

## Research Article

# Strong Convergence Theorems for Zeros of Bounded Maximal Monotone Nonlinear Operators

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An iteration process studied by Chidume and Zegeye 2002 is proved to converge *strongly* to a solution of the equation  $Au = 0$  where  $A$  is a bounded  $m$ -accretive operator on certain real Banach spaces  $E$  that include  $L_p$  spaces  $2 \leq p < \infty$ . The iteration process does not involve the computation of the resolvent at any step of the process and does not involve the projection of an initial vector onto the intersection of two convex subsets of  $E$ , setbacks associated with the classical *proximal point algorithm* of Martinet 1970, Rockafellar 1976 and its modifications by various authors for approximating of a solution of this equation. The ideas of the iteration process are applied to approximate fixed points of uniformly continuous pseudocontractive maps.

## 1. Introduction

Consider the following problem:

$$\text{find } u \in H \quad \text{such that } 0 \in Au, \quad (1.1)$$

where  $H$  is a real Hilbert space and  $A$  is a *maximal monotone operator* (defined below) on  $H$ . One of the classical algorithms for approximating a solution of (1.1), assuming existence, is the so-called *proximal point algorithm* introduced by Martinet [1] and studied further by Rockafellar [2] and a host of other authors. Specifically, given  $x_k \in H$ , an approximation of a solution of (1.1), the proximal point algorithm generates the next iterate  $x_{k+1}$  by solving the following equation:

$$x_{k+1} = \left( I + \frac{1}{\lambda_k} A \right)^{-1} (x_k) + e_k, \quad (1.2)$$

where  $\lambda_k > 0$  is a regularizing parameter. If the sequence  $\{\lambda_k\}_{k=1}^{\infty}$  is bounded from above, then the resulting sequence  $\{x_k\}_{k=1}^{\infty}$  of proximal point iterates converges *weakly* to a solution of (1.1), provided that a solution exists (Rockafellar [2]).

Rockafellar then posed the following question.

(Q1) Does the proximal point algorithm always converge *strongly*?

This question was resolved in the negative by Güler [3] who produced a proper closed convex function  $g$  in the infinite-dimensional Hilbert space  $l_2$  for which the proximal point algorithm converges *weakly* but *not strongly*. This naturally raises the following question.

(Q2) Can the proximal point algorithm be modified to guarantee *strong convergence*?

Before we comment on this question, we make the following observation. The proximal point algorithm (1.2) is not at all convenient to use in any possible application. This is because *at each step of the iteration process*, one has to compute  $(I + (1/\lambda_k)A)^{-1}(x_k)$  and this is certainly not convenient. Consequently, while thinking of modifications of the proximal point algorithm that will guarantee strong convergence, the following question is, perhaps, more important than Q2.

(Q3) Can an iteration process be developed which will not involve the computation of  $(I + (1/\lambda_k)A)^{-1}$  at each step of the iteration process and which will guarantee *strong convergence* to a solution of (1.1)?

With respect to Q2, Solodov and Svaiter [4] were the first to propose a modification of the proximal point algorithm, which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows.

*Algorithm.* Choose any  $x^0 \in H$  and  $\sigma \in [0, 1)$ . At iteration  $k$ , having  $x_k$ , choose  $\mu_k > 0$ , and find  $(y_k, v_k)$  an inexact solution of  $0 \in Tx + \mu_k(x - x_k)$ , with tolerance  $\sigma$ . Define

$$\begin{aligned} C_k &:= \left\{ z \in H \mid \langle z - y^k, v^k \rangle \leq 0 \right\}, \\ Q_k &:= \left\{ z \in H \mid \langle z - x^k, x^0 - x^k \rangle \leq 0 \right\}. \end{aligned} \tag{1.3}$$

Take

$$x_{k+1} = P_{C_k \cap Q_k}(x^0). \tag{1.4}$$

The authors themselves noted ([4], p.195) that "... at each iteration, there are two subproblems to be solved ...": (i) find an inexact solution of the proximal point algorithm, and (ii) find the projection of  $x^0$  onto  $C_k \cap Q_k$ , the intersection of the two halfspaces. They also acknowledged that these two subproblems constitute a serious drawback in their algorithm. This method of Solodov and Svaiter is the so-called CQ-method which has been studied by various authors.

For more on the computation of an approximate solution and the projection step, both of which must be performed at each step, the reader may consult [4].

Kamimura and Takahashi [5], extended the work of Solodov and Svaiter [4] to the framework of Banach spaces that are both uniformly convex and uniformly smooth.

Xu [6] noted that "... Solodov and Svaiter's algorithm, though strongly convergent, does need more computing time due to the projection in the second subproblem ..."

He then proposed another modification of the proximal point algorithm, which does not increase computing time by much compared to the algorithm of Solodov and Svaiter.

Xu [7] proposed and studied the following algorithm:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(I + c_n T)^{-1}(x_n) + e_n, \quad n \geq 0. \tag{1.5}$$

He proved that (1.5) converges strongly provided that the sequences  $\{\alpha_n\}$  and  $\{c_n\}$  of real numbers and the sequence  $\{e_n\}$  of errors are chosen appropriately. He argued that once  $u_n := (I + c_n T)^{-1}(x_n) + e_n$  has been calculated, the calculation of the mean  $\alpha_n x_0 + (1 - \alpha_n)u_n$  is much easier than the projection of  $x_0$  onto  $C_n \cap Q_n$  mentioned earlier, and so his algorithm seems simpler than that of Solodov and Svaiter [4]. But the algorithm (1.5) of Xu still has the serious setback associated with the classical proximal point algorithm: the computation of  $(I + c_n T)^{-1}(x_n)$  at each step of the iteration process.

