le k \le \infty,\ x_n = nh,\ 0 \le n \le M,\ Mh = \pi\}. \tag{2.19}$$

Using difference scheme accurate to first order for the approximate solutions of the initial-boundary-value problem for the delay parabolic equation (2.17), we get the following system of equations:

$$\begin{gathered} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + (0.1)\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{h^2} = 0, \\ mN+1 \le k \le (m+1)N, \quad m = 0,1,\ldots,\ 1 \le n \le M-1, \\ u_n^k = e^{-t_k}\sin x_n, \quad -N \le k \le 0,\ 0 \le n \le M, \\ u_0^k = u_M^k = 0, \quad k \ge 0. \end{gathered} \tag{2.20}$$

In this first step, applying difference scheme accurate to first order, we obtain a system of equations in matrix form

$$\begin{gathered} AU_{n+1}^m + BU_n^m + CU_{n-1}^m = R\varphi_n^m, \quad 1 \le n \le M-1,\ m = 0,1,\ldots, \\ U_0^m = \tilde{0}, \qquad U_M^m = \tilde{0}, \end{gathered} \tag{2.21}$$

where A, B, C are $(N+1)\times(N+1)$ matrices defined by

$$A=\begin{bmatrix} 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&a&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&a&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&a&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&0&a&\cdot&\cdot&\cdot&0&0&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&0&0&\cdot&\cdot&\cdot&a&0&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&a&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&a&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&a \end{bmatrix}_{(N+1)\times(N+1)},$$
$$B=\begin{bmatrix} 1&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ b&c&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&b&c&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&0&b&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&0\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&c&0\\ 0&0&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&0&\cdot&\cdot&\cdot&b&c \end{bmatrix}_{(N+1)\times(N+1)}. \tag{2.22}$$

$C=A$, R is $(N+1)\times(N+1)$ identity matrix and φ_n^m, U_s^m are $(N+1)\times 1$ column vectors as

$$\varphi_n^m=\begin{bmatrix}\varphi_n^{mN}\\ \varphi_n^{mN+1}\\ \vdots\\ \varphi_n^{(m+1)N}\end{bmatrix}_{(N+1)\times(1)},\qquad U_s^m=\begin{bmatrix}U_s^{mN}\\ U_s^{mN+1}\\ \vdots\\ U_s^{(m+1)N}\end{bmatrix}_{(N+1)\times(1)}\qquad \text{for } s=n\pm 1, n, \tag{2.23}$$

where u_n^{mN} is given for any $m=0,1,\ldots,$

$$\begin{gathered}\varphi_n^k=-(0.1)\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{h^2},\\ mN+1\le k\le(m+1)N,\quad m=0,1,\ldots,\ 1\le n\le M-1,\\ u_n^k=e^{-t_k}\sin x_n,\quad -N\le k\le 0.\end{gathered} \tag{2.24}$$

Here, we denote

$$a=-\frac{1}{h^2},\qquad b=-\frac{1}{\tau},\qquad c=\frac{1}{\tau}+\frac{2}{h^2}. \tag{2.25}$$

So, we have second-order difference equation (2.21) with matrix coefficients. To solve this difference equation, we have applied a procedure of modified Gauss elimination method. Hence, we obtain a solution of the matrix equation in the following form:

$$\begin{gathered} U_j^m = \alpha_{j+1} U_{j+1}^m + \beta_{j+1}^m, \quad j = M-1, \ldots, 2, 1, \\ U_M^m = 0, \end{gathered} \tag{2.26}$$

where α_j ($j = 1, \ldots, M$) are $(N+1) \times (N+1)$ square matrices and β_j^m ($j = 1, \ldots, M$) are $(N+1) \times 1$ column matrices defined by

$$\begin{gathered} \alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1} A, \\ \beta_{j+1}^m = \left(B + C\alpha_j\right)^{-1} \left(R\varphi_j^m - C\beta_j\right), \end{gathered} \tag{2.27}$$

where $j = 1, \ldots, M-1$, α_1 is the $(N+1) \times (N+1)$ zero matrix, and β_1^m is the $(N+1) \times 1$ zero matrix.

Second, using the second order of accuracy difference scheme for the approximate solutions of problem (2.17) and applying formulae

$$\begin{gathered} \frac{2u(0) - 5u(h) + 4u(2h) - u(3h)}{h^2} - u''(0) = O\left(h^2\right), \\ \frac{2u(1) - 5u(1-h) + 4u(1-2h) - u(1-3h)}{h^2} - u''(1) = O\left(h^2\right), \end{gathered} \tag{2.28}$$

we obtain the following system of equations:

$$\begin{gathered} \frac{u_n^k - u_n^{k-1}}{\tau} - \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \frac{\tau}{2}\left(\frac{u_{n+2}^k - 4u_{n+1}^k + 6u_n^k - 4u_{n-1}^k + u_{n-2}^k}{h^4}\right) \\ +(0.1)\left\{\frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{2h^2} + \frac{u_{n+1}^{k-1-N} - 2u_n^{k-1-N} + u_{n-1}^{k-1-N}}{2h^2}\right. \\ -\frac{\tau}{2}\left[\frac{u_{n+2}^{k-N} - 4u_{n+1}^{k-N} + 6u_n^{k-N} - 4u_{n-1}^{k-N} + u_{n-2}^{k-N}}{2h^4}\right. \\ \left.\left. + \frac{u_{n+2}^{k-1-N} - 4u_{n+1}^{k-1-N} + 6u_n^{k-1-N} - 4u_{n-1}^{k-1-N} + u_{n-2}^{k-1-N}}{2h^4}\right]\right\} = 0 \\ mN + 1 \leq k \leq (m+1)N, \quad m = 0, 1, \ldots, \quad 2 \leq n \leq M-2, \\ u_n^k = e^{-t_k} \sin x_n, \quad -N \leq k \leq 0, \ 0 \leq n \leq M, \\ u_1^k = \frac{4}{5}u_2^k - \frac{1}{5}u_3^k, \quad k \geq 0, \\ u_{M-1}^k = \frac{4}{5}u_{M-2}^k - \frac{1}{5}u_{M-3}^k, \quad k \geq 0, \\ u_0^k = u_M^k = 0, \quad k \geq 0. \end{gathered} \tag{2.29}$$

In the second step, we apply second-order difference scheme to get the system of linear equations in matrix form

$$\begin{gathered} AU_{n+2}^{m}+BU_{n+1}^{m}+CU_{n}^{m}+DU_{n-1}^{m}+EU_{n-2}^{m}=R\varphi_{n}^{m}, \\ m=0,1,\ldots, \quad 2\leq n\leq M-2, \\ U_{0}^{m}=\tilde{0}, \quad U_{M}^{m}=\tilde{0}, \\ U_{1}^{m}=\frac{4}{5}U_{2}^{m}-\frac{1}{5}U_{3}^{m}, \\ U_{M-1}^{m}=\frac{4}{5}U_{M-2}^{m}-\frac{1}{5}U_{M-3}^{m}, \end{gathered} \tag{2.30}$$

where A, B, C are $(N+1)\times(N+1)$ matrices defined by

$$\begin{gathered} A=\begin{bmatrix} 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&x&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&x&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&x&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&0&x&\cdot&\cdot&\cdot&0&0&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&0&0&\cdot&\cdot&\cdot&x&0&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&x&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&x&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&x \end{bmatrix}_{(N+1)\times(N+1)}, \\ B=\begin{bmatrix} 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&y&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&y&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&y&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&0&y&\cdot&\cdot&\cdot&0&0&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&0&0&\cdot&\cdot&\cdot&y&0&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&y&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&y&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&y \end{bmatrix}_{(N+1)\times(N+1)}, \\ C=\begin{bmatrix} 1&0&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ z&t&0&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&z&t&0&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&z&t&0&\cdot&\cdot&\cdot&0&0&0&0\\ 0&0&0&z&t&\cdot&\cdot&\cdot&0&0&0&0\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ 0&0&0&0&0&\cdot&\cdot&\cdot&z&t&0&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&z&t&0\\ 0&0&0&0&0&\cdot&\cdot&\cdot&0&0&z&t \end{bmatrix}_{(N+1)\times(N+1)}, \end{gathered} \tag{2.31}$$

