

## Research Article

# General Solutions of Two Quadratic Functional Equations of Pexider Type on Orthogonal Vectors

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Based on the studies on the Hyers-Ulam stability and the orthogonal stability of some Pexider-quadratic functional equations, in this paper we find the general solutions of two quadratic functional equations of Pexider type. Both equations are studied in restricted domains: the first equation is studied on the restricted domain of the orthogonal vectors in the sense of Rätz, and the second equation is considered on the orthogonal vectors in the inner product spaces with the usual orthogonality.

## 1. Introduction

Stability problems for some functional equations have been extensively investigated by several authors, and in particular one of the most important functional equation studied in this topic is the quadratic functional equation,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.1)$$

(Skof [1], Cholewa [2], Czerwik [3], Rassias [4], among others).

Recently, many articles have been devoted to the study of the stability or orthogonal stability of quadratic functional equations of Pexider type on the restricted domain of orthogonal vectors in the sense of Rätz.

We remind the definition of orthogonality space (see [5]). The pair  $(X, \perp)$  is called orthogonality space in the sense of Rätz if  $X$  is a real vector space with  $\dim X \geq 2$  and  $\perp$  is a binary relation on  $X$  with the following properties:

- (i)  $x \perp 0, 0 \perp x$  for all  $x \in X$ ,
- (ii) if  $x, y \in X - \{0\}$ ,  $x \perp y$ , then the vectors are linearly independent,

- (iii) if  $x, y \in X$ ,  $x \perp y$ , then  $\alpha x \perp \beta y$  for all  $\alpha, \beta \in R$ ,
- (iv) let  $P$  be a 2-dimensional subspace of  $X$ . If  $x \in P$  then there exists  $y_0 \in P$  such that  $x \perp y_0$  and  $x + y_0 \perp \lambda x - y_0$ .

The relation  $\perp$  is called symmetric if  $x \perp y$  implies that  $y \perp x$ .

An example of orthogonality in the sense of Rätz is the ordinary orthogonality on an inner product space  $(H, (\cdot, \cdot))$  given by  $x \perp y \Leftrightarrow (x, y) = 0$ .

In the class of real functionals  $f, g, h$  defined on an orthogonality space in the sense of Rätz,  $f, g, h : (X, \perp) \rightarrow R$ , a first version of the quadratic equation of Pexider type is

$$f(x + y) + f(x - y) = 2g(x) + 2h(y) \quad (1.2)$$

and its relative conditional form is

$$x \perp y \implies f(x + y) + f(x - y) = 2g(x) + 2h(y). \quad (1.3)$$

Although the Hyers-Ulam stability of the conditional quadratic functional equation (1.3) has been studied by Moslehian [6], we do not know the characterization of the solutions of the conditional equation (1.3).

In the same class of functions,  $f, g, h, k : (X, \perp) \rightarrow R$ , another version of the quadratic functional equation of Pexider type is

$$f(x + y) + g(x - y) = h(x) + k(y), \quad (1.4)$$

and its relative conditional form is

$$x \perp y \implies f(x + y) + g(x - y) = h(x) + k(y). \quad (1.5)$$

Equation (1.4) has been solved by Ebanks et al. [7]; its stability has been studied, among others, by Jung and Sahoo [8] and Yang [9] and its orthogonal stability has been studied by Mirzavaziri and Moslehian [10], but also in this case we do not know the general solutions of (1.5).

Based on those studies, we intend to consider the above-mentioned functional equations (1.3) and (1.5) on the restricted domain of orthogonal vectors in order to present the characterization of their general solutions.

Throughout the paper, the orthogonality  $\perp$  in the sense of Rätz is assumed to be symmetric.

## 2. The Conditional Equation $x \perp y \implies f(x + y) + f(x - y) = 2g(x) + 2h(y)$ in Orthogonality Spaces in the Sense of Rätz

In the class of real functionals  $f, g, h$  defined on an orthogonality space in the sense of Rätz,  $f, g, h : (X, \perp) \rightarrow R$ , let us consider the conditional equation (1.3).

We describe its solutions first assuming that  $f$  is an odd functional, then an even functional, finally, using the decomposition of the functionals  $f, g, h$  into their even and odd parts, we describe the general solutions.

