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Research Article

N_θ-Ward Continuity

Huseyin Cakalli

Department of Mathematics, Maltepe University, Marmara Education Village, 34857 Istanbul, Turkey

Correspondence should be addressed to Huseyin Cakalli, hcakalli@maltepe.edu.tr

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A function f is continuous if and only if f preserves convergent sequences; that is, $(f(\alpha_n))$ is a convergent sequence whenever (α_n) is convergent. The concept of N_θ-ward continuity is defined in the sense that a function f is N_θ-ward continuous if it preserves N_θ-quasi-Cauchy sequences; that is, $(f(\alpha_n))$ is an N_θ-quasi-Cauchy sequence whenever (α_n) is N_θ-quasi-Cauchy. A sequence (α_k) of points in $\mathbf{R}$, the set of real numbers, is N_θ-quasi-Cauchy if $\lim_{r\to\infty}(1/h_r)\sum_{k\in I_r}|\Delta\alpha_k| = 0$, where $\Delta\alpha_k = \alpha_{k+1} - \alpha_k$, $I_r = (k_{r-1}, k_r]$, and $\theta = (k_r)$ is a lacunary sequence, that is, an increasing sequence of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$. A new type compactness, namely, N_θ-ward compactness, is also, defined and some new results related to this kind of compactness are obtained.

1. Introduction

It is well known that a real function f is continuous if and only if, for each point α_0 in the domain, $\lim_{n\to\infty} f(\alpha_n) = f(\alpha_0)$ whenever $\lim_{n\to\infty}\alpha_n = \alpha_0$. This is equivalent to the statement that $(f(\alpha_n))$ is a convergent sequence whenever (α_n) is. This is also equivalent to the statement that $(f(\alpha_n))$ is a Cauchy sequence whenever (α_n) is Cauchy provided that the domain of the function is either whole $\mathbf{R}$ or a bounded and closed subset of $\mathbf{R}$ where $\mathbf{R}$ is the set of real numbers. These well known results for continuity for real functions in terms of sequences suggested to introduce and study new types of continuities such as slowly oscillating continuity [1], quasi-slowly oscillating continuity [2], δ-quasi-slowly oscillating continuity [3], forward continuity [4], statistical ward continuity [5] which enabled some authors to obtain some characterizations of uniform continuity in terms of sequences in the sense that a function preserves either quasi-Cauchy sequences or slowly oscillating sequences (see [6–8]).

The purpose of this paper is to introduce a new kind of continuity and a new type of compactness, namely, N_θ-ward continuity and N_θ-ward compactness, respectively, in the senses that a function f is N_θ-ward continuous if f preserves N_θ-quasi-Cauchy sequences, and a subset A of $\mathbf{R}$ is N_θ-ward compact if any sequence of points in A has an N_θ-quasi-Cauchy subsequence and to investigate relations among this kind of continuity, compactness, and some other types of continuities.

2. Preliminaries

We will use boldface letters $\boldsymbol{\alpha}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for sequences $\boldsymbol{\alpha} = (\alpha_k)$, $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, and $\mathbf{z} = (z_n), \ldots$ of points in $\mathbf{R}$ for the sake of abbreviation. s and c will denote the set of all sequences and the set of convergent sequences of points in $\mathbf{R}$.

A subset of $\mathbf{R}$ is compact if and only if it is closed and bounded. A subset A of $\mathbf{R}$ is bounded if $|a| \leq M$ for all $a \in A$ where M is a positive real constant number. This is equivalent to the statement that any sequence of points in A has a Cauchy subsequence. The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (α_n) of points in $\mathbf{R}$ is quasi-Cauchy if $(\Delta\alpha_n)$ is a null sequence where $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$. These sequences were named as quasi-Cauchy by Burton and Coleman [8, page 328], while they were called as forward convergent to 0 sequences in [4, page 226].