Lehdili and Moudafi [8] considered the technique of the proximal map and the Tikhonov regularization to introduce the so-called Prox-Tikhonov method which generates the sequence  $\{x_n\}$  by the following algorithm:

$$x_{n+1} = J_{\lambda_n}^{T_n} x_n, \quad n \geq 0, \tag{1.6}$$

where  $T_n := \mu_n T + T$ ,  $\mu_n > 0$  is viewed as a Tikhonov regularization of  $T$  and  $J_{\lambda_n}^{T_n} := (I + \lambda_n T_n)^{-1}$ . Using the notation of variational distance, Lehdili and Moudafi [8] proved convergence theorems for the algorithm (1.6) and its perturbed version, under appropriate conditions on the sequences  $\{\lambda_n\}$  and  $\{\mu_n\}$ .

It is known that Tikhonov regularization is not generally effective if an appropriate regularization parameter is not chosen, especially for ill-posed problems. For example, in order to use the discrepancy principle, it is necessary to have information about the noise. Also, in the case of generalized cross validation, efficient implementation for Tikhonov regularization requires computing the singular value decomposition of the matrix, which, for large-scale problems, may be formidable.

Xu [6] studied the algorithm (1.6). He used the technique of nonexpansive mappings to get convergence theorems for the perturbed version of the algorithm (1.6), under much relaxed conditions on the sequences  $\{\lambda_n\}$  and  $\{\mu_n\}$ . Here again, the algorithm (1.6) has the drawback of the classical proximal point algorithm:  $J_{\lambda_n}^{T_n} x_n$  has to be computed at each step.

Another modification of the proximal point algorithm, perhaps the most significant, which yields strong convergence, is implicitly contained in the following theorem of Reich.

**Theorem 1.1** (Reich [9]). *Let  $E$  be a  $q$ -uniformly smooth real Banach space. Let  $A : E \rightarrow E$  be accretive with  $D(A) = E$ , and suppose that  $A$  satisfies the range condition:  $\mathcal{R}(I + rA) = E$  for all  $r > 0$ . Let  $J_t x := (I + tA)^{-1}x$ ,  $t > 0$ , be the resolvent of  $A$ , and assume that  $A^{-1}(0)$  is nonempty. Then for each  $x \in E$ ,  $\lim_{t \rightarrow \infty} J_t x \in A^{-1}(0)$ .*

We first make the following observations about this theorem.

- (i) The  $q$ -uniformly smooth real Banach spaces include the  $L_p$  spaces,  $1 < p < \infty$ . In particular, all Hilbert spaces and  $L_p$  spaces,  $2 \leq p < \infty$ , are 2-uniformly smooth.

- (ii) Any  $m$ -accretive operator satisfies the *range condition*. The converse is not necessarily true. Hence, *range condition* is weaker than  $m$ -accretive.

Now, let  $\{\lambda_n\}$  be a sequence in  $(0, 1)$  such that  $\lambda_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Set  $t_n = \lambda_n^{-1}$ . Then, using the notation of Theorem 1.1, define

$$J_{t_n}x := (I + t_n A)^{-1}x. \quad (1.7)$$

By the theorem, for arbitrary  $x_0 \in E$ ,  $\lim_{n \rightarrow \infty} J_{t_n}x_0$  exists, call it  $x^*$  say, and  $x^* \in A^{-1}(0)$ . We can now define the sequence  $\{x_n\}$  as follows:

$$x_0 \in E, \quad x_{n+1} = \left( I + \frac{1}{\lambda_n} A \right)^{-1} x_0, \quad n \geq 1, \quad (1.8)$$

as a modification of the proximal point algorithm, which yields *strong convergence* to a solution of the equation  $Au = 0$ , assuming existence. Clearly, (1.8) is easier to use and requires less computation time than any of (1.2), (1.5), or (1.6), or using the (CQ)-method of Solodov and Svaiter.

We have seen, in response to Q2, that *all* modifications of the classical proximal point algorithm to obtain strong convergence so far studied inherited the drawback of the algorithm: the computation of  $(I + c_n T)^{-1}$  at each step of the process.

We remark, however, that the proximal point algorithm can be still useful in some special cases. For example, the algorithm was recently successfully used in signal processing and in image restoration where the proximal mappings are fairly evaluated.

We now turn our attention to the consideration of the more important question Q3. It is our purpose in this paper to prove that an iteration process studied by Chidume and Zegye [10] converges strongly to a solution of the equation  $Au = 0$  where  $A$  is a bounded  $m$ -accretive (defined below) operator even on certain real Banach spaces much more general than Hilbert spaces. The iteration process will not involve the computation of  $(I + c_n T)^{-1}(x_n)$  at any stage and will not involve the computation of two convex subsets at each step and the projection of an initial vector to their intersection.

Before we prove our convergence theorems, we need the following definitions and preliminaries.

Let  $E$  be a real normed linear space with dual  $E^*$ . For  $q > 1$ , we denote by  $J_q$  the generalized duality mapping from  $E$  to  $2^{E^*}$  defined by

$$J_q(x) := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\| \|f^*\|, \|f^*\| = \|x\|^{q-1} \right\}, \quad (1.9)$$

where  $\langle \cdot, \cdot \rangle$  denotes the generalized duality pairing.  $J_2$  is denoted by  $J$  and is the so-called *normalized duality map*. If  $E^*$  is strictly convex, then  $J_q$  is single-valued (see, e.g., Xu [11]). A mapping  $A$  with domain  $D(A)$  and range  $R(A)$  in  $E$  is said to be *strongly  $\phi$ -accretive* if, for any  $x, y \in D(A)$ , there exist  $j_q(x - y) \in J_q(x - y)$  and a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$  with  $\phi(0) = 0$  such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \phi(\|x - y\|) \|x - y\|^{q-1}. \quad (1.10)$$

The mapping  $A$  is called *generalized  $\Phi$ -accretive* if, for any  $x, y \in D(A)$ , there exist  $j_q(x - y) \in J_q(x - y)$  and a strictly increasing function  $\Phi : [0, \infty) \rightarrow [0, \infty)$  with  $\Phi(0) = 0$  such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \Phi(\|x - y\|). \quad (1.11)$$

It is well known that the class of generalized  $\Phi$ -accretive mappings includes the class of strongly  $\phi$ -accretive operators as a special case (one sets  $\Phi(s) = s\phi(s)$  for all  $s \in [0, \infty)$ ).