**Theorem 2.1.** *Let  $f, g, h : (X, \perp) \rightarrow R$  be real functionals satisfying (1.3).*

*If  $f$  is an odd functional, then the solutions of (1.3) are given by*

$$\begin{aligned} f(x) &= A(x), \\ g(x) &= A(x) + g(0), \\ h(x) &= h(0), \end{aligned} \tag{2.1}$$

where  $A : (X, \perp) \rightarrow R$  is an additive function, that is,  $A$  is solution of  $A(x + y) = A(x) + A(y)$  for all  $(x, y) \in X^2$ .

*If  $f$  is an even functional, then the solutions of (1.3) are given by*

$$\begin{aligned} f(x) &= Q(x) + f(0), \\ g(x) &= Q(x) + g(0), \\ h(x) &= Q(x) + h(0), \end{aligned} \tag{2.2}$$

where  $Q : (X, \perp) \rightarrow R$  is an orthogonally quadratic function, that is, solution of  $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$  for  $x \perp y$ .

*Proof.* Let us first consider  $f$  an odd functional. Letting  $x = 0$  and  $y = 0$  in (1.3), by  $f(0) = 0$  for the oddness of  $f$ , we obtain

$$g(0) + h(0) = 0. \tag{2.3}$$

Now, putting  $(x, 0)$  in place of  $(x, y)$  in (1.3), we have  $f(x) = g(x) + h(0)$ , then putting again  $(0, x)$  in place of  $(x, y)$  we get  $g(0) + h(x) = 0$  for all  $x \in X$ , since  $f$  is odd. The first equation gives

$$g(x) = f(x) + g(0) \tag{2.4}$$

from (2.3), and the last equation proves that

$$h(x) = h(0) \tag{2.5}$$

using (2.3) again.

From the above results, (1.3) may be rewritten in the following way:  $f(x+y) + f(x-y) = 2f(x)$  for all  $x \perp y$ . Hence by Lemma 3.1, [6], we have  $f(x) - f(0) = A(x)$  where  $A : X \rightarrow R$  is an orthogonally additive functional. But since  $f(0) = 0$  and from [5, Theorem 5], we deduce that  $A$  is everywhere additive.

Consider now  $f$  an even functional. Substituting in (1.3)  $(0, 0)$  in place of  $(x, y)$ , we obtain

$$g(0) + h(0) = f(0). \quad (2.6)$$

Now writing (1.3) with  $(x, y)$  replaced, respectively, first by  $(x, 0)$ , then by  $(0, y)$ , we get

$$f(x) = g(x) + h(0), \quad (2.7)$$

$$f(y) = g(0) + h(y), \quad (2.8)$$

for all  $x, y \in X$ , since  $f$  is even. From (1.3), using (2.7), (2.8), and (2.6), we obtain

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) - 2f(0). \quad (2.9)$$

Hence, setting  $Q(t) = f(t) - f(0)$ , we infer  $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$  for  $x \perp y$ , that is,  $Q$  is an orthogonally quadratic functional. So,  $f(x) = Q(x) + f(0)$ , and from (2.7), using (2.6),  $g(x) = Q(x) + f(0) - h(0) = Q(x) + g(0)$ , and from (2.8),  $h(x) = Q(x) + f(0) - g(0) = Q(x) + h(0)$ .

The theorem is so proved.  $\square$

**Lemma 2.2.** *Let  $f, g, h : (X, \perp) \rightarrow R$  be real functionals satisfying (1.3).*

*Then both the even parts and the odd parts of  $f, g, h$ , namely,  $f_e, g_e, h_e$  and  $f_o, g_o, h_o$ , satisfy (1.3).*

*Proof.* Denoting by  $f_e, g_e, h_e$  and  $f_o, g_o, h_o$  the even and odd parts, respectively, of  $f, g, h$ , we have from (1.3)

$$f_e(x + y) + f_o(x + y) + f_e(x - y) + f_o(x - y) = 2g_e(x) + 2g_o(x) + 2h_e(y) + 2h_o(y), \quad \text{for } x \perp y. \quad (2.10)$$

From the homogeneity of the orthogonality relation (property (iii)), we have  $x \perp y \Rightarrow -x \perp -y$ , so that, by (1.3), choosing  $-x, -y$ , we get

$$f_e(x + y) - f_o(x + y) + f_e(x - y) - f_o(x - y) = 2g_e(x) - 2g_o(x) + 2h_e(y) - 2h_o(y), \quad \text{for } x \perp y. \quad (2.11)$$

Adding and then subtracting (2.10) and (2.11), we easily prove the lemma.