$B = D$, $E = A$, R is $(N+1) \times (N+1)$ identity matrix, and φ_n^m, U_s^m are $(N+1) \times 1$ column vectors as

$$\varphi_n^m = \begin{bmatrix} \varphi_n^{mN} \\ \varphi_n^{mN+1} \\ \vdots \\ \varphi_n^{(m+1)N} \end{bmatrix}_{(N+1)\times(1)}, \qquad U_s^m = \begin{bmatrix} U_s^{mN} \\ U_s^{mN+1} \\ \vdots \\ U_s^{(m+1)N} \end{bmatrix}_{(N+1)\times(1)} \quad \text{for } s = n \pm 1, n \pm 2, n, \tag{2.32}$$

where u_n^{mN} is given for any $m = 0, 1, \ldots,$

$$\begin{gathered} \varphi_n^k = -(0.1)\left\{ \frac{u_{n+1}^{k-N} - 2u_n^{k-N} + u_{n-1}^{k-N}}{2h^2} + \frac{u_{n+1}^{k-1-N} - 2u_n^{k-1-N} + u_{n-1}^{k-1-N}}{2h^2} \right. \\ -\frac{\tau}{2}\left[\frac{u_{n+2}^{k-N} - 4u_{n+1}^{k-N} + 6u_n^{k-N} - 4u_{n-1}^{k-N} + u_{n-2}^{k-N}}{2h^4} \right. \\ \left.\left. + \frac{u_{n+2}^{k-1-N} - 4u_{n+1}^{k-1-N} + 6u_n^{k-1-N} - 4u_{n-1}^{k-1-N} + u_{n-2}^{k-1-N}}{2h^4} \right]\right\}, \\ mN + 1 \le k \le (m+1)N, \quad m = 0, 1, \ldots, \ 1 \le n \le M-1, \\ u_n^k = e^{-t_k} \sin x_n, \quad -N \le k \le 0. \end{gathered} \tag{2.33}$$

Here, we denote

$$\begin{gathered} x = \frac{\tau}{2h^4}, \qquad y = -\frac{1}{h^2} - \frac{2\tau}{h^4}, \\ z = -\frac{1}{\tau}, \qquad t = \frac{1}{\tau} + \frac{2}{h^2} + \frac{3\tau}{h^4}, \end{gathered} \tag{2.34}$$

Hence, we have second-order difference equation (2.30) with matrix coefficients. For the solution of this matrix equation, we use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$\begin{gathered} U_j^m = \alpha_{j+1} U_{j+1}^m + \beta_{j+1} U_{j+2}^m + \gamma_{j+1}^m, \quad j = M-2, \ldots, 2, 1, 0 \\ U_M^m = 0, \\ U_{M-1}^m = \left[(\beta_{M-2} + 5I) - 4(I - \alpha_{M-2})\alpha_{M-1} \right]^{-1} \left[4(I - \alpha_{M-2})\gamma_{M-1}^m - \gamma_{M-2}^m \right], \end{gathered} \tag{2.35}$$

where α_j $(j = 2, \ldots, M-2)$ and β_j $(j = 2, \ldots, M-2)$ are $(N+1) \times (N+1)$ square matrices and γ_j^m $(j = 2, \ldots, M-2)$ are column matrices defined by

$$\begin{gathered} \alpha_{j+1} = -\left(C + D\alpha_j + E\beta_{j-1} + E\alpha_{j-1}\alpha_j\right)^{-1}\left(B + D\beta_j + E\alpha_{j-1}\beta_j\right), \\ \beta_{j+1} = -\left(C + D\alpha_j + E\beta_{j-1} + E\alpha_{j-1}\alpha_j\right)^{-1}(A), \\ \gamma_{j+1}^m = -\left(C + D\alpha_j + E\beta_{j-1} + E\alpha_{j-1}\alpha_j\right)^{-1}\left(R\varphi_j^m - D\gamma_j^m - E\alpha_{j-1}\gamma_j^m - E\gamma_{j-1}^m\right), \end{gathered} \tag{2.36}$$

Table 1: Comparison of the errors of different difference schemes in $t \in [0,1]$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (2.20)	0.00599088	0.00286092	0.00139629
Difference scheme (2.29)	0.00076265	0.00020317	0.00005148

Table 2: Comparison of the errors of different difference schemes in $t \in [1,2]$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (2.20)	0.07324151	0.03693061	0.01845008
Difference scheme (2.29)	0.00077055	0.00020541	0.00005206

Table 3: Comparison of the errors of different difference schemes in $t \in [2,3]$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (2.20)	0.03520081	0.01749862	0.00872690
Difference scheme (2.29)	0.00067084	0.00017845	0.00004521

Table 4: Comparison of the errors of different difference schemes in $t \in [3,4]$.

Method	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (2.20)	0.01687166	0.00844577	0.00421761
Difference scheme (2.29)	0.00045527	0.00012107	0.00003067

where $j = 2, \ldots, M-2$, α_1 is $(N+1) \times (N+1)$ zero matrix, and β_1 is $(N+1) \times (N+1)$ zero matrix, γ_1^m and γ_2^m are $(N+1) \times 1$ zero matrices.

We give the results of the numerical analysis. The numerical solutions are recorded for different values of N and M and u_n^k represent the numerical solutions of these difference schemes at (t_k, x_n). Tables 1, 2, 3, and 4 are constructed for $N = M = 20, 40, 80$ in $t \in [0,1]$, $t \in [1,2]$, $t \in [2,3]$, $t \in [3,4]$, respectively, and the error is computed by the following formula:

$$E_M^N = \max_{\substack{-N \le k \le N \\ 1 \le n \le M-1}} \left| u(t_k, x_n) - u_n^k \right|. \tag{2.37}$$

Thus, by using the second order of accuracy difference scheme, the accuracy of solution increases faster than the first order of accuracy difference scheme.