Let  $N(A) := \{x \in E : Ax = 0\} \neq \emptyset$ . The mapping  $A$  is called *strongly quasi-accretive* if for all  $x \in E$ ,  $x^* \in N(A)$ , there exists  $k \in (0, 1)$  such that

$$\langle Ax - Ax^*, j_q(x - x^*) \rangle \geq k\|x - x^*\|^q. \quad (1.12)$$

$A$  is called *strongly  $\phi$ -quasi-accretive* if, for all  $x \in E$ ,  $x^* \in N(A)$ , there exists  $\phi$  such that

$$\langle Ax - Ax^*, j_q(x - x^*) \rangle \geq \phi(\|x - x^*\|)\|x - x^*\|^{q-1}. \quad (1.13)$$

Finally,  $A$  is called *generalized  $\Phi$ -quasi-accretive* if, for all  $x \in E$ ,  $x^* \in N(A)$ , there exists  $j_q(x - x^*) \in J(x - x^*)$  such that

$$\langle Ax - Ax^*, j_q(x - x^*) \rangle \geq \Phi(\|x - x^*\|). \quad (1.14)$$

It is well known that the class of generalized  $\Phi$ -quasi-accretive mappings is the largest class (among those defined above) for which the equation  $Au = 0$  has a *unique* solution.

A mapping  $A$  with domain  $D(A)$  and range  $\mathcal{R}(A)$  in  $E$  is called *accretive* if and only if for all  $x, y \in D(A)$ , the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \quad \forall s > 0. \quad (1.15)$$

As a consequence of a result of Kato [12], it follows that  $A$  is accretive if and only if for each  $x, y \in D(A)$ , there exists  $j_q(x - y) \in J_q(x - y)$  such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq 0. \quad (1.16)$$

It follows from inequality (1.15) that  $A$  is accretive if and only if  $(I + sA)$  is *expansive*, and consequently its inverse  $(I + sA)^{-1}$  exists and is *nonexpansive* (i.e.,  $\|(I + sA)^{-1}x - (I + sA)^{-1}y\| \leq \|x - y\|$ , for all  $x, y \in \mathcal{R}(I + sA)$ ) as a mapping from  $\mathcal{R}(I + sA)$  into  $D(A)$ , where  $\mathcal{R}(I + sA)$  denotes the range of  $(I + sA)$ . The range of  $(I + sA)$  needs not be the whole of  $E$ . This leads to the following definitions. A mapping  $A$  is said to be *m-accretive* if  $A$  is accretive and the range of  $(I + sA)$  is all of  $E$  for *some*  $s > 0$ . The operator  $A$  is said to satisfy the *range condition* if  $\mathcal{R}(I + sA) = E$  for *all*  $s > 0$ . It can be shown that if  $\mathcal{R}(I + sA) = E$  for some  $s > 0$ , then it holds for all  $s > 0$ . We prove this important fact, which will be used in the sequel in Lemma 2.5. Hence, *m-accretive condition* implies *range condition*. In Hilbert space, accretive operators are called *monotone*.

Let  $E$  be a real Banach space and  $A : E \rightarrow E$  a map. Assume that the equation  $Au = 0$  has a solution. Iterative methods for approximating such a solution have been of interest

to numerous researchers in nonlinear operator theory. Bruck [13] considered an iteration process of the Mann-type and proved that the sequence of the process converges strongly to a solution of the inclusion  $0 \in Au$  in a real Hilbert space where  $A$  is a maximal monotone map, provided the initial vector is taken in a neighbourhood of a solution of this inclusion. Chidume [14] extended this result to  $L^p$  spaces,  $p \geq 2$ . These results of Bruck and Chidume are not convenient in any possible application because the neighbourhood of a solution in which the initial vector must be chosen is not known precisely. Other early results involved mappings  $A$  that are *Lipschitz* and *strongly accretive* (see, e.g., Browder and Petryshyn [15, 16], Chidume [17, 18], Chidume and Moore [19, 20], Deng [21, 22], Zhou [23], Zhou and Jia [24], Weng [25], Xu and Yin [26], Xu and Roach [27] and a host of other authors).

Numerous papers were later published, extending these results to the class of mappings  $A$  that are *Lipschitz* and *strongly  $\phi$ -accretive* (see e.g., Chang et al [28], Zhou [23] and the references contained therein). Recently, some authors have proved convergence theorems for the solution of  $Au = 0$ , where  $A$  is assumed to be a *Lipschitz* and *generalized  $\Phi$ -accretive* map.

A few papers have recently been published establishing convergence theorems for the solution of  $Au = 0$ , where  $A$  is a *uniformly continuous* and *generalized  $\Phi$ -quasi-accretive* map (see, e.g., Gu [29], Chang et al. [30], and the references contained in them). Related results on general variational inequalities can be found in the paper of Aslam Noor [31].

Recall that the class of *generalized  $\Phi$ -quasi accretive* maps is the largest class for which the equation  $Au = 0$  has a *unique* solution. Most papers have been devoted to this. It is well known that for the general *accretive* operator  $A$ , the solution of this equation, whenever it exists, is generally not unique. In this case, the technique used in approximating a solution of the equation when it exists and is unique does not carry over to the case when it exists and is not unique.

The accretive mappings were introduced independently in 1967 by Browder [32] and Kato [12]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in real Banach spaces. It is known (see, e.g., Zeidler [33]) that many physically significant problems can be modelled in terms of an initial-value problem of the form

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0, \quad (1.17)$$

where  $A$  is an accretive map on an appropriate real Banach space. Typical examples of such evolution equations are found in models involving the heat, wave, or Schrödinger equations (see [34]). Observe that in the model (1.17), if the solution  $u$  is independent of time (i.e., at the equilibrium state of the system), then  $du/dt = 0$  and (1.17) reduces to

$$Au = 0, \quad (1.18)$$

whose solutions then correspond to the equilibrium state of the system described by (1.17). To approximate a solution of (1.18), Browder converted (1.18) to a fixed point problem as follows. He called an operator  $T := I - A$  *pseudocontractive* if  $A$  is *accretive* and  $I$  is the identity map on the space. Then,  $u^*$  is a solution of (1.18) if and only if it is a fixed point of  $T$ . Consequently, a lot of effort has been devoted to approximating fixed points of *pseudocontractive* maps.