From Lemma 2.2 and Theorem 2.1, we may easily prove the following theorem.  $\square$

**Theorem 2.3.** *The general solution  $f, g, h : (X, \perp) \rightarrow R$  of the functional equation (1.3) is given by*

$$\begin{aligned} f(x) &= A(x) + Q(x) + f(0), \\ g(x) &= A(x) + Q(x) + g(0), \\ h(x) &= Q(x) + h(0), \end{aligned} \quad (2.12)$$

where  $A : (X, \perp) \rightarrow R$  is an additive function and  $Q : (X, \perp) \rightarrow R$  is an orthogonally quadratic function.

In the case of an inner product space  $(H, (\cdot, \cdot))$  ( $\dim H > 2$ ) which is a particular orthogonality space in the sense of Rätz, with the ordinary orthogonality given by  $x \perp y \Leftrightarrow (x, y) = 0$ , we have the characterization of the orthogonally quadratic mappings from [11, Theorem 2]. Hence we have the following corollary.

**Corollary 2.4.** *Let  $H$  be an inner product space with  $\dim H > 2$  and  $f, g, h : (H, (\cdot, \cdot)) \rightarrow R$ . The general solution of the functional equation (1.3) is given by*

$$\begin{aligned} f(x) &= A(x) + Q(x) + f(0), \\ g(x) &= A(x) + Q(x) + g(0), \\ h(x) &= Q(x) + h(0), \end{aligned} \tag{2.13}$$

where  $A : (H, (\cdot, \cdot)) \rightarrow R$  is an additive function and  $Q : (H, (\cdot, \cdot)) \rightarrow R$  is a quadratic function.

### 3. The Conditional Equation $x \perp y \Rightarrow f(x + y) + g(x - y) = h(x) + k(y)$ in Inner Product Spaces

Consider now  $H$  an inner product space with  $\dim H > 2$  and the usual orthogonality given by  $x \perp y \Leftrightarrow (x, y) = 0$ . In the class of real functionals  $f, g, h, k$  defined on  $H$ , we consider the conditional equation (1.5).

First prove the following lemma.

**Lemma 3.1.** *Let  $f, g, h, k : H \rightarrow R$  be solutions of (1.5); then*

$$h(x) = A(x) + Q(x) + h(0), \tag{3.1}$$

where  $A : H \rightarrow R$  is an additive function and  $Q : H \rightarrow R$  is a quadratic function.

*Proof.* Replacing in (1.5)  $(x, y)$  by  $(0, 0)$ , then by  $(x, 0)$  and finally by  $(0, y)$ , we obtain

$$\begin{aligned} \text{(i)} \quad & f(0) + g(0) = h(0) + k(0), \\ \text{(ii)} \quad & f(x) + g(x) = h(x) + k(0), \\ \text{(iii)} \quad & f(y) + g(-y) = h(0) + k(y). \end{aligned}$$

Hence (1.5) may be rewritten as

$$f(x + y) + g(x - y) = f(x) + f(y) + g(x) + g(-y) - f(0) - g(0). \tag{3.2}$$

So that, setting  $F(t) = f(t) - f(0)$  and  $G(t) = g(t) - g(0)$ , we infer

$$F(x + y) + G(x - y) = F(x) + F(y) + G(x) + G(-y). \tag{3.3}$$

Now, substituting  $-y$  in (3.3) in place of  $y$ , we have

$$F(x - y) + G(x + y) = F(x) + F(-y) + G(x) + G(y). \quad (3.4)$$

Adding (3.3) and (3.4), we get

$$F(x + y) + F(x - y) + G(x + y) + G(x - y) = 2F(x) + F(y) + F(-y) + 2G(x) + G(y) + G(-y). \quad (3.5)$$