3. Homotopy Analysis Method

In this section, we consider homotopy analysis method for the solution of problem (1.1). We study the initial-boundary-value problem for the delay parabolic equation (1.1). To illustrate the basic idea of homotopy analysis method (HAM) developed by Liao (see, e.g., [29–35]), the following differential equation is considered:

$$N[u(t,x)] = f(t,x), \tag{3.1}$$

where N is a linear operator for problem (1.1), t and x denote independent variables, $u(t,x)$ is an unknown function, and $f(t,x)$ is a known analytical function. Liao constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(t,x;q)-u_0(t,x)] = q\hbar\{N[\phi(t,x;q)]-f(t,x)\}, \tag{3.2}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(t,x)$ is an initial guess of $u(t,x)$, and $\phi(t,x;q)$ is an unknown function. When $q = 0$ and $q = 1$, it holds

$$\phi(t,x;0) = u_0(t,\mathrm{x}), \qquad \phi(t,x;1) = u(t,x), \tag{3.3}$$

respectively. As q increases from 0 to 1, the solution $\phi(t,x;q)$ varies from the initial guess $u_0(t,x)$ to the solution $u(t,x)$. Expanding $\phi(t,x;q)$ in Taylor series with respect to q, we get

$$\phi(t,x;q) = u_0(t,x) + \sum_{m=1}^{+\infty} u_m(t,x)q^m, \tag{3.4}$$

where

$$u_m(t,x) = \frac{1}{m!}\frac{\partial^m \phi(t,x;q)}{\partial q^m}\bigg|_{q=0}, \tag{3.5}$$

when the initial guess $u_0(t,x)$, the auxiliary linear operator L and the auxiliary parameter $\hbar$ are chosen properly, the series (3.4) converges at $q = 1$. We get

$$u(t,x) = u_0(t,x) + \sum_{m=1}^{+\infty} u_m(t,x). \tag{3.6}$$

Then define the vectors

$$\overrightarrow{u_n} = \{u_0(t,x), u_1(t,x), \dots, u_n(t,x)\}. \tag{3.7}$$

Differentiating the zero-order deformation equation (3.2) m times with respect to the embedding parameter q and dividing them by $m!$, we obtain the mth-order deformation equation

$$L[u_m(t,x) - \chi_m u_{m-1}(t,x)] = \hbar \Re_m(\vec{u}_{m-1}), \tag{3.8}$$

where

$$\begin{gathered}\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}(N[\phi(t,x;q)]-f(t,x))}{\partial q^{m-1}}\bigg|_{q=0},\\ \chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}\end{gathered} \tag{3.9}$$

with the initial condition

$$u_m(0,x)=0,\quad m\geq 1. \tag{3.10}$$

High-order deformation equation (3.8) is governed by the linear operator L. $\Re_m(\vec{u}_{m-1})$ can be represented by $u_1(t,x), u_m(t,x), u_2(t,x), \ldots, u_{m-1}(t,x)$ and high-order deformation equation can be solved consecutively. The Nth-order approximation of $u(t,x)$ is given by

$$u(t,x)\approx u_0(t,x)+\sum_{m=1}^{N}u_m(t,x). \tag{3.11}$$

3.1. Homotopy Analysis Solution

For the approximate solution of the delay parabolic differential equation with the Dirichlet condition, we consider the delay parabolic equation (2.17) and rewrite the equation for $t\in[0,1]$ in the following form:

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}=(0.1)\ e^{-(t-1)}\sin x,\quad 0<t\leq 1,\ 0<x<\pi,\\ u(0,x)=\sin x,\quad 0\leq x\leq \pi\\ u(t,0)=u(t,\pi)=0,\quad 0\leq t\leq 1.\end{gathered} \tag{3.12}$$

To solve the initial-boundary-value problem (3.12) by means of HAM, we choose the initial approximation

$$u_0(t,x)=\sin x, \tag{3.13}$$

and the linear operator

$$L[\phi(t,x;q)]=\frac{\partial\phi(t,x;q)}{\partial t}, \tag{3.14}$$

with the property

$$L[c]=0, \tag{3.15}$$

where c is constant of integration. From (3.12), we define a linear operator as

$$N[\phi(t,x;q)]=\frac{\partial\phi(t,x;q)}{\partial t}-\frac{\partial^2\phi(t,x;q)}{\partial x^2}. \tag{3.16}$$

Firstly, we construct the zero-order deformation equation

$$(1-q)L[\phi(t,x;q)-u_0(t,x)]=q\hbar\{N[\phi(t,x;q)]-f(t,x)\}, \tag{3.17}$$

when $q = 0$ and $q = 1$,

$$\phi(t, x; 0) = u_0(t, x) = u(0, x), \phi(t, x; 1) = u(t, x). \tag{3.18}$$

Then, we get mth-order deformation equations (3.8) for $m \geq 1$ with the initial conditions

$$u_m(0, x) = 0, \tag{3.19}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(t, x)}{\partial t} - \frac{\partial^2 u_{m-1}(t, x)}{\partial x^2} - (1 - \chi_m)(0.1)e^{-(t-1)} \sin x. \tag{3.20}$$

The solution of the mth-order deformation equations (3.20) for $m \geq 1$ is

$$u_m(t, x) = \chi_m u_{m-1}(t, x) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{3.21}$$

From (3.12) and (3.21), we obtain

$$\begin{aligned}
u_0(t, x) &= \sin x, \\
u_1(t, x) &= -\hbar(0.1)e \sum_{k=1}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \sin x + \hbar t \sin x, \\
u_2(t, x) &= \hbar(\hbar + 1)t \sin x - \hbar(\hbar + 1) \sum_{k=2}^{\infty} \frac{(-1)^k t^{k-1}}{(k-1)!} \sin x \\
&\quad + \hbar^2(0.1)e \sum_{k=2}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \sin x + \hbar^2 \frac{t^2}{2!} \sin x, \\
u_3(t, x) &= \hbar(\hbar + 1)t \sin x + 3\hbar^2(\hbar + 1) \frac{t^2}{2!} \sin x - \hbar(\hbar + 1)^2 \sum_{k=3}^{\infty} \frac{(-1)^{k+1} t^{k-2}}{(k-2)!} \sin x \\
&\quad - 2\hbar^2(\hbar + 1) \sum_{k=3}^{\infty} \frac{(-1)^{k+1} t^{k-1}}{(k-1)!} \sin x - \hbar^3(0.1)e \sum_{k=3}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \sin x + \hbar^3 \frac{t^3}{3!} \sin x, \\
&\ \ \vdots \\
u_n(t, x) &= f_1(\hbar + 1, t, x) + (-1)^n \hbar^n (0.1)e \sum_{k=n}^{\infty} \frac{(-1)^{k+1} t^k}{k!} \sin x + \hbar^n \frac{t^n}{n!} \sin x, \\
&\ \ \vdots
\end{aligned} \tag{3.22}$$

and so on. Then for $\hbar = -1$, we get

$$\begin{aligned}
u_0(t,x) &= \sin x, \\
u_1(t,x) &= (0.1)e\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^k}{k!}\sin x - t\sin x, \\
u_2(t,x) &= (0.1)e\sum_{k=2}^{\infty}\frac{(-1)^{k+1}t^k}{k!}\sin x + \frac{t^2}{2!}\sin x, \\
u_3(t,x) &= (0.1)e\sum_{k=3}^{\infty}\frac{(-1)^{k+1}t^k}{k!}\sin x - \frac{t^3}{3!}\sin x, \\
&\vdots \\
u_n(t,x) &= (0.1)e\sum_{k=n}^{\infty}\frac{(-1)^{k+1}t^k}{k!} + \frac{(-1)^n t^n}{n!}\sin x, \\
&\vdots
\end{aligned} \tag{3.23}$$

and so on.

From (3.6), when we take $\hbar = -1$, the solution of (3.12) can be obtained as

$$u(t,x) = ((0.1)et + 1)\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\sin x. \tag{3.24}$$

Equation (3.24) has the closed form

$$u(t,x) = ((0.1)et + 1)e^{-t}\sin x, \tag{3.25}$$

which is the exact solution of (3.12).