Let  $K$  be a nonempty subset of  $E$ . We reiterate that a mapping  $T : K \rightarrow K$  is called *pseudocontractive* if  $A := (I - T) : E \rightarrow E$  is accretive. Consequently, solutions of  $Au = 0$  (when they exist) for accretive operators  $A$  correspond to fixed points of pseudocontractions  $T$ . Every nonexpansive map is Lipschitz pseudocontractive.

In the late 1960s and early 1970s, the well-known *Mann iteration process* [35] was successfully employed to approximate fixed points of nonexpansive maps under suitable assumptions. All attempts to apply it to approximate fixed points of *Lipschitz pseudocontractions* proved abortive. In 1974, Ishikawa [36] introduced an iteration scheme involving two parameters  $\alpha_n$  and  $\beta_n$  for approximating a fixed point of a *Lipschitz pseudocontraction* defined on a *compact convex* subset of a real Hilbert space. He proved strong convergence of the sequence generated by his scheme. An example of  $\alpha_n$  and  $\beta_n$  satisfying his condition is  $\alpha_n = \beta_n = n^{-1/2}$ ,  $n \geq 1$  (see, e.g., Berinde [37], Chidume [38], Ishikawa [36]). It is still an open question whether or not this theorem of Ishikawa can be extended to real Banach spaces more general than Hilbert spaces. Since 1974, three other iteration methods have been introduced and studied and have been successfully employed to approximate fixed points of *Lipschitz pseudocontractive* mappings in certain Banach spaces more general than Hilbert spaces.

One of these three iteration processes was introduced and studied by Schu [39]. The recursion formula studied involved the use of two real sequences  $\lambda_n, \mu_n$  which are required to have the so-called *Property (A)*.

Let  $\alpha_n \in (0, \infty)$ ,  $\mu_n \in (0, 1)$  for all nonnegative integers  $n$ . Then,  $(\{\alpha_n\}, \{\mu_n\})$  is said to have *Property (A)* if and only if the following conditions hold:

- (i)  $\{\alpha_n\}$  is decreasing and  $\{\mu_n\}$  is strictly increasing;
- (ii) there is a sequence  $\{\beta_n\} \subseteq \mathbb{N}$ , strictly increasing such that

- (a)  $\lim_{n \rightarrow \infty} \beta_n(1 - \mu_n) = \infty$ ,
- (b)  $\lim_{n \rightarrow \infty} (1 - \mu_{(n+\beta_n)}) / (1 - \mu_n) = 1$ ,
- (c)  $\lim_{n \rightarrow \infty} (\alpha_n - \alpha_{(n+\beta_n)}) / (1 - \mu_n) = 0$ .

Schu proved his convergence theorem in real Hilbert spaces. Chidume [40] extended it to real Banach spaces possessing *weakly sequentially continuous duality maps* (e.g.,  $l_p$  spaces,  $1 < p < \infty$ ). However, it is known that  $L_p$  spaces,  $1 < p < \infty$ ,  $p \neq 2$ , do not possess this property.

Another iteration scheme for approximating fixed points of *Lipschitz pseudocontractive* mappings was implicitly introduced by Bruck [13], who actually applied the scheme, still in real Hilbert spaces, to approximate a solution of the inclusion  $0 \in Au$ , where  $A$  is an *m-accretive* operator. The recursion formula studied by Bruck involved two real sequences  $\lambda_n$  and  $\theta_n$  which are required to be *acceptably paired*.

Two real sequences  $\{\lambda_n\}$  and  $\{\theta_n\}$  are called *acceptably paired* if they satisfy the following conditions:  $\{\theta_n\}$  is decreasing,  $\lim_{n \rightarrow \infty} \theta_n = 0$ , and there exists a strictly increasing sequence  $\{n(i)\}_{i=1}^{\infty}$  of positive integers such that

- (i)  $\liminf_i (\theta_{n(i)} \sum_{j=n(i)}^{n(i+1)} \lambda_j) > 0$ ,
- (ii)  $\lim_i [\theta_{n(i)} - \theta_{n(i+1)}] \sum_{j=n(i)}^{n(i+1)} \lambda_j = 0$ ,
- (iii)  $\lim_i \sum_{j=n(i)}^{n(i+1)} \lambda_j^2 = 0$ .

An example of such sequences given in Bruck [13] is

$$\lambda_n = n^{-1}, \theta_n = (\log \log n)^{-1}, \quad n(i) = i^i. \quad (1.19)$$

The idea of sequences with *Property (A)* or *acceptably paired* are due to Halpern [41]. Reich [9, 42] also studied the recursion formula studied by Bruck for *Lipschitz accretive* operators on real uniformly convex Banach spaces with a *duality map that is weakly sequentially continuous at zero*, but with  $\lambda_n$  and  $\theta_n$  not necessarily being *acceptably paired*.

Motivated by the papers of Reich [9, 42], Chidume and Zegeye [10] introduced and studied a perturbation of the Mann recurrence relation (see Theorem CZ below) to approximate zeros of *Lipschitz accretive* maps in real Banach spaces much more general than real Hilbert spaces. They proved the following theorem.

**Theorem CZ** (Chidume and Zegeye, [10]). *Let  $E$  be a reflexive real Banach space with a uniformly Gâteaux differentiable norm. Let  $A : E \rightarrow E$  be a Lipschitz accretive operator and let  $N(A) := \{x \in E : Ax = 0\} \neq \emptyset$ . Suppose that every nonempty closed convex and bounded subset of  $E$  has the fixed point property for nonexpansive self-mappings. Let a sequence  $\{x_n\}$  be generated from arbitrary  $x_1 \in E$  by*

$$x_{n+1} = x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \quad (1.20)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$ ,
- (2)  $\lambda_n(1 + \theta_n) < 1$ ,  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ,
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{(n-1)})/\theta_n) - 1)/\lambda_n \theta_n = 0$ .