It is known that a sequence (α_n) of points in $\mathbf{R}$ is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim_{n}} \max_{n+1 \leq k \leq [\lambda n]} |\alpha_k - \alpha_n| = 0, \tag{2.1}$$

where $[\lambda n]$ denotes the integer part of λn (see [9, Definition 2 page 947]). Any Cauchy sequence is slowly oscillating, and any slowly oscillating sequence is quasi-Cauchy. There are quasi-Cauchy sequences which are not Cauchy. For example, the sequence $(\sqrt{n})$ is quasi-Cauchy, but not Cauchy. Any subsequence of a Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences, and fails for slowly oscillating sequences as well. A counter example for the case, quasi-Cauchy, is again the sequence $(a_n) = (\sqrt{n})$ with the subsequence $(a_{n^2}) = (n)$. A counter example for the case slowly oscillating is the sequence $(\log_{10} n)$ with the subsequence (n). Furthermore we give more examples without neglecting: the sequences $(\sum_{k=1}^{\infty} 1/n), (\ln n), (\ln(\ln n)), (\ln(\ln(\ln n))), \ldots, (\ln(\ln(\ln(\cdots(\ln n)\cdots))))$ and combinations like that are all slowly oscillating, but not Cauchy. The bounded sequence $(\cos(6\log(n+1)))$ is slowly oscillating, but not Cauchy. The sequences $(\cos(\pi\sqrt{n}))$ and $(\sum_{k=1}^{k=n}(1/k)(\sum_{j=1}^{j=k}(1/j)))$ are quasi-Cauchy, but not slowly oscillating.

By a method of sequential convergence, or briefly a method, we mean a linear function G defined on a subspace of s, denoted by c_G, into $\mathbf{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\boldsymbol{\alpha} \in c_G$ and $G(\boldsymbol{\alpha}) = \ell$ [10]. In particular, lim denotes the limit function $\lim \boldsymbol{\alpha} = \lim_n \alpha_n$ on the space c of convergent sequences of points in $\mathbf{R}$. A method G is called regular if $c \subset c_G$; that is, every convergent sequence $\boldsymbol{\alpha} = (\alpha_n)$ is G-convergent with $G(\boldsymbol{\alpha}) = \lim \boldsymbol{\alpha}$. A point ℓ in $\mathbf{R}$ is in the G-sequential closure of a subset A of $\mathbf{R}$ if there is a sequence $\mathbf{x} = (x_n)$ of points in A such that $G(\mathbf{x}) = \ell$. A subset A is called *G-sequentially closed* if it contains all of the points in its G-sequential closure.

Consider an infinite matrix $\mathtt{A} = (a_{nk})_{n,k=1}^{\infty}$ of real numbers. Then, for any sequence $\mathbf{x} = (x_n)$ the sequence $\mathtt{A}\mathbf{x}$ is defined as

$$\mathtt{A}\mathbf{x} = \left(\sum_{k=1}^{\infty} a_{nk} x_k\right)_n \tag{2.2}$$

provided that each of the series converges. A sequence $\mathbf{x}$ is called $\mathtt{A}$-convergent (or $\mathtt{A}$-summable) to ℓ if $\mathtt{A}\mathbf{x}$ exists and is convergent with

$$\lim \mathtt{A}\mathbf{x} = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}x_k = \ell. \tag{2.3}$$

Then ℓ is called the $\mathtt{A}$-limit of $\mathbf{x}$. We have thus defined a method of sequential convergence, that is, $G(\mathbf{x}) = \lim \mathtt{A}\mathbf{x}$, called a matrix method or a summability matrix.

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence. A sequence (α_k) of points in $\mathbf{R}$ is called statistically convergent to an element ℓ of $\mathbf{R}$ if for each ε

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k \le n : |\alpha_k - \ell| \ge \varepsilon\}\right| = 0, \tag{2.4}$$

and this is denoted by st-$\lim_{k\to\infty}\alpha_k = \ell$ (see [11–15]). This defines a method of sequential convergence, that is, $G(\boldsymbol{\alpha}) :=$ st-$\lim_{k\to\infty}\alpha_k$.

Now we recall the concepts of ward compactness, and slowly oscillating compactness: a subset A of $\mathbf{R}$ is called ward compact if whenever (α_n) is a sequence of points in A, there is a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of (α_n) [4]. A subset A of $\mathbf{R}$ is called slowly oscillating compact if whenever (α_n) is a sequence of points in A, there is a slowly oscillating subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of (α_n) [1].