Second, we consider the solution of the delay parabolic equation (2.17) for $t \in [1,2]$ and rewrite this equation in the following form:

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = (0.1)e^{-(t-1)}[(0.1)e(t-1)+1]\sin x, \quad 1 < t \le 2,\ 0 < x < \pi, \\
u(1,x) = e^{-1}[(0.1)e + 1]\sin x, \quad 0 \le x \le \pi \\
u(t,0) = u(t,\pi) = 0, \quad 1 \le t \le 2.
\end{gathered} \tag{3.26}$$

Now, we choose the initial approximation

$$u_0(t,x) = e^{-1}[(0.1)e + 1]\sin x. \tag{3.27}$$

We take the linear operator (3.14) with the property (3.15), and we define the operator (3.16) from (3.26).

Firstly, we construct the zero-order deformation equation (3.2) and then, we obtain mth-order deformation equations (3.8) for $m \geq 1$ with the initial conditions

$$u_m(1, x) = 0, \tag{3.28}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(t, x)}{\partial t} - \frac{\partial^2 u_{m-1}(t, x)}{\partial x^2} - (1 - \chi_m)(0.1)e^{-(t-1)}[(0.1)e(t-1) + 1] \sin x. \tag{3.29}$$

The solution of the mth-order deformation equations (3.29) for $m \geq 1$ is

$$u_m(t, x) = \chi_m u_{m-1}(t, x) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{3.30}$$

From (3.26) and the mth-order deformation equations (3.30), we get

$$\begin{aligned}
u_0(t, x) &= e^{-1}[(0.1)e + 1] \sin x, \\
u_1(t, x) &= \hbar e^{-1}[(0.1)e + 1] \sin x(t-1) \\
&\quad - \hbar(0.1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-1)^k}{k!} - \hbar(0.1)^2 e\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)(k-1)!}\sin x, \\
u_2(t, x) &= \hbar(\hbar+1)e^{-1}[(0.1)e + 1](t-1) \sin x \\
&\quad - \hbar(\hbar+1)(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k}(t-1)^{k-1}}{(k-1)!}\sin x - \hbar(\hbar+1)(0.1)^2 e\sum_{k=2}^{\infty}\frac{(-1)^{k}(t-1)^{k}}{k(k-2)!}\sin x \\
&\quad + \hbar^2 e^{-1}[(0.1)e+1]\sin x\frac{(t-1)^2}{2!} + \hbar^2(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-1)^{k}}{k!}\sin x \\
&\quad + \hbar^2(0.1)^2 e\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-2)!}\sin x, \\
u_3(t, x) &= \hbar(\hbar+1)^2 e^{-1}[(0.1)e+1]\sin x(t-1) \\
&\quad + 2\hbar^2(\hbar+1)e^{-1}[(0.1)e+1]\sin x\frac{(t-1)^2}{2!} \\
&\quad + \hbar(\hbar+1)^2(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-1)^{k-2}}{(k-2)!}\sin x - \hbar(\hbar+1)^2(0.1)^2 e\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-1)^{k}}{(k-1)(k-3)!}\sin x \\
&\quad - 2\hbar^2(\hbar+1)(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k}(t-1)^{k-1}}{(k-1)!}\sin x - \hbar(\hbar+1)(0.1)^2 e\sum_{k=3}^{\infty}\frac{(-1)^{k}(t-1)^{k}}{k(k-1)(k-3)!}\sin x \\
&\quad + \hbar^2(\hbar+1)(0.1)^2 e\sum_{k=3}^{\infty}\frac{(-1)^{k}(t-1)^{k}}{k(k-1)(k-2)!}\sin x + \hbar^3 e^{-1}[(0.1)e+1]\sin x\frac{(t-1)^3}{3!}
\end{aligned}$$

$$
\begin{aligned}
&- \hbar^3 (0.1) \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x - \hbar^3 (0.1)^2 e \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-1)(k-3)!} \sin x, \\
&\vdots \\
u_n(t,x) &= f_2(\hbar+1,t,x) + \hbar^n e^{-1}[(0.1)e+1] \sin x \frac{(t-1)^n}{n!} \\
&- \hbar^n (0.1) \sum_{k=n}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x \\
&+ (-1)^n \hbar^n (0.1)^2 e \sum_{k=n}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!} \sin x, \\
&\vdots
\end{aligned}
\tag{3.31}
$$

and so on. When we choose $\hbar = -1$, we obtain

$$
\begin{aligned}
u_0(t,x) &= e^{-1}[(0.1)e+1] \sin x, \\
u_1(t,x) &= -e^{-1}[(0.1)\mathrm{e}+1] \sin x (t-1) \\
&+ (0.1) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x + (0.1)^2 e \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)(k-1)!} \sin x, \\
u_2(t,x) &= e^{-1}[(0.1)e+1] \sin x \frac{(t-1)^2}{2!} \\
&+ (0.1) \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x + (0.1)^2 e \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-2)!} \sin x, \\
u_3(t,x) &= -e^{-1}[(0.1)e+1] \sin x \frac{(t-1)^3}{3!} \\
&+ (0.1) \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x + (0.1)^2 e \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-1)(k-3)!} \sin x, \\
&\vdots \\
u_n(t,x) &= (-1)^n e^{-1}[(0.1)e+1] \sin x \frac{(t-1)^n}{n!} \\
&+ (0.1) \sum_{k=n}^{\infty} \frac{(-1)^{k+1}(t-1)^k}{k!} \sin x \\
&+ (0.1)^2 e \sum_{k=n}^{\infty} \frac{(-1)^{k+1}(t-1)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!} \sin x, \\
&\vdots
\end{aligned}
\tag{3.32}
$$

and so on.

From (3.6), the solution of (3.26) for $\hbar = -1$ can be obtained as

$$u(t,x) = \left(\frac{[(0.1)e(t-1)]^2}{2!} + (0.1)et + 1\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\sin x. \tag{3.33}$$

Equation (3.33) has the closed form

$$u(t,x) = \left(\frac{[(0.1)e(t-1)]^2}{2!} + (0.1)et + 1\right)e^{-t}\sin x, \tag{3.34}$$

which is the exact solution of the (3.26).

Now, we consider the solution of the delay parabolic equation (2.17) for $t \in [2,3]$ and rewrite the equation in the following form:

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = (0.1)e^{-(t-1)}\left[\frac{[(0.1)e(t-2)]^2}{2!} + (0.1)e(t-1) + 1\right]\sin x, \\ &\qquad\qquad 2 < t \leq 3, \quad 0 < x < \pi, \\ &u(2,x) = e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x, \quad 0 \leq x \leq \pi \\ &u(t,0) = u(t,\pi) = 0, \quad 2 \leq t \leq 3. \end{aligned} \tag{3.35}$$

The initial approximation is

$$u_0(t,x) = e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x. \tag{3.36}$$

We take the linear operator (3.14) with the property (3.15). From parabolic equation (3.35), we define a linear operator (3.16) and obtain the zero-order deformation equation (3.2). Thus, we get the mth-order deformation equations (3.8) for $m \geq 1$ with the initial conditions

$$u_m(2,x) = 0, \tag{3.37}$$

where

$$\begin{aligned} \Re_m(\vec{u}_{m-1}) = {} & \frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} - (1-\chi_m) \\ & \times (0.1)e^{-(t-1)}\left[\frac{[(0.1)e(t-2)]^2}{2!} + (0.1)e(t-1) + 1\right]\sin x. \end{aligned} \tag{3.38}$$