Then,  $\{x_n\}$  converges strongly to a solution of  $Au = 0$ .

An example of sequences  $\lambda_n$  and  $\theta_n$  satisfying Theorem CZ is  $\lambda_n = n^{-a}$ ,  $\theta_n = n^{-b}$ ,  $n \geq 1$ ,  $0 < b < a$ , and  $a + b < 1$ .

It is clear that these parameters are much simpler than the requirement that they have *Property (A)* or are *acceptably paired*.

It is our purpose in this paper to provide affirmative answer to Q3 when  $A$  is bounded and satisfies the range condition. We show that the iteration process (1.20) converges strongly even in real Banach spaces much more general than Hilbert spaces to a solution of the equation  $Au = 0$  (assuming existence) for arbitrary initial vector  $x_0 \in E$ . These spaces include  $L_p$  spaces,  $2 \leq p < \infty$ . Moreover, it is clear that the recursion formula (1.20) is simpler than that of the *proximal point algorithm* and involves direct applications of  $A$ . Furthermore, the regularization parameters  $\lambda_n, \theta_n$  are easily chosen at the beginning of the iteration process. Computation of  $(I + c_n T)^{-1}$  is not required at any stage, and computation of two convex sets and projection of  $x_0$  onto their intersection will not be required. Finally, we derive some applications of our theorems to approximate fixed points of *uniformly continuous pseudocontractions*. We achieve this by means of an incisive result recently proved by C. E. Chidume and C. O. Chidume [43].

## 2. Preliminaries

Let  $X$  be a real normed linear space of dimension  $\geq 2$ . The *modulus of smoothness* of  $X$  is defined by

$$\rho_X(\tau) := \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}, \quad \tau > 0. \quad (2.1)$$

If there exist a constant  $c > 0$  and a real number  $q > 1$ , such that  $\rho_X(\tau) \leq c\tau^q$ , then  $X$  is said to be *q-uniformly smooth*. Typical examples of such spaces are the  $L^p$ ,  $\ell_p$ , and  $W_p^m$  spaces for  $1 < p < \infty$ , where

$$L^p \text{ (or } \ell_p) \text{ or } W_p^m \text{ is } \begin{cases} 2\text{-uniformly smooth} & \text{if } 2 \leq p < \infty, \\ p\text{-uniformly smooth} & \text{if } 1 < p < 2. \end{cases} \quad (2.2)$$

In the sequel we will need the following results.

**Theorem 2.1** (Xu [11]). *Let  $q > 1$  and  $E$  be a real Banach space. Then the following are equivalent.*

- (i)  $E$  is *q-uniformly smooth*.
- (ii) There exists a constant  $d_q > 0$  such that for all  $x, y \in E$ ,

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x) \rangle + d_q\|y\|^q. \quad (2.3)$$

For the remainder of this paper,  $c_q$  and  $d_q$  will denote the constants appearing in Theorem 2.1.

**Lemma 2.2.** *Let  $E$  be a real normed linear space. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y) \rangle \quad \forall j(x+y) \in J(x+y), \quad \forall x, y \in E. \quad (2.4)$$

**Lemma 2.3** (Xu [7]). *Let  $\{a_n\}$  be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0, \quad (2.5)$$

where (i)  $\{\alpha_n\} \subset (0, 1)$ ,  $\sum \alpha_n = \infty$ ; (ii)  $\limsup \sigma_n \leq 0$ ; (iii)  $\gamma_n \geq 0$ , ( $n \geq 0$ ),  $\sum \gamma_n < \infty$ . Then,  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.4** (C. E. Chidume and C. O. Chidume [43]). *Let  $X$  and  $Y$  be real normed linear spaces and let  $T : X \rightarrow Y$  be a uniformly continuous map. For arbitrary  $r > 0$  and fixed  $x^* \in X$ , let*

$$B_X(x^*, r) := \{x \in X : \|x - x^*\|_X \leq r\}. \quad (2.6)$$

Then  $T(B(x^*, r))$  is bounded.

Since this result is yet to appear, we reproduce its short proof here.

*Proof.* By the uniform continuity of  $T$  and by taking  $\epsilon = 1$ , there exists  $\delta > 0$  such that for all  $x, y \in X$ ,

$$\|x - y\| < \delta \implies \|Tx - Ty\| < 1. \quad (2.7)$$

For  $r > 0$ , let  $z \in B(x^*, r)$  be arbitrary. Choose  $n_0 \in \mathbb{N}$  fixed such that  $r < n_0\delta$ . Set

$$\begin{aligned} z_0 = x^*, z_1 = x^* + \frac{z - x^*}{n_0}, z_2 = x^* + \frac{2(z - x^*)}{n_0}, z_3 = x^* + \frac{3(z - x^*)}{n_0}, \dots, z_k = x^* + \frac{k(z - x^*)}{n_0}, \\ z_{k+1} = x^* + \frac{(k+1)(z - x^*)}{n_0}, \dots, z_{n_0-1} = x^* + \frac{(n_0-1)(z - x^*)}{n_0}, z_{n_0} = z. \end{aligned} \quad (2.8)$$

Then,

$$\|z_{k+1} - z_k\| = \frac{\|z - x^*\|}{n_0} \leq \frac{r}{n_0} < \delta. \quad (2.9)$$

By the uniform continuity of  $T$ ,  $\|Tz_{k+1} - Tz_k\| < 1$ . Furthermore,

$$\begin{aligned} \|Tz\| = \|Tz_{n_0}\| &\leq \|Tz_{n_0} - Tz_{n_0-1}\| + \|Tz_{n_0-1} - Tz_{n_0-2}\| + \dots + \|Tz_1 - Tz_0\| \\ &+ \|Tz_0\| \leq n_0 + \|Tz_0\| \leq n_0 + \|Tx^*\|. \end{aligned} \quad (2.10)$$

Hence,  $T(B(x^*, R))$  is bounded. □

We now prove the following lemma, which will be used in the sequel.