A function f is called G-sequentially continuous at $u \in \mathbf{R}$ if, given a sequence $\boldsymbol{\alpha} = (\alpha_n)$ of points in $\mathbf{R}$, $G(\boldsymbol{\alpha}) = u$ implies that $G(f(\boldsymbol{\alpha})) = f(u)$.

Recently, Cakalli (see [16, page 594], [17]) gave a sequential definition of compactness, which is a generalization of ordinary sequential compactness, as in the following: a subset A of $\mathbf{R}$ is G-sequentially compact if for any sequence (α_k) of points in A there exists a subsequence $\mathbf{z}$ of the sequence such that $G(\mathbf{z}) \in A$. His idea enables us obtaining new kinds of compactness via most of the nonmatrix sequential convergence methods as well as all matrix sequential convergence methods.

3. N_θ-Quasi-Cauchy Sequences

A lacunary sequence $\theta = (k_r)$ is an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r : k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r. Sums of the form $\sum_{k_{r-1}+1}^{k_r}|\alpha_k|$ frequently occur, and will often be written for convenience as $\sum_{k\in I_r}|\alpha_k|$. Throughout this paper, we will assume that $\liminf_r q_r > 1$.

The notion of N_θ convergence was introduced and studied by Freedman et al. in [18]. Basarir and Altundag studied Δ-N_θ-asymptotically equivalent sequences in [19]. Using the idea of Sember and Raphael, Fridy and Orhan introduced lacunary statistical convergence (see [20, 21]).

A sequence (α_k) of points in $\mathbf{R}$ is called N_θ-convergent to an element ℓ of $\mathbf{R}$ if

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\alpha_k - \ell| = 0, \tag{3.1}$$

and it is denoted by N_θ-$\lim \alpha_k = \ell$. This defines a method of sequential convergence, that is, $G(\boldsymbol{\alpha}) := N_\theta$-$\lim \alpha_k$. Any convergent sequence is N_θ-convergent, but the converse is not always true. Throughout the paper N_θ will denote the set of N_θ convergent sequences of points in $\mathbf{R}$.

For example, limit of the sequence of the ratios of Fibonacci numbers converge, to the golden mean. This property ensures the regularity of lacunary sequential method obtained via the sequence of Fibonacci numbers; that is, $\theta = (k_r)$ is the lacunary sequence defined by writing $k_0 = 0$ and $k_r = F_{r+2}$ where (F_r) is the Fibonacci sequence, that is, $F_1 = 1$, $F_2 = 1$, and $F_r = F_{r-1} + F_{r-2}$ for $r \geq 3$.

Now we modify the definition of G-sequential compactness to the special case, $G = N_\theta$ [16] as in the following: a subset A, of $\mathbf{R}$ is called N_θ-sequentially compact if whenever (α_n) is a sequence of points in A there is an N_θ-convergent subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of (α_n) whose N_θ-limit is in A.

Adopting the technique in the proof of the necessity of Theorem 6 in [22], we see that the sequential method N_θ is subsequential. It follows from [16, Corollary 5, page 597] that a subset A of $\mathbf{R}$ is sequentially compact if and only if it is N_θ-sequentially compact. A subset A of $\mathbf{R}$ is closed and bounded if and only if it is N_θ-sequentially compact. A subset of $\mathbf{R}$ is G-sequentially compact if and only if it is N_θ-sequentially compact for any regular subsequential method G.

In connection with N_θ-convergent sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbf{R}$:

$$\begin{aligned} (N_\theta) &: (\alpha_n) \in N_\theta \Longrightarrow (f(\alpha_n)) \in N_\theta, \\ (N_\theta c) &: (\alpha_n) \in N_\theta \Longrightarrow (f(\alpha_n)) \in c, \\ (c) &: (\alpha_n) \in c \Longrightarrow (f(\alpha_n)) \in c, \\ (cN_\theta) &: (\alpha_n) \in c \Longrightarrow (f(\alpha_n)) \in N_\theta. \end{aligned} \tag{3.2}$$

We see that (N_θ) is N_θ-sequential continuity of f, and (c) is the ordinary continuity of f. It is easy to see that $(N_\theta c)$ implies (N_θ), and (N_θ) does not imply $(N_\theta c)$; (N_θ) implies (cN_θ), and (cN_θ) does not imply (N_θ); $(N_\theta c)$ implies (c), and (c) does not imply $(N_\theta c)$; and (c) is equivalent to (cN_θ).