The solution of the mth-order deformation equations (3.38) for $m \geq 1$ is

$$u_m(t,x) = \chi_m u_{m-1}(t,x) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{3.39}$$

From (3.35) and (3.39), we obtain

$$u_0(t,x) = e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x,$$

$$u_1(t,x) = \hbar e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x(t-2)$$
$$- \hbar e^{-1}(0.1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x - \hbar(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)(k-1)!}\sin x$$
$$- \hbar(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x - \hbar e\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k-1)!}\sin x,$$

$$u_2(t,x) = \hbar(\hbar+1)e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x(t-2)$$
$$- \hbar(\hbar+1)e^{-1}(0.1)\sum_{k=2}^{\infty}\frac{(-1)^k(t-2)^{k-1}}{(k-1)!}\sin x - \hbar(\hbar+1)(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^k(t-2)^{k-1}}{(k-1)!}\sin x$$
$$- \hbar(\hbar+1)(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^k(t-2)^k}{k(k-2)!}\sin x - \hbar(\hbar+1)\sum_{k=2}^{\infty}\frac{(-1)^k(t-2)^{k+1}}{(k+1)(k-2)!}\sin x$$
$$+ \hbar^2 e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x\frac{(t-2)^2}{2!}$$
$$+ \hbar^2 e^{-1}(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + \hbar^2(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-2)!}\sin x$$
$$+ \hbar^2(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + \hbar^2 e\frac{(0.1)^3}{2!}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)(k-2)!}\sin x,$$

$$u_3(t,x) = \hbar(\hbar+1)^2 e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x(t-2)$$
$$+ 2\hbar^2(\hbar+1)e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x\frac{(t-2)^2}{2!}$$
$$- \hbar(\hbar+1)^2(0.1)e^{-1}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k-2}}{(k-2)!}\sin x$$
$$- \hbar(\hbar+1)^2(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k-2}}{(k-2)!}\sin x$$
$$- 3\hbar(\hbar+1)(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k-1}}{(k-1)!}\sin x$$

$$
\begin{aligned}
&- \hbar^2(\hbar+1)(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k-1}}{(k-1)!}\sin x\\
&- 2\hbar^2(\hbar+1)e^{-1}(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k-1}}{(k-1)!}\sin x\\
&- \hbar(\hbar+1)^2\frac{(0.1)^3}{2!}e\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k(k-3)!}\sin x\\
&- 2\hbar^2(\hbar+1)(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k(k-1)(k-3)!}\sin x\\
&- 2\hbar^2(\hbar+1)\frac{(0.1)^3}{2!}e\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-3)!}\sin x\\
&+ \hbar^3e^{-2}\left[\frac{[(0.1)e]^2}{2!}+2(0.1)e+1\right]\sin x\frac{(t-2)^3}{3!}\\
&- \hbar^3e^{-1}(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k!}\sin x-\hbar^3(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x\\
&- \hbar^3(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k!}\sin x-\hbar^3e\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)k(k-3)!}\sin x,\\
&\ \vdots\\
u_n(t,x) &= f_3(\hbar+1,t,x)+\hbar^n e^{-2}\left[\frac{[(0.1)e]^2}{2!}+2(0.1)e+1\right]\sin x\frac{(t-2)^n}{n!}\\
&+ (-1)^n\hbar^n e^{-1}(0.1)\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k!}\sin x\\
&+ (-1)^n\hbar^n(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x\\
&+ (-1)^n\hbar^n(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k}}{k!}\sin x\\
&+ (-1)^n\hbar^n e\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)k\cdots(k-(n-3))(k-n)!}\sin x,\\
&\ \vdots
\end{aligned}
\tag{3.40}
$$

and so on. When we choose $\hbar = -1$, we obtain

$$
\begin{aligned}
u_0(t,x) &= e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x,\\
u_1(t,x) &= -e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x(t-2)\\
&\quad + e^{-1}(0.1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + (0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)(k-1)!}\sin x\\
&\quad + (0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + e\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k-1)!}\sin x,\\
u_2(t,x) &= e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x\frac{(t-2)^2}{2!}\\
&\quad + e^{-1}(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + (0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-2)!}\sin x\\
&\quad + (0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + e\frac{(0.1)^3}{2!}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)(k-2)!}\sin x,\\
u_3(t,x) &= -e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x\frac{(t-2)^3}{3!}\\
&\quad + e^{-1}(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + (0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x\\
&\quad + (0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x + e\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)k(k-3)!}\sin x,\\
&\ \vdots\\
u_n(t,x) &= (-1)^n e^{-2}\left[\frac{[(0.1)e]^2}{2!} + 2(0.1)e + 1\right]\sin x\frac{(t-2)^n}{n!}\\
&\quad + e^{-1}(0.1)\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x\\
&\quad + (0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x\\
&\quad + (0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^k}{k!}\sin x\\
&\quad + e\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-2)^{k+2}}{(k+2)(k+1)k\cdots(k-(n-3))(k-n)!}\sin x,\\
&\ \vdots
\end{aligned}
\tag{3.41}
$$

and so on.

From (3.6), the solution of (3.35) for $\hbar = -1$ is

$$u(t,x) = \left(\frac{[(0.1)e(t-2)]^3}{3!} + \frac{[(0.1)e(t-1)]^2}{2!} + (0.1)et + 1\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\sin x. \tag{3.42}$$

This series has the closed form

$$u(t,x) = \left(\frac{[(0.1)e(t-2)]^3}{3!} + \frac{[(0.1)e(t-1)]^2}{2!} + (0.1)et + 1\right)e^{-t}\sin x, \tag{3.43}$$

which is the exact solution of the (3.35).

Finally, we consider the solution of the delay parabolic equation (2.17) for $t \in [3,4]$ and rewrite this equation in the following form:

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = (0.1)e^{-(t-1)}\left[\frac{[(0.1)e(t-3)]^3}{3!} + \frac{[(0.1)e(t-2)]^2}{2!} + (0.1)e(t-1) + 1\right]\sin x, \\ 3 < t \le 4, \quad 0 < x < \pi, \\ u(3,x) = e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x, \quad 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 3 \le t \le 4.\end{gathered} \tag{3.44}$$

We take the initial approximation

$$u_0(t,x) = e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x \tag{3.45}$$

and the linear operator (3.14) with the property (3.15). From (3.44), we define linear operator (3.16).

We construct the zero-order deformation equation (3.2) and the mth-order deformation equations (3.8) for $m \ge 1$ with the initial conditions

$$u_m(3,x) = 0, \tag{3.46}$$

where

$$\begin{aligned}\Re_m(\vec{u}_{m-1}) = {} & \frac{\partial u_{m-1}(t,x)}{\partial t} - \frac{\partial^2 u_{m-1}(t,x)}{\partial x^2} - (1-\chi_m) \\ & \times (0.1)e^{-(t-1)}\left(\frac{[(0.1)e(t-3)]^3}{3!} + \frac{[(0.1)e(t-2)]^2}{2!} + (0.1)e(t-1) + 1\right)\sin x.\end{aligned} \tag{3.47}$$

The solution of the mth-order deformation equations (3.47) for $m \geq 1$ is

$$u_m(t,x) = \chi_m u_{m-1}(t,x) + \hbar L^{-1}[\Re_m(\vec{u}_{m-1})]. \tag{3.48}$$