**Lemma 2.5.** For  $q > 1$ , let  $E$  be a  $q$ -uniformly smooth real Banach space and  $A : E \rightarrow E$  be a map with  $D(A) = E$ . Suppose that  $A$  is  $m$ -accretive, that is, (i) for all  $u, v \in E$ ,  $\exists j(u-v) \in J_q(u-v)$  such that  $\langle Au - Av, j(u-v) \rangle \geq 0$ . (ii)  $\mathcal{R}(I + s_0A) = E$  for some  $s_0 > 0$ . Then  $A$  satisfies the range condition, that is,  $\mathcal{R}(I + sA) = E$  for all  $s > 0$ .

*Proof.* Assume that there exists some  $s_0 > 0$  such that  $\mathcal{R}(I + s_0A) = E$ . It is known that since  $A$  is accretive, the map  $(I + s_0A) : E \rightarrow E$  is invertible and, moreover,  $J_{s_0} := (I + s_0A)^{-1}$  is non-expansive, that is,

$$\|J_{s_0}u - J_{s_0}v\| \leq \|u - v\| \quad \forall u, v \in E. \quad (2.11)$$

*Claim.*  $\mathcal{R}(I + sA) = E$  for any  $s > s_0/2$ . Indeed, let  $s > s_0/2$  and  $w \in E$ , we solve the equation

$$u + sAu = w. \quad (2.12)$$

We observe that  $u \in E$  solves (2.12) if and only if

$$u + s_0 Au = \frac{s_0}{s} w + \left(1 - \frac{s_0}{s}\right) u \quad (2.13)$$

or equivalently

$$u = J_{s_0} \left( \frac{s_0}{s} w + \left(1 - \frac{s_0}{s}\right) u \right). \quad (2.14)$$

Observing that  $|1 - s_0/s| < 1$ , it follows from the Banach fixed point theorem that (2.14) has a unique solution. This proves the claim.

Since  $A$  is  $m$ -accretive,  $\mathcal{R}(I + s_0 A) = E$  for some  $s_0 > 0$ . Using the claim, it follows that  $\mathcal{R}(I + sA) = E$  for any  $s > s_0/2$ . By induction, we have that  $\mathcal{R}(I + sA) = E$  for any  $s > s_0/2^n$  and for any  $n \geq 1$ . So, the conclusions follows.  $\square$

### 3. Approximation of Zeros of Bounded $m$ -Accretive Operators

We now prove the following theorem.

**Theorem 3.1.** *Let  $E$  be a 2-uniformly smooth real Banach space, and let  $A : E \rightarrow E$  be a bounded  $m$ -accretive map. For arbitrary  $x_1 \in E$ , define the sequence  $\{x_n\}$  iteratively by*

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \quad (3.1)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$ , and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

*Proof.* Let  $x^* \in E$  be a solution of the equation  $Ax = 0$ . There exists  $r > 0$  sufficiently large such that  $x_1 \in B(x^*, r/2)$ . Define  $B := \overline{B(x^*, r)}$ . Since  $A$  is bounded, it follows that  $A(B)$  is bounded. So,

$$M_0 := \sup \left\{ \|Ax + \theta(x - x_1)\|^2 : x \in B, 0 < \theta \leq 1 \right\} + 1 < \infty. \quad (3.2)$$

Let

$$M_2 := d_2 M_0, \quad \gamma_0 := \min \left\{ 1, \frac{r^2}{4M_2} \right\}, \quad (3.3)$$

where  $d_2 = d_q$  in Theorem 2.1 with  $q = 2$ .

*Step 1.* We prove that  $\{x_n\}$  is bounded. Indeed, it suffices to show that  $x_n$  is in  $B$  for all  $n \geq 1$ . The proof is by induction. By construction,  $x_1 \in B$ . Suppose that  $x_n \in B$  for some  $n \geq 1$ . We prove that  $x_{n+1} \in B$ .

Using inequality (2.3) of Theorem 2.1 with  $q = 2$ , we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|x_n - x^* - \lambda_n(Ax_n + \theta_n(x_n - x_1))\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\lambda_n \langle Ax_n + \theta_n(x_n - x_1), j(x_n - x^*) \rangle \\ &\quad + d_2 \lambda_n^2 \|Ax_n + \theta_n(x_n - x_1)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\lambda_n \langle Ax_n + \theta_n(x_n - x_1), j(x_n - x^*) \rangle + \lambda_n^2 M_2. \end{aligned} \quad (3.4)$$

Using the fact that  $A$  is accretive, we obtain

$$\langle Ax_n + \theta_n(x_n - x_1), j(x_n - x^*) \rangle \geq \theta_n \|x_n - x^*\|^2 + \theta_n \langle x^* - x_1, j(x_n - x^*) \rangle. \quad (3.5)$$

Therefore, we have the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - 2\lambda_n \theta_n) \|x_n - x^*\|^2 - 2\lambda_n \theta_n \langle x^* - x_1, j(x_n - x^*) \rangle + \lambda_n^2 M_2 \\ &\leq (1 - 2\lambda_n \theta_n) \|x_n - x^*\|^2 + 2\lambda_n \theta_n \|x^* - x_1\| \|x_n - x^*\| + \lambda_n^2 M_2 \\ &\leq (1 - 2\lambda_n \theta_n) \|x_n - x^*\|^2 + 2\lambda_n \theta_n \left( \frac{1}{2} \|x^* - x_1\|^2 + \frac{1}{2} \|x_n - x^*\|^2 \right) + \lambda_n^2 M_2. \end{aligned} \quad (3.6)$$

Thus,

$$\|x_{n+1} - x^*\|^2 \leq (1 - \lambda_n \theta_n) \|x_n - x^*\|^2 + \lambda_n \theta_n \|x^* - x_1\|^2 + \lambda_n^2 M_2. \quad (3.7)$$

So, using the induction assumption, the fact that  $x_1 \in B(x^*, r/2)$  and the condition  $\lambda_n \leq \gamma_0 \theta_n$ , we obtain

$$\|x_{n+1} - x^*\|^2 \leq \left(1 - \frac{1}{2} \lambda_n \theta_n\right) r^2 \leq r^2. \quad (3.8)$$

Therefore,  $x_{n+1} \in B$ . Thus by induction,  $\{x_n\}$  is bounded.