If a function f is N_θ-sequentially continuous at a point α_0, then it is continuous at α_0. If a function f is N_θ-sequentially continuous on a subset A of $\mathbf{R}$, then it is statistically continuous on A. We obtain from [16, Theorem 7, page 597] that N_θ-sequentially continuous image of any N_θ-sequentially compact subset of $\mathbf{R}$ is N_θ-sequentially compact.

In [23] a nonempty subset A of a $\mathbf{R}$ is called *G-sequentially connected* if there are no nonempty and disjoint G-sequentially closed subsets U and V such that $A \subseteq U \bigcup V$, and $A \cap U$ and $A \cap V$ are nonempty. As far as G-sequentially connectedness is considered, we see that N_θ-sequentially continuous image of any N_θ-sequentially connected subset of $\mathbf{R}$ is N_θ-sequentially connected, so N_θ-sequentially continuous image of any interval is an interval. Furthermore it can be easily seen that a subset of $\mathbf{R}$ is N_θ-sequentially connected if and only if it is connected in the ordinary sense, and so it is an interval.

Definition 3.1. A sequence (α_n) of points in $\mathbf{R}$ is called N_θ-quasi-Cauchy if $(\Delta\alpha_n)$ is N_θ-convergent to 0. ΔN_θ^0 will denote the set of all N_θ-quasi-Cauchy sequences of points in $\mathbf{R}$.

We note that N_θ-quasi-Cauchy sequences were studied in [19] in a different point of view.

Now we give the definition of N_θ-ward compactness.

Definition 3.2. A subset A of $\mathbf{R}$ is called N_θ-ward compact if whenever (α_n) is a sequence of points in A, there is an N_θ-quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (a_{n_k})$ of (α_n).

Theorem 3.3. *A subset A of $\mathbf{R}$ is bounded if and only if it is N_θ-ward compact.*

Proof. Let A be any bounded subset of $\mathbf{R}$ and let (α_n) be any sequence of points in A. (α_n) is also a sequence of points in $\overline{A}$ where $\overline{A}$ denotes the closure of A. As $\overline{A}$ is sequentially compact, there is a convergent subsequence (α_{n_k}) of (α_n) (no matter the limit is in A or not). This subsequence is N_θ-convergent since N_θ-method is regular. Hence (α_{n_k}) is N_θ-quasi-Cauchy. Thus (a) implies (b). To prove that (b) implies (a), suppose that A is unbounded. If it is unbounded above, then one can construct a sequence (α_n) of numbers in A such that $\alpha_{n+1} > 1+\alpha_n$ for each positive integer n. Then the sequence (α_n) does not have any N_θ-quasi-Cauchy subsequence, so A is not N_θ-ward compact. If A is bounded above and unbounded below, then similarly we obtain that A is not N_θ-ward compact. This completes the proof of the theorem. □

It easily follows from the preceding theorem that a closed subset of $\mathbf{R}$ is N_θ-ward compact if and only if it is N_θ-sequentially compact and a closed subset of $\mathbf{R}$ is N_θ-ward compact if and only if it is statistically ward compact.

A sequence $\alpha = (\alpha_n)$ is δ-quasi-Cauchy if $\lim_{k\to\infty}\Delta^2\alpha_n = 0$ where $\Delta^2\alpha_n = a_{n+2}-2a_{n+1}+\alpha_n$ [3]. A subset A of $\mathbf{R}$ is called δ-ward compact if whenever $\boldsymbol{\alpha} = (\alpha_n)$ is a sequence of points in A, there is a subsequence $\mathbf{z} = (z_k) = (\alpha_{n_k})$ of $\boldsymbol{\alpha}$ with $\lim_{k\to\infty}\Delta^2 z_k = 0$. It follows from the previous theorem that any N_θ-ward compact subset of $\mathbf{R}$ is δ-ward compact.

We see that for any regular subsequential method G defined on $\mathbf{R}$, if a subset A of $\mathbf{R}$ is G-sequentially compact, then it is N_θ-ward compact. But the converse is not always true.