From (3.44) and (3.48), we get

$$\begin{aligned}
u_0(t,x) &= e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x, \\
u_1(t,x) &= \hbar e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x(t-3) \\
&\quad - \hbar e^{-2}(0.1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x - \hbar e^{-1}(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)(k-1)!}\sin x \\
&\quad - 2\hbar e^{-1}(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x - \hbar\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k-1)!}\sin x \\
&\quad - \hbar(0.1)^3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)(k-1)!}\sin x - \hbar\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x \\
&\quad - \hbar e\frac{(0.1)^4}{3!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k-1)!}\sin x, \\
u_2(t,x) &= \hbar(\hbar+1)e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x(t-3) \\
&\quad - \hbar(\hbar+1)(0.1)\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^{k-1}}{(k-2)!}\sin x - \hbar(\hbar+1)(0.1)^2e^{-1}\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^k}{k!}\sin x \\
&\quad - 2\hbar(\hbar+1)(0.1)^2e^{-1}\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^{k-1}}{(k-1)!}\sin x \\
&\quad - \hbar(\hbar+1)\frac{(0.1)^3}{2!}\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^{k-1}}{(k-1)!}\sin x \\
&\quad - \hbar(\hbar+1)(0.1)^3\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^k}{k(k-2)!}\sin x - \hbar(\hbar+1)(0.1)^3\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^{k+1}}{(k+1)(k-2)!}\sin x \\
&\quad - \hbar(\hbar+1)\frac{(0.1)^4}{3!}e\sum_{k=2}^{\infty}\frac{(-1)^k(t-3)^{k+2}}{(k+2)(k-2)!}\sin x \\
&\quad + \hbar^2e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right]\sin x\frac{(t-3)^2}{2!} \\
&\quad + \hbar^2e^{-2}(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x + \hbar^2e^{-1}(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-2)!}\sin x
\end{aligned}$$

$$+ 2\hbar^2 e^{-1}(0.1)^2 \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-3)^k}{k!} \sin x + \hbar^2 \frac{(0.1)^3}{2!} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)(k-2)!} \sin x$$

$$+ \hbar^2 (0.1)^3 \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-2)!} \sin x + \hbar^2 \frac{(0.1)^3}{2!} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-3)^k}{k!} \sin x$$

$$+ \hbar^2 e \frac{(0.1)^4}{3!} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k+2)(k-2)!} \sin x,$$

$$u_3(t,x) = \hbar(\hbar+1)^2 e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right] \sin x (t-3)$$

$$- \hbar(\hbar+1)^2 (0.1) e^{-2} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-2}}{(k-2)!} \sin x$$

$$- \hbar(\hbar+1)^2 (0.1)^2 e^{-1} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-1}}{(k-1)(k-3)!} \sin x$$

$$- 2\hbar(\hbar+1)^2 (0.1)^2 e^{-1} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-2}}{(k-2)!} \sin x$$

$$- \hbar(\hbar+1)^2 \frac{(0.1)^3}{2!} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k}}{k(k-3)!} \sin x$$

$$- \hbar(\hbar+1)^2 (0.1)^3 \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-1}}{(k-1)(k-3)!} \sin x$$

$$- \hbar(\hbar+1)^2 \frac{(0.1)^3}{2!} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-2}}{(k-2)!} \sin x$$

$$- \hbar(\hbar+1)^2 e \frac{(0.1)^4}{3!} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)(k-3)!} \sin x$$

$$+ 2\hbar^2(\hbar+1) e^{-3}\left[\frac{[(0.1)e]^3}{3!} + \frac{[2(0.1)e]^2}{2!} + 3(0.1)e + 1\right] \sin x \frac{(t-3)^2}{2!}$$

$$- 2\hbar^2(\hbar+1) e^{-2}(0.1) \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-1}}{(k-1)!} \sin x$$

$$- \hbar^2(\hbar+1) e^{-1}(0.1)^2 \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k}}{k(k-1)(k-3)!} \sin x$$

$$- 4\hbar^2(\hbar+1) e^{-1}(0.1)^2 \sum_{k=3}^{\infty} \frac{(-1)^{k+1}(t-3)^{k-1}}{(k-1)!} \sin x$$

$$- 2\hbar^2(\hbar+1)\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-3)!}\sin x$$

$$- 2\hbar^2(\hbar+1)(0.1)^3\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k}}{k(k-1)(k-3)!}\sin x$$

$$- 2\hbar^2(\hbar+1)\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k-1}}{(k-1)!}\sin x$$

$$- 2\hbar^2(\hbar+1)e\frac{(0.1)^4}{3!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)(k-3)!}\sin x$$

$$+ \hbar^3e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x\frac{(t-3)^3}{3!}$$

$$- \hbar^3e^{-2}(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k}}{k!}\sin x-\hbar^3e^{-1}(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x$$

$$- 2\hbar^3e^{-1}(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k}}{k!}\sin x-\hbar^3\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)k(k-3)!}\sin x$$

$$- \hbar^3(0.1)^3\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x-\hbar^3\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k}}{k!}\sin x$$

$$- \hbar^3e\frac{(0.1)^4}{3!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k+2)(k+1)(k-3)!}\sin x,$$

$$\vdots$$

$$u_n(t,x) = f_4(\hbar+1,t,x)+\hbar^ne^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x\frac{(t-3)^n}{n!}$$

$$+ (-1)^n\hbar^ne^{-2}(0.1)\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k}}{k!}\sin x$$

$$+ (-1)^n\hbar^ne^{-1}(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x$$

$$+ (-1)^n\hbar^n2e^{-1}(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{\mathrm{k}}}{k!}\sin x$$

$$+ (-1)^n\hbar^n\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)k(k-3)\cdots(k-(n-3))(k-n)!}\sin x$$

$$+ (-1)^n\hbar^n(0.1)^3\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x$$

$$
\begin{aligned}
&+(-1)^n\hbar^n\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+(-1)^n\hbar^n e\frac{(0.1)^4}{3!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+2)(k+1)k\cdots(k-(n-4))(k-n)!}\sin x,\\
&\vdots
\end{aligned}
\tag{3.49}
$$

and so on. For $\hbar=-1$, we get

$$
\begin{aligned}
u_0(t,x)={}&e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x,\\
u_1(t,x)={}&-e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x(t-3)\\
&+e^{-2}(0.1)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+e^{-1}(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)(k-1)!}\sin x\\
&+2e^{-1}(0.1)^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k-1)!}\sin x\\
&+(0.1)^3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)(k-1)!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+e\frac{(0.1)^4}{3!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k-1)!}\sin x,\\
u_2(t,x)={}&e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x\frac{(t-3)^2}{2!}\\
&+e^{-2}(0.1)\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+e^{-1}(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-2)!}\sin x\\
&+2e^{-1}(0.1)^2\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)(k-2)!}\sin x\\
&+(0.1)^3\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-2)!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+e\frac{(0.1)^4}{3!}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k+2)(k-2)!}\sin x,\\
u_3(t,x)={}&-e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x\frac{(t-3)^3}{3!}\\
&+e^{-2}(0.1)\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+e^{-1}(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x
\end{aligned}
$$

$$
\begin{aligned}
&+2e^{-1}(0.1)^2\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)k(k-3)!}\sin x\\
&+(0.1)^3\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)(k-3)!}\sin x+\frac{(0.1)^3}{2!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+e\frac{(0.1)^4}{3!}\sum_{k=3}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+3)(k+2)(k+1)(k-3)!}\sin x,\\
&\vdots\\
u_n(t,x)&=(-1)^n e^{-3}\left[\frac{[(0.1)e]^3}{3!}+\frac{[2(0.1)e]^2}{2!}+3(0.1)e+1\right]\sin x\frac{(t-3)^n}{n!}\\
&+e^{-2}(0.1)\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+e^{-1}(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x\\
&+2e^{-1}(0.1)^2\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+2}}{(k+2)(k+1)k(k-3)\cdots(k-(n-3))(k-n)!}\sin x\\
&+(0.1)^3\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+1}}{(k+1)k(k-1)\cdots(k-(n-2))(k-n)!}\sin x\\
&+\frac{(0.1)^3}{2!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^k}{k!}\sin x\\
&+e\frac{(0.1)^4}{3!}\sum_{k=n}^{\infty}\frac{(-1)^{k+1}(t-3)^{k+3}}{(k+2)(k+1)k\cdots(k-(n-4))(k-n)!}\sin x,\\
&\vdots
\end{aligned}
\tag{3.50}
$$

and so on.