*Step 2.* We prove that  $\{x_n\}$  converges to a solution of  $Ax = 0$ . Since  $A$  is  $m$ -accretive, using Lemma 2.5 and Theorem 1.1, there exists a unique sequence  $\{y_n\}$  in  $E$  satisfying the following properties:

$$\theta_n(y_n - x_1) + Ay_n = 0 \quad \forall n \geq 1, \quad (i)$$

$$y_n \longrightarrow y^* \quad \text{with } Ay^* = 0. \quad (ii)$$

Indeed, applying Theorem 1.1, with  $t = 1/\theta_n$ , then the sequence  $\{y_n\}$ , with

$$y_n = \left( I + \frac{1}{\theta_n} A \right)^{-1} x_1 \quad (3.9)$$

has the desired properties.

*Claim.*  $\|x_{n+1} - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\{x_n\}$  and  $\{y_n\}$  are bounded and  $A$  is bounded, there exists some positive constant  $M$  such that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &= \|x_n - y_n - \lambda_n(Ax_n + \theta_n(x_n - x_1))\|^2 \\ &\leq \|x_n - y_n\|^2 - 2\lambda_n \langle Ax_n + \theta_n(x_n - x_1), j(x_n - y_n) \rangle \\ &\quad + d_2 \lambda_n^2 \|Ax_n + \theta_n(x_n - x_1)\|^2 \\ &\leq \|x_n - y_n\|^2 - 2\lambda_n \langle Ax_n + \theta_n(x_n - x_1), j(x_n - y_n) \rangle + \lambda_n^2 M. \end{aligned} \quad (3.10)$$

Using (23) and the fact that  $A$  is accretive, we have

$$\begin{aligned} \langle Ax_n + \theta_n(x_n - x_1), j(x_n - y_n) \rangle &= \langle Ax_n - Ay_n, j(x_n - y_n) \rangle + \theta_n \|x_n - y_n\|^2 \\ &\quad + \langle Ay_n + \theta_n(y_n - x_1), j(x_n - y_n) \rangle \\ &\geq \frac{\theta_n}{2} \|x_n - y_n\|^2. \end{aligned} \quad (3.11)$$

Therefore,

$$\|x_{n+1} - y_n\|^2 \leq (1 - \lambda_n \theta_n) \|x_n - y_n\|^2 + \lambda_n^2 M. \quad (3.12)$$

Using again the fact that  $A$  is accretive, we obtain

$$\|y_{n-1} - y_n\|^2 \leq \left\| y_{n-1} - y_n + \frac{1}{\theta_n} (Ay_{n-1} - Ay_n) \right\|^2. \quad (3.13)$$

From (i), observing that

$$y_{n-1} - y_n + \frac{1}{\theta_n} (Ay_{n-1} - Ay_n) = \frac{\theta_n - \theta_{n-1}}{\theta_n} (y_{n-1} - x_1), \quad (3.14)$$

it follows that

$$\|y_{n-1} - y_n\| \leq \frac{\theta_{n-1} - \theta_n}{\theta_n} \|y_{n-1} - x_1\|. \quad (3.15)$$

By Lemma 2.2, we have

$$\begin{aligned} \|x_n - y_n\|^2 &= \|(x_n - y_{n-1}) + (y_{n-1} - y_n)\|^2 \\ &\leq \|x_n - y_{n-1}\|^2 + 2 \langle y_{n-1} - y_n, j(x_n - y_n) \rangle. \end{aligned} \quad (3.16)$$

Using Schwartz's inequality, we obtain:

$$\|x_n - y_n\|^2 \leq \|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\|\|x_n - y_n\|. \quad (3.17)$$

Using (3.12), (3.15), (3.17), and the fact that  $\{x_n\}$  and  $\{y_n\}$  are bounded, we have,

$$\|x_{n+1} - y_n\|^2 \leq (1 - \lambda_n \theta_n) \|x_n - y_{n-1}\|^2 + C \left( \frac{\theta_{n-1} - \theta_n}{\theta_n} \right) + \lambda_n^2 M. \quad (3.18)$$

for some positive constant  $C$ . Thus, by Lemma 2.3,  $x_{n+1} - y_n \rightarrow 0$ . Using (ii), it follows that  $x_n \rightarrow y^*$  and  $Ay^* = 0$ . This completes the proof.  $\square$

**Corollary 3.2.** *Let  $E$  be a 2-uniformly smooth real Banach space, and let  $A : E \rightarrow E$  be a uniformly continuous  $m$ -accretive map. For arbitrary  $x_1 \in E$ , define the sequence  $\{x_n\}$  iteratively by*

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \quad (3.19)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

*Proof.* Since  $A$  is uniformly continuous, then, by Lemma 2.4,  $A$  is bounded. So the result follows from Theorem 3.1.  $\square$

**Corollary 3.3.** *Let  $H$  be real Hilbert space, and let  $A : H \rightarrow H$  be a bounded maximal monotone operator. For arbitrary  $x_1 \in H$ , define the sequence  $\{x_n\}$  iteratively by*

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \quad (3.20)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

**Corollary 3.4.** *Let  $H$  be real Hilbert space, and let  $A : H \rightarrow H$  be a uniformly continuous maximal monotone operator. For arbitrary  $x_1 \in H$ , define the sequence  $\{x_n\}$  iteratively by*

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \quad (3.21)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

**Corollary 3.5.** Let  $E = L_p$  (or  $l_p$ ) space,  $(2 \leq p < \infty)$ , and let  $A : E \rightarrow E$  be a bounded  $m$ -accretive map. For arbitrary  $x_1 \in E$ , define the sequence  $\{x_n\}$  iteratively by

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \tag{3.22}$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

**Corollary 3.6.** Let  $E = L_p$  (or  $l_p$ ) space,  $(2 \leq p < \infty)$ , and let  $A : E \rightarrow E$  be a uniformly continuous  $m$ -accretive map. For arbitrary  $x_1 \in E$ , define the sequence  $\{x_n\}$  iteratively by

$$x_{n+1} := x_n - \lambda_n Ax_n - \lambda_n \theta_n (x_n - x_1), \quad n \geq 1, \tag{3.23}$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (2)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (3)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Suppose that the equation  $Ax = 0$  has a solution. Then, there exists a constant  $\gamma_0 > 0$  such that if  $\lambda_n \leq \gamma_0 \theta_n$  for all  $n \geq 1$ ,  $\{x_n\}$  converges strongly to a solution of the equation  $Ax = 0$ .