Now we give the definition of N_θ-ward continuity in the following.

Definition 3.4. A function defined on a subset A of $\mathbf{R}$ is called N_θ-ward continuous if it preserves N_θ-quasi-Cauchy sequences; that is, $(f(\alpha_k))$ is an N_θ-quasi-Cauchy sequence whenever (α_k) is.

Sum of two N_θ-ward continuous functions is N_θ-ward continuous, but product of N_θ-ward continuous functions need not be N_θ-ward continuous.

In connection with N_θ-quasi-Cauchy sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbf{R}$:

$$\begin{aligned}
(\delta N_\theta) &: (\alpha_n) \in \Delta N_\theta^0 \Longrightarrow \left(f(\alpha_n)\right) \in \Delta N_\theta^0, \\
(\delta N_\theta c) &: (\alpha_n) \in \Delta N_\theta^0 \Longrightarrow \left(f(\alpha_n)\right) \in c, \\
(c) &: (\alpha_n) \in c \Longrightarrow \left(f(\alpha_n)\right) \in c, \\
(c\delta N_\theta) &: (\alpha_n) \in c \Longrightarrow \left(f(\alpha_n)\right) \in \Delta N_\theta^0, \\
(N_\theta) &: (\alpha_n) \in N_\theta \Longrightarrow \left(f(\alpha_n)\right) \in N_\theta^0.
\end{aligned} \tag{3.3}$$

We see that (δN_θ) is N_θ-ward continuity of f, (N_θ) is N_θ-sequential continuity of f, and (c) is the ordinary continuity of f. It is easy to see that $(\delta N_\theta c)$ implies (δN_θ), and (δN_θ) does not imply $(\delta N_\theta c)$; (δN_θ) implies $(c\delta N_\theta)$, and $(c\delta N_\theta)$ does not imply (δN_θ); $(\delta N_\theta c)$ implies (c), and (c) does not imply $(\delta N_\theta c)$; (N_θ) clearly implies (c) as we have seen in Section 3.

Now we give the implication that (δN_θ) implies (N_θ); that is, any N_θ-ward continuous function is N_θ-sequentially continuous.

Theorem 3.5. *If f is N_θ-ward continuous on a subset A of $\mathbf{R}$, then it is N_θ-sequentially continuous on A.*

Proof. Assume that f is an N_θ-ward continuous function on a subset A of $\mathbf{R}$. Let (α_n) be any N_θ-convergent sequence with $N_\theta - \lim_{k\to\infty}\alpha_k = \alpha_0$. Then the sequence

$$(\alpha_1, \alpha_0, \alpha_2, \alpha_0, \ldots, \alpha_{n-1}, \alpha_0, \alpha_n, \alpha_0, \ldots) \tag{3.4}$$

is also N_θ-convergent to α_0. Hence it is N_θ-quasi-Cauchy. As f is N_θ-ward continuous, the sequence

$$(f(\alpha_1), f(\alpha_0), f(\alpha_2), f(\alpha_0), \ldots, f(\alpha_{n-1}), f(\alpha_0), f(\alpha_n), f(\alpha_0), \ldots) \tag{3.5}$$

is N_θ-quasi-Cauchy. It follows from this that the sequence $(f(\alpha_n))$ N_θ-converges to $f(\alpha_0)$. This completes the proof of the theorem.

The converse is not always true for the function $f(x) = x^2$ is an example since the sequence $(\sqrt{n})$ is N_θ-quasi-Cauchy while $(f(\sqrt{n})) = (n)$ is not. □

Corollary 3.6. *If f is N_θ-ward continuous on a subset A of $\mathbf{R}$, then it is continuous on A.*

Proof. The proof immediately follows from the preceding theorem, so it is omitted. □

Corollary 3.7. *If f is N_θ-ward continuous on a subset A of $\mathbf{R}$, then it is statistically continuous on A.*

It is well known that any continuous function on a compact subset A of $\mathbf{R}$ is uniformly continuous on A. It is also true for a regular subsequential method G that any N_θ-ward continuous function on a G-sequentially compact subset A of $\mathbf{R}$ is also uniformly continuous on A (see [6]). Furthermore, for N_θ-ward continuous functions defined on an N_θ-ward compact subset of $\mathbf{R}$, we have the following.