So, defining the functional  $S : H \rightarrow R$  by

$$S(t) = F(t) + G(t), \quad (3.6)$$

the above equation becomes

$$x \perp y \implies S(x + y) + S(x - y) = 2S(x) + S(y) + S(-y). \quad (3.7)$$

From [11, Theorem 3], we have

$$S(x) = A(x) + Q(x), \quad (3.8)$$

where  $A : H \rightarrow R$  is an additive function and  $Q : H \rightarrow R$  is a quadratic function. From (3.6), we have,  $F(x) + G(x) = A(x) + Q(x)$ , that is,  $f(x) - f(0) + g(x) - g(0) = A(x) + Q(x)$ . Using (ii) and (i), the left-hand side of the above equation may be written in the following way:  $h(x) + k(0) - f(0) - g(0) = h(x) + k(0) - h(0) - k(0) = h(x) - h(0)$ ; hence we get  $h(x) = A(x) + Q(x) + h(0)$ . The theorem is so proved.  $\square$

Our aim is now to characterize the general solutions of (1.5): this is obtained using the decomposition of the functionals  $f, g, h, k$  into their even and odd parts. Using the same approach of Lemma 2.2, we easily prove the following lemma.

**Lemma 3.2.** *Let  $f, g, h, k : H \rightarrow R$  be real functionals satisfying (1.5).*

*Then both the even parts and the odd parts of  $f, g, h, k$ , namely,  $f_e, g_e, h_e, k_e$  and  $f_o, g_o, h_o, k_o$ , satisfy (1.5), that is,*

$$x \perp y \implies f_o(x + y) + g_o(x - y) = h_o(x) + k_o(y), \quad (3.9)$$

$$x \perp y \implies f_e(x + y) + g_e(x - y) = h_e(x) + k_e(y). \quad (3.10)$$

Now consider (3.9): the characterization of its solutions is given by the following theorem.

**Theorem 3.3.** Let  $f_o, g_o, h_o, k_o : H \rightarrow R$  be real odd functionals satisfying (3.9); then the solutions of (3.9) are given by

$$\begin{aligned} f_o(x) &= \frac{A(x) + B(x)}{2}, \\ g_o(x) &= \frac{A(x) - B(x)}{2}, \\ h_o(x) &= A(x), \\ k_o(x) &= B(x), \end{aligned} \tag{3.11}$$

where  $A : H \rightarrow R$  and  $B : H \rightarrow R$  are additive functions.

*Proof.* Substituting in (3.9) first  $(0, x)$ , then  $(x, 0)$  in place of  $(x, y)$ , and by  $h_o(0) = 0$  and  $k_o(0) = 0$  by the oddness of the functions, we obtain

$$\begin{aligned} f_o(x) - g_o(x) &= k_o(x), \\ f_o(x) + g_o(x) &= h_o(x). \end{aligned} \tag{3.12}$$

Adding and then subtracting the above equations, we get

$$\begin{aligned} 2f_o(x) &= h_o(x) + k_o(x), \\ 2g_o(x) &= h_o(x) - k_o(x). \end{aligned} \tag{3.13}$$

By (3.1),  $h_o(x) = A(x)$ , hence from the above equations,

$$2f_o(x) = A(x) + k_o(x), \tag{3.14}$$

$$2g_o(x) = A(x) - k_o(x). \tag{3.15}$$

Consider now  $x, y \in H$  with  $x \perp y$ . Writing (3.14) with  $x + y$  instead of  $x$  and (3.15) with  $x - y$  instead of  $x$ , we get

$$\begin{aligned} 2f_o(x + y) &= A(x + y) + k_o(x + y), \\ 2g_o(x - y) &= A(x - y) - k_o(x - y). \end{aligned} \tag{3.16}$$

Adding the above equations, from (3.9), the additivity of  $A$  and  $h_o(x) = A(x)$ , we obtain

$$k_o(x + y) - k_o(x - y) = 2k_o(y) \tag{3.17}$$

for  $x \perp y$ . By the symmetry of the orthogonality relation, we get, changing  $x$  and  $y$  and from the oddness of the function,

$$k_o(x + y) + k_o(x - y) = 2k_o(x), \tag{3.18}$$

hence  $k_0(x + y) = k_0(x) + k_0(y)$  for  $x \perp y$ . By [5, Theorem 5],  $k_0$  is an additive function; consequently, there exists an additive function  $B : H \rightarrow R$  such that  $k_0(x) = B(x)$  for all  $x \in H$ . Now (3.14) and (3.15) give  $f_o(x) = (A(x) + B(x))/2$  and  $g_o(x) = (A(x) - B(x))/2$ , so the theorem is proved.  $\square$

Finally, consider equation (3.10): the characterization of its solutions is given by the following theorem.