From (3.6), the solution of (3.44) is obtained as

$$
u(t,x)=\left(\frac{[(0.1)e(t-3)]^4}{4!}+\frac{[(0.1)e(t-2)]^3}{3!}+\frac{[(0.1)e(t-1)]^2}{2!}+(0.1)et+1\right)\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{k!}\sin x.
\tag{3.51}
$$

Table 5: The absolute error at $x = \pi/2$ when $\hbar = -1$.

t	u_{exact}	u_{app}	u_{err}
0.5	0.68896672324764	0.68718517400418	0.00178154924346
1.5	0.31617066069370	0.31528438731741	0.00088627337628
2.5	0.14472535694258	0.14432199982347	0.00040335711910
3.5	0.06624155329591	0.06619299448877	0.00004855880714

Table 6: Comparison of the absolute error in HAM when $\hbar = -1$ and the errors of different difference schemes at $(0.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	0.68718517400418	0.00178154924346
Difference scheme (2.20) for $N = M = 48$	0.69069236574147	0.00172564249383
Difference scheme (2.29) for $N = M = 10$	0.69067805057383	0.00171132732619

Table 7: Comparison of the absolute error in HAM when $\hbar = -1$ and the errors of different difference schemes at $(1.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	0.31528438731741	0.00088627337628
Difference scheme (2.20) for $N = M = 148$	0.31705389317070	0.00088323247700
Difference scheme (2.29) for $N = M = 18$	0.31707957467449	0.00090891398079

This series has the closed form

$$u(t,x) = \left(\frac{[(0.1)e(t-3)]^4}{4!} + \frac{[(0.1)e(t-2)]^3}{3!} + \frac{[(0.1)e(t-1)]^2}{2!} + (0.1)et + 1 \right) e^{-t} \sin x, \tag{3.52}$$

which is the exact solution of (3.44).

We give the HAM solutions in $t \in [0,1]$, $t \in [1,2]$, $t \in [2,3]$, $t \in [3,4]$. We use four terms for evaluating the approximate solution $u_{\text{app}} = \sum_{k=0}^{3} u_k(t,x)$. According to the $\hbar$-curve of $u_{xt}(0,0)$, the solution series is convergent when $-1.48 \le \hbar \le 0.48$, $-1.41 \le \hbar \le 2.10$, $-1.19 \le \hbar \le 0.12$, and $-1.02 \le \hbar \le 0$, respectively, in $t \in [0,1]$, $t \in [1,2]$, $t \in [2,3]$, $t \in [3,4]$. We take $\hbar = -1$ to determine how much the approximate solution is accurate and compute the absolute errors $u_{\text{err}} = |u_{\text{exact}} - u_{\text{app}}|$ at the points $(0.5, \pi/2)$, $(1.5, \pi/2)$, $(2.5, \pi/2)$, $(3.5, \pi/2)$ in Table 5.

4. Conclusion

The numerical solutions of first order of difference scheme (2.20) and second order of difference scheme (2.29) for different values of N and M and the approximate solutions obtained by HAM for $N = 3$ in (3.11) when $\hbar = -1$ are given at the same points $(0.5, \pi/2)$, $(1.5, \pi/2)$, $(2.5, \pi/2)$, $(3.5, \pi/2)$ in Tables 6, 7, 8, and 9, respectively. The absolute errors computed show that, with homotopy analysis method, the results are more accurate for the parabolic delay equation (2.17).

Although HAM seems to be more rapid than finite difference method, the series solutions obtained by HAM are convergence only for the regions determined by convergence

Table 8: Comparison of the absolute error in HAM when $\hbar = -1$ and the errors of different difference schemes at $(2.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	0.14432199982347	0.00040335711910
Difference scheme (2.20) for $N = M = 290$	0.14512731459568	0.00040195765309
Difference scheme (2.29) for $N = M = 24$	0.14512723912249	0.00047857754134

Table 9: Comparison of the absolute error in HAM when $\hbar = -1$ and the errors of different difference schemes at $(3.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	0.06619299448877	0.00004855880714
Difference scheme (2.20) for $N = M = 480$	0.06640694878963	0.00016953954937
Difference scheme (2.29) for $N = M = 58$	0.06628727563690	0.00004572234990

Table 10: Comparison of the absolute error in HAM when $\hbar = -2$ and the errors of different difference schemes at $(0.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	1.82382103940268	1.13485431615504
Difference scheme (2.20) for $N = M = 4$	0.71754584429923	0.02857912105159
Difference scheme (2.29) for $N = M = 4$	0.64396829164838	0.04499843159926

Table 11: Comparison of the absolute error in HAM when $\hbar = -2.1$ and the errors of different difference schemes at $(1.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	−0.71640688651704	1.0325774721074
Difference scheme (2.20) for $N = M = 4$	0.26625468031687	0.04991598037683
Difference scheme (2.29) for $N = M = 4$	0.26908021797566	0.04709044271804

Table 12: Comparison of the absolute error in HAM when $\hbar = 1.5$ and the errors of different difference schemes at $(2.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	1.85743968284120	1.71271432589862
Difference scheme (2.20) for $N = M = 4$	0.09250455294144	0.05222080400114
Difference scheme (2.29) for $N = M = 4$	0.11418378483060	0.03054157211198

control parameter $\hbar$. So, convergence region is limited for HAM. The comparison of two methods of finite difference and homotopy analysis shows that latter is more rapid and more accurate in the cases that series solutions are convergence. When we take $\hbar$ out of the convergence region determined by $\hbar$ curves, it is shown that finite difference method is faster and more accurate than HAM. The approximate solutions obtained by HAM for different values of $\hbar$ chosen from out of the convergence region of the series solutions and the numerical solutions of first and second order of difference schemes (2.20) and (2.29) for $N = M = 4$ are given in Tables 10, 11, 12, and 13, respectively at the same points $(0.5, \pi/2)$, $(1.5, \pi/2)$, $(2.5, \pi/2)$, $(3.5, \pi/2)$.

Table 13: Comparison of the absolute error in HAM when $\hbar = 2$ and the errors of different difference schemes at $(3.5, \pi/2)$.

Method	u_{app}	u_{err}
HAM for $N = 3$	1.33954603011737	1.27330447682146
Difference scheme (2.20) for $N = M = 4$	0.02904127716967	0.03720027612623
Difference scheme (2.29) for $N = M = 4$	0.04842804202239	0.01781351127352

Despite HAM, by finite difference method, we can guarantee the convergence in the whole domain that (2.17) is defined in. Therefore finite difference method is more efficient than HAM.