Theorem 3.8. *Let A be an N_θ-ward compact subset A of $\mathbf{R}$ and let $f : A \to \mathbf{R}$ be an N_θ-ward continuous function on A. Then f is uniformly continuous on A.*

Proof. Suppose that f is not uniformly continuous on A so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there are $x, y \in E$ with $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon_0$. For each positive integer n, there exist α_n and β_n such that $|\alpha_n - \beta_n| < 1/n$, and $|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0$. Since A is N_θ-ward compact, there exists an N_θ-quasi-Cauchy subsequence (α_{n_k}) of the sequence (α_n). It is clear that the corresponding subsequence (β_{n_k}) of the sequence (β_n) is also N_θ-quasi-Cauchy, since $(\beta_{n_{k+1}} - \beta_{n_k})$ is a sum of three N_θ-null sequences, that is,

$$\beta_{n_{k+1}} - \beta_{n_k} = \left(\beta_{n_{k+1}} - \alpha_{n_{k+1}}\right) + \left(\alpha_{n_{k+1}} - \alpha_{n_k}\right) + \left(\alpha_{n_k} - \beta_{n_k}\right). \tag{3.6}$$

On the other hand, it follows from the equality $\alpha_{n_{k+1}} - \beta_{n_k} = \alpha_{n_{k+1}} - \alpha_{n_k} + \alpha_{n_k} - \beta_{n_k}$ that the sequence $(\alpha_{n_{k+1}} - \beta_{n_k})$ is N_θ-convergent to 0. Hence the sequence

$$\left(a_{n_1}, \beta_{n_1}, \alpha_{n_2}, \beta_{n_2}, \alpha_{n_3}, \beta_{n_3}, \ldots, \alpha_{n_k}, \beta_{n_k}, \ldots\right) \tag{3.7}$$

is N_θ-quasi-Cauchy. But the transformed sequence

$$\left(f(\alpha_{n_1}), f\left(\beta_{n_1}\right), f(\alpha_{n_2}), f\left(\beta_{n_2}\right), f(\alpha_{n_3}), f\left(\beta_{n_3}\right), \ldots, f(\alpha_{n_k}), f\left(\beta_{n_k}\right), \ldots\right) \tag{3.8}$$

is not N_θ-quasi-Cauchy. Thus f does not preserve N_θ-quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Corollary 3.9. *If a function f is N_θ-ward continuous on a bounded subset A of* **R**, *then it is uniformly continuous on A.*

Proof. The proof follows from the preceding theorem and Theorem 3.3. □

Theorem 3.10. *N_θ-ward continuous image of any N_θ-ward compact subset of* **R** *is N_θ-ward compact.*

Proof. Assume that f is an N_θ-ward continuous function on a subset A of **R** and E is an N_θ-ward compact subset of A. Let (β_n) be any sequence of points in $f(E)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in E$ for each positive integer n. N_θ-ward compactness of E implies that there is a subsequence $(\gamma_k) = (\alpha_{n_k})$ of (α_n) with $N_\theta \lim_{k\to\infty} \Delta\gamma_k = 0$. Write $(t_k) = (f(\gamma_k))$. As f is N_θ-ward continuous, $(f(\gamma_k))$ is N_θ-quasi-Cauchy. Thus we have obtained a subsequence (t_k) of the sequence $(f(\alpha_n))$ with $N_\theta - \lim_{k\to\infty} \Delta t_k = 0$. Thus $f(E)$ is N_θ-ward compact. This completes the proof of the theorem. □

Corollary 3.11. *N_θ-ward continuous image of any compact subset of* **R** *is N_θ-ward compact.*

The proof follows from the preceding theorem.

Corollary 3.12. *N_θ-ward continuous image of any bounded subset of* **R** *is bounded.*

The proof follows from Theorems 3.3 and 3.10.

Corollary 3.13. *N_θ-ward continuous image of a G-sequentially compact subset of* **R** *is N_θ-ward compact for any regular subsequential method G.*

For a further study, we suggest to investigate N_θ-quasi-Cauchy sequences of fuzzy points and N_θ-ward continuity for the fuzzy functions (see [24] for the definitions and related concepts in fuzzy setting). However due to the change in settings, the definitions and methods of proofs will not always be analogous to those of the present work.