#### 4. Approximation of Fixed Points of Uniformly Continuous Pseudocontractive Operators

We will make use of the following result.

**Lemma 4.1** (Reich [9], Morales and Jung [44], Takahashi and Ueda [45]). Let  $K$  be closed convex subset of a reflexive Banach space  $E$  with a uniformly Gâteaux differentiable norm. Let  $T : K \rightarrow K$  be continuous pseudocontractive mapping with  $F(T) \neq \emptyset$ . Suppose that every closed convex and bounded subset of  $K$  has the fixed point property for nonexpansive self-mappings. Then for  $u \in K$ , the path  $t \rightarrow y_t \in K$ ,  $t \in (0, 1]$ , satisfying,  $y_t = (1-t)Ty_t + tu$ , converges strongly to a fixed point  $Qu$  of  $T$  as  $t \rightarrow 0$ , where  $Q$  is the unique sunny nonexpansive retract from  $K$  onto  $F(T)$ .

We now prove the following theorem.

**Theorem 4.2.** *Let  $K$  be a nonempty closed convex subset of a 2-uniformly smooth real Banach space  $E$ . Let  $T : K \rightarrow K$  be a uniformly continuous pseudocontractive map with  $F(T) \neq \emptyset$ . Let a sequence  $\{x_n\}$  be generated from arbitrary  $x_1 \in K$  by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall n \geq 1, \quad (4.1)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lambda_n(1 + \theta_n) < 1$ ;
- (2)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (3)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (4)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Then,  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

*Proof.* Since  $E$  is a 2-uniformly smooth real Banach space, it has uniformly Gâteaux differentiable norm and every closed bounded convex nonempty subset of  $E$  has the fixed point property for nonexpansive self-mappings. Set  $A := (I - T)$ . Then  $A$  is uniformly continuous and accretive. Now, using the fact  $T$  is uniformly continuous and pseudocontractive, it follows from Lemma 4.1 that there exists a unique sequence  $(y_n)$ , with  $y_n := y_{t_n}$ ,  $t_n := \theta_n/(1 + \theta_n)$  satisfying the following properties:

$$\begin{aligned} \theta_n(y_n - x_1) + A y_n &= 0 \quad \forall n \geq 1, \\ y_n &\longrightarrow y^* \quad \text{with } A y^* = 0. \end{aligned} \quad (4.2)$$

Since  $A$  is uniformly continuous, then from Lemma 2.4 it is bounded. As in the proof of Theorem 3.1,  $\{x_n\}$  converges strongly to  $y^*$ , with  $A y^* = 0$ . So,  $\{x_n\}$  converges strongly to a fixed point of  $T$ .  $\square$

**Corollary 4.3.** *Let  $K$  be a nonempty closed convex subset of a 2-uniformly smooth real Banach space  $E$ . Let  $T : K \rightarrow K$  be a Lipschitz pseudocontractive map with  $F(T) \neq \emptyset$ . Let a sequence  $\{x_n\}$  be generated from arbitrary  $x_1 \in K$  by*

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall n \geq 1, \quad (4.3)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lambda_n(1 + \theta_n) < 1$ ;
- (2)  $\lim \theta_n = 0$ ; and  $\{\theta_n\}$  is decreasing;
- (3)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (4)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Then,  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

**Corollary 4.4.** Let  $E = L_p$  (or  $l_p$ ) space, ( $2 \leq p < \infty$ ), and let  $K$  be a nonempty closed convex subset of  $E$ . Let  $T : K \rightarrow K$  be a uniformly continuous pseudocontractive map with  $F(T) \neq \emptyset$ . Let a sequence  $\{x_n\}$  be generated from arbitrary  $x_1 \in K$  by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall n \geq 1, \quad (4.4)$$

where  $\{\lambda_n\}$  and  $\{\theta_n\}$  are sequences in  $(0, 1)$  satisfying the following conditions:

- (1)  $\lambda_n(1 + \theta_n) < 1$ ;
- (2)  $\lim \theta_n = 0$  and  $\{\theta_n\}$  is decreasing;
- (3)  $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$ ,  $\lambda_n = o(\theta_n)$ ;
- (4)  $\lim_{n \rightarrow \infty} ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n = 0$ ,  $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$ .

Then,  $\{x_n\}$  converges strongly to a fixed point of  $T$ .

*Remark 4.5.* Addition of bounded error terms to any of the recursion formulas studied in this paper yields no further generalizations.

*Remark 4.6.* Real sequences that satisfy the hypotheses of Theorems 3.1 are  $\lambda_n = n^{-a}$  and  $\theta_n = n^{-b}$ ,  $n \geq 1$  with  $0 < b < a$ ,  $1/2 < a < 1$ , and  $a + b < 1$ . We verify that these choices satisfy in particular, the first part of condition (3) of Theorem 3.1. In fact, using the fact that  $(1 + x)^s \leq 1 + sx$ , for  $x > -1$  and  $0 < s < 1$ , we have

$$\begin{aligned} 0 \leq ((\theta_{n-1}/\theta_n) - 1)/\lambda_n \theta_n &= \left[ \left(1 + \frac{1}{n}\right)^b - 1 \right] \cdot (n+1)^{a+b} \\ &\leq b \cdot \frac{(n+1)^{a+b}}{n} = b \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^{1-(a+b)}} \rightarrow 0 \end{aligned} \quad (4.5)$$

as  $n \rightarrow \infty$ .

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