**Theorem 3.4.** *Let  $f_e, g_e, h_e, k_e : H \rightarrow R$  be real even functionals satisfying (3.10); then there exist a quadratic function  $Q : H \rightarrow R$  and a function  $\varphi : [0, \infty) \rightarrow R$  such that*

$$\begin{aligned} f_e(x) &= \frac{Q(x) + \varphi(\|x\|) + h_e(0) + k_e(0)}{2}, \\ g_e(x) &= \frac{Q(x) - \varphi(\|x\|) + h_e(0) + k_e(0)}{2}, \\ h_e(x) &= Q(x) + h_e(0), \\ k_e(x) &= Q(x) + k_e(0). \end{aligned} \tag{3.19}$$

*Proof.* From Lemma 3.1, we first notice that

$$h_e(x) = Q(x) + h_e(0). \tag{3.20}$$

Substituting now in (3.10) first  $(x, 0)$  then  $(0, x)$  instead of  $(x, y)$ , we obtain, respectively

$$\begin{aligned} f_e(x) + g_e(x) &= h_e(x) + k_e(0), \\ f_e(x) + g_e(x) &= h_e(0) + k_e(x). \end{aligned} \tag{3.21}$$

Consequently, by subtraction and from (3.20), we have

$$k_e(x) = Q(x) + k_e(0). \tag{3.22}$$

Substitution of (3.20) and (3.22) in (3.10) gives

$$f_e(x + y) + g_e(x - y) = Q(x) + Q(y) + h_e(0) + k_e(0). \tag{3.23}$$

Then, we substitute  $-y$  in place of  $y$  in (3.23) and have

$$f_e(x - y) + g_e(x + y) = Q(x) + Q(y) + h_e(0) + k_e(0) \tag{3.24}$$

for all  $x \perp y$ . Hence, for  $y = 0$  in (3.24), we obtain

$$f_e(x) + g_e(x) = Q(x) + h_e(0) + k_e(0). \tag{3.25}$$



Subtracting now (3.23) and (3.24), we get  $f_e(x + y) + g_e(x - y) - f_e(x - y) - g_e(x + y) = 0$  for all  $x \perp y$ . Consider  $u, v \in H$  with  $\|u\| = \|v\|$ : it follows that  $(u + v)/2 \perp (u - v)/2$ , hence in the above equation we may replace  $x, y$  with  $(u + v)/2, (u - v)/2$ , respectively. We obtain  $f_e(u) + g_e(v) - f_e(v) - g_e(u) = 0$ , that is,  $f_e(u) - g_e(u) = f_e(v) - g_e(v)$  for all  $u, v \in H$  with  $\|u\| = \|v\|$ . Thus the function  $f_e(t) - g_e(t)$  is constant on each sphere with center 0, and  $\varphi : [0, \infty) \rightarrow R$  is well defined by

$$\varphi(\|x\|) = f_e(x) - g_e(x). \quad (3.26)$$

Hence (3.25) and (3.26) lead to

$$\begin{aligned} f_e(x) &= \frac{Q(x) + \varphi(\|x\|) + h_e(0) + k_e(0)}{2}, \\ g_e(x) &= \frac{Q(x) - \varphi(\|x\|) + h_e(0) + k_e(0)}{2}, \end{aligned} \quad (3.27)$$

which finishes the proof.  $\square$

Finally, the general solution of (1.5) is characterized by the following theorem.

**Theorem 3.5.** *Let  $f, g, h, k : H \rightarrow R$  be real functionals satisfying (1.5); then there exist additive functions,  $B : H \rightarrow R$ , a quadratic function  $Q : H \rightarrow R$ , and a function  $\varphi : [0, \infty) \rightarrow R$  such that*

$$\begin{aligned} f(x) &= \frac{A(x) + B(x) + Q(x) + \varphi(\|x\|) + h(0) + k(0)}{2}, \\ g(x) &= \frac{A(x) - B(x) + Q(x) - \varphi(\|x\|) + h(0) + k(0)}{2}, \\ h(x) &= A(x) + Q(x) + h(0), \\ k(x) &= B(x) + Q(x) + k(0). \end{aligned} \quad (3.28)$$

Conversely, the above functionals satisfy (1.5).

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