References

[1] H. Cakalli, "Slowly oscillating continuity," *Abstract and Applied Analysis*, vol. 2008, Article ID 485706, 5 pages, 2008.

[2] İ. Çanak and M. Dik, "New types of continuities," *Abstract and Applied Analysis*, vol. 2010, Article ID 258980, 6 pages, 2010.

[3] H. Cakalli, "δ-quasi-Cauchy sequences," *Mathematical and Computer Modelling*, vol. 53, no. 1-2, pp. 397–401, 2011.

[4] H. Cakalli, "Forward continuity," *Journal of Computational Analysis and Applications*, vol. 13, no. 2, pp. 225–230, 2011.

[5] H. Cakalli, "Statistical ward continuity," *Applied Mathematics Letters*, vol. 24, no. 10, pp. 1724–1728, 2011.

[6] H. Cakalli, "Statistical quasi-Cauchy sequences," *Mathematical and Computer Modelling*, vol. 54, no. 5-6, pp. 1620–1624, 2011.

[7] R. W. Vallin, "Creating slowly oscillating sequences and slowly oscillating continuous functions," *Acta Mathematica Universitatis Comenianae*, vol. 80, no. 1, pp. 71–78, 2011.

[8] D. Burton and J. Coleman, "Quasi-Cauchy sequences," *American Mathematical Monthly*, vol. 117, no. 4, pp. 328–333, 2010.

[9] F. Dik, M. Dik, and İ. Çanak, "Applications of subsequential Tauberian theory to classical Tauberian theory," *Applied Mathematics Letters*, vol. 20, no. 8, pp. 946–950, 2007.

[10] J. Connor and K.-G. Grosse-Erdmann, "Sequential definitions of continuity for real functions," *The Rocky Mountain Journal of Mathematics*, vol. 33, no. 1, pp. 93–121, 2003.

[11] J. A. Fridy, "On statistical convergence," *Analysis*, vol. 5, no. 4, pp. 301–313, 1985.

[12] G. Di Maio and L. D. R. Kočinac, "Statistical convergence in topology," *Topology and Its Applications*, vol. 156, no. 1, pp. 28–45, 2008.

[13] H. Cakalli, "A study on statistical convergence," *Functional Analysis, Approximation and Computation*, vol. 1, no. 2, pp. 19–24, 2009.

[14] A. Caserta, G. Di Maio, and L. D. R. Kočinac, "Statistical convergence in function spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 420419, 11 pages, 2011.

[15] A. Caserta and L. D. R. Kočinac, "On statistical exhaustiveness," *Applied Mathematics Letters*. In press.

[16] H. Cakalli, "Sequential definitions of compactness," *Applied Mathematics Letters*, vol. 21, no. 6, pp. 594–598, 2008.

[17] H. Cakalli, "On G-continuity," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 313–318, 2011.

[18] A. R. Freedman, J. J. Sember, and M. Raphael, "Some Cesàro-type summability spaces," *Proceedings of the London Mathematical Society*, vol. 37, no. 3, pp. 508–520, 1978.

[19] M. Basarir and S. Altunda, "On Δ-N_θ-lacunary statistical asymptotically equivalent sequences," *Univerzitet u Nišu. Prirodno-Matematički Fakultet*, vol. 22, no. 1, pp. 161–172, 2008.

[20] J. A. Fridy and C. Orhan, "Lacunary statistical convergence," *Pacific Journal of Mathematics*, vol. 160, no. 1, pp. 43–51, 1993.

[21] J. A. Fridy and C. Orhan, "Lacunary statistical summability," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 497–504, 1993.

[22] H. Cakalli, "Lacunary statistical convergence in topological groups," *Indian Journal of Pure and Applied Mathematics*, vol. 26, no. 2, pp. 113–119, 1995.

[23] H. Cakalli, "Sequential definitions of connectedness," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 461–465, 2012.

[24] H. Cakalli and P. Das, "Fuzzy compactness via summability," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1665–1669, 2009.