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References

[1] A. N. Al-Mutib, "Stability properties of numerical methods for solving delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 10, no. 1, pp. 71–79, 1984.

[2] A. Bellen, "One-step collocation for delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 10, no. 3, pp. 275–283, 1984.

[3] A. Bellen, Z. Jackiewicz, and M. Zennaro, "Stability analysis of one-step methods for neutral delay-differential equations," *Numerische Mathematik*, vol. 52, no. 6, pp. 605–619, 1988.

[4] K. L. Cooke and I. Győri, "Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments," *Computers & Mathematics with Applications*, vol. 28, no. 1–3, pp. 81–92, 1994, Advances in difference equations.

[5] L. Torelli, "Stability of numerical methods for delay differential equations," *Journal of Computational and Applied Mathematics*, vol. 25, no. 1, pp. 15–26, 1989.

[6] A. F. Yeniçerioğlu and S. Yalçinbaş, "On the stability of the second-order delay differential equations with variable coefficients," *Applied Mathematics and Computation*, vol. 152, no. 3, pp. 667–673, 2004.

[7] A. F. Yeniçerioğlu, "Stability properties of second order delay integro-differential equations," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3109–3117, 2008.

[8] A. F. Yeniçerioğlu, "The behavior of solutions of second order delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1278–1290, 2007.

[9] A. Ashyralyev and H. Akca, "On difference shemes for semilinear delay differential equations with constant delay," in *Proceedings of the Conference TSU: Actual Problems of Applied Mathematics, Physics and Engineering*, pp. 18–27, Ashgabat, Turkmenistan, 1999.

[10] A. Ashyralyev, H. Akça, and U. Guray, "Second order accuracy difference scheme for approximate solutions of delay differential equations," *Functional Differential Equations*, vol. 6, no. 3-4, pp. 223–231, 1999.

[11] A. Ashyralyev and H. Akça, "Stability estimates of difference schemes for neutral delay differential equations," *Nonlinear Analysis*, vol. 44, no. 4, pp. 443–452, 2001.

[12] A. Ashyralyev, H. Akça, and F. Yeniçerioğlu, "Stability properties of difference schemes for neutral differential equations," in *Differential Equations and Applications*, vol. 3, pp. 27–34, Nova Science, Hauppauge, NY, USA, 2004.

[13] J. Liu, P. Dong, and G. Shang, "Sufficient conditions for inverse anticipating synchronization of unidirectional coupled chaotic systems with multiple time delays," in *Proceedings of the Chinese Control and Decision Conference (CCDC '10)*, pp. 751–756, IEEE, 2010.

[14] S. Mohamad, H. Akça, and V. Covachev, "Discrete-time Cohen-Grossberg neural networks with transmission delays and impulses," *Tatra Mountains Mathematical Publications*, vol. 43, pp. 145–161, 2009.

[15] A. Ashyralyev and P. E. Sobolevskii, "On the stability of the linear delay differential and difference equations," *Abstract and Applied Analysis*, vol. 6, no. 5, pp. 267–297, 2001.
[16] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, vol. 148 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 2004.
[17] G. Di Blasio, "Delay differential equations with unbounded operators acting on delay terms," *Nonlinear Analysis*, vol. 52, no. 1, pp. 1–18, 2003.
[18] M. A. Bazarov, "On the structure of fractional spaces," in *Proceedings of the 27th All-Union Scientific Student Conference "The Student and Scientific-Technological Progress" (Novosibirsk, 1989)*, pp. 3–7, Novosibirsk University, Novosibirsk, Russia, 1989.
[19] A. O. Ashyralyev and P. E. Sobolevskiĭ, "The theory of interpolation of linear operators and the stability of difference schemes," *Doklady Akademii Nauk SSSR*, vol. 275, no. 6, pp. 1289–1291, 1984.
[20] U. Erdogan and T. Ozis, "A smart nonstandard finite difference scheme for second order nonlinear boundary value problems," *Journal of Computational Physics*, vol. 230, no. 17, pp. 6464–6474, 2011.
[21] A. Ashyralyev and P. E. Sobolevskiĭ, *Well-Posedness of Parabolic Difference Equations*, vol. 69 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1994.
[22] A. Ashyralyev, "High-accuracy stable difference schemes for well-posed NBVP," *Operator Theory*, vol. 191, no. 2, pp. 229–252, 2009.
[23] H. A. Alibekov and P. E. Sobolevskiĭ, "The stability of difference schemes for parabolic equations," *Doklady Akademii Nauk SSSR*, vol. 232, no. 4, pp. 737–740, 1977 (Russian).
[24] H. A. Alibekov and P. E. Sobolevskiĭ, "Stability and convergence of high-order difference schemes of approximation for parabolic equations," *Akademiya Nauk Ukrainskoĭ SSR*, vol. 31, no. 6, pp. 627–634, 1979.
[25] H. A. Alibekov and P. E. Sobolevskiĭ, "Stability and convergence of difference schemes of a high order of approximation for parabolic partial differential equations," *Akademiya Nauk Ukrainskoĭ SSR*, vol. 32, no. 3, pp. 291–300, 1980.
[26] A. A. Samarskii and E. S. Nikolaev, *Numerical Methods for Grid Equations. Vol. II*, Birkhäuser, Basel, Switzerland, 1989.
[27] H. Akça, V. B. Shakhmurov, and G. Arslan, "Differential-operator equations with bounded delay," *Nonlinear Times and Digest*, vol. 2, no. 2, pp. 179–190, 1995.
[28] H. Akca and V. Covachev, "Spatial discretization of an impulsive Cohen-Grossberg neural network with time-varying and distributed delays and reaction-diffusion terms," *Analele Stiintifice Ale Universitatii Ovidius Constanta-Seria Matematica*, vol. 17, no. 3, pp. 15–26, 2009.
[29] S. J. Liao, "Homotopy analysis method: a new analytical technique for nonlinear problems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 2, no. 2, pp. 95–100, 1997.
[30] S. J. Liao, "On the general boundary element method," *Engineering Analysis with Boundary Elements*, vol. 21, no. 1, pp. 39–51, 1998.
[31] S. Liao, "General boundary element method: an application of homotopy analysis method," *Communications in Nonlinear Science & Numerical Simulation*, vol. 3, no. 3, pp. 159–163, 1998.
[32] S. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.
[33] S. J. Liao, "Boundary element method for general nonlinear differential operators," *Engineering Analysis with Boundary Elements*, vol. 20, no. 2, pp. 91–99, 1997.
[34] S. J. Liao, "On the general Taylor theorem and its applications in solving non-linear problems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 2, no. 3, pp. 135–140, 1997.
[35] S. Liao, *Beyond Perturbation*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, 2004.
[36] H. Khan, S. J. Liao, R. N. Mohapatra, and K. Vajravelu, "An analytical solution for a nonlinear timedelay model in biology," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 7, pp. 3141–3148, 2009.
[37] Y. Y. Wu, S. J. Liao, and X. Y. Zhao, "Some notes on the general boundary element method for highly nonlinear problems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 10, no. 7, pp. 725–735, 2005.
[38] A. Ashyralyev and D. Agirseven, "Finite difference method for delay parabolic equations," in *Proceedings of the International Conference on Numerical Analysis and Applied Mathematics: Numerical Analysis And Applied Mathematics (ICNAAM '11)*, vol. 1389 of *AIP Conference Proceedings*, pp. 573–576, 2011.