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Research Article

Solvability of Three-Point Boundary Value Problems at Resonance with a p-Laplacian on Finite and Infinite Intervals

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Three-point boundary value problems of second-order differential equation with a p-Laplacian on finite and infinite intervals are investigated in this paper. By using a new continuation theorem, sufficient conditions are given, under the resonance conditions, to guarantee the existence of solutions to such boundary value problems with the nonlinear term involving in the first-order derivative explicitly.

1. Introduction

This paper deals with the three-point boundary value problem of differential equation with a p-Laplacian

$$\begin{gathered}(\Phi_p(x'))' + f(t, x, x') = 0, \quad 0 < t < T, \\ x(0) = x(\eta), \qquad x'(T) = 0,\end{gathered} \tag{1.1}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \in (0, T)$ is a constant, $T \in (0, +\infty]$, and $x'(T) = \lim_{t\to T^-} x'(t)$.

Boundary value problems (BVPs) with a p-Laplacian have received much attention mainly due to their important applications in the study of non-Newtonian fluid theory, the turbulent flow of a gas in a porous medium, and so on [1–10]. Many works have been done to discuss the existence of solutions, positive solutions subject to Dirichlet, Sturm-Liouville, or nonlinear boundary value conditions.

In recent years, many authors discussed, solvability of boundary value problems at resonance, especially the multipoint case [3, 11–15]. A boundary value problem of differential equation is said to be at resonance if its corresponding homogeneous one has nontrivial solutions. For (1.1), it is easy to see that the following BVP

$$\begin{aligned} &(\Phi_p(x'))' = 0, \quad 0 < t < T, \\ &x(0) = x(\eta), \qquad x'(T) = 0 \end{aligned} \tag{1.2}$$

has solutions $\{x \mid x = a,\ a \in R\}$. When $a \neq 0$, they are nontrivial solutions. So, the problem in this paper is a BVP at resonance. In other words, the operator L defined by $Lx = (\Phi_p(x'))'$ is not invertible, even if the boundary value conditions are added.

For multi-point BVP at resonance without p-Laplacians, there have been many existence results available in the references [3, 11–15]. The methods mainly depend on the coincidence theory, especially Mawhin continuation theorem. At most linearly increasing condition is usually adopted to guarantee the existence of solutions, together with other suitable conditions imposed on the nonlinear term.

On the other hand, for BVP at resonance with a p-Laplacian, very little work has been done. In fact, when $p \neq 2$, $\Phi_p(x)$ is not linear with respect to x, so Mawhin continuation theorem is not valid for some boundary conditions. In 2004, Ge and Ren [3, 4] established a new continuation theorem to deal with the solvability of abstract equation $Mx = Nx$, where M, N are nonlinear maps; this theorem extends Mawhin continuation theorem. As an application, the authors discussed the following three-point BVP at resonance

$$\begin{aligned} &(\Phi_p(u'))' + f(t,u) = 0, \quad 0 < t < 1, \\ &u(0) = 0 = G(u(\eta), u(1)), \end{aligned} \tag{1.3}$$

where $\eta \in (0,1)$ is a constant and G is a nonlinear operator. Through some special direct-sum-spaces, they proved that (1.3) has at least one solution under the following condition.

There exists a constant $D > 0$ such that $f(t,D) < 0 < f(t,-D)$ for $t \in [0,1]$ and $G(x,D) < 0 < G(x,-D)$ or $G(x,D) > 0 > G(x,-D)$ for $|x| \leqslant D$.

The above result naturally prompts one to ponder if it is possible to establish similar existence results for BVP at resonance with a p-Laplacian under at most linearly increasing condition and other suitable conditions imposed on the nonlinear term.

Motivated by the works mentioned above, we aim to study the existence of solutions for the three-point BVP (1.1). The methods used in this paper depend on the new Ge-Mawhin's continuation theorem [3] and some inequality techniques. To generalize at most linearly increasing condition to BVP at resonance with a p-Laplacian, a small modification is added to the new Ge-Mawhin's continuation theorem. What we obtained in this paper is applicable to BVP of differential equations with nonlinear term involving in the first-order derivative explicitly. Here we note that the techniques used in [3] are not applicable to such case. An existence result is also established for the BVP at resonance on a half-line, which is new for multi-point BVPs on infinite intervals [16, 17].

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we discuss the existence of solutions for BVP (1.1) when T is a real constant, which we call the finite case. In Section 4, we establish an existence result for the bounded solutions to BVP (1.1) when $T = +\infty$, which we call the infinite case. Some explicit examples are also given in the last section to illustrate our main results.

2. Preliminaries

For the convenience of the readers, we provide here some definitions and lemmas which are important in the proof of our main results. Ge-Mawhin's continuation theorem and the modified one are also stated in this section.

Lemma 2.1. *Let $\Phi_p(s) = |s|^{p-2}s$, $p > 1$. Then Φ_p satisfies the properties.*

(1) *Φ_p is continuous, monotonically increasing, and invertible. Moreover $\Phi_p^{-1} = \Phi_q$ with $q > 1$ a real number satisfying $1/p + 1/q = 1$;*

(2) *for any $u, v \geqslant 0$,*

$$\begin{gathered} \Phi_p(u+v) \leqslant \Phi_p(u) + \Phi_p(v), \quad \text{if } p < 2, \\ \Phi_p(u+v) \leqslant 2^{p-2}\left(\Phi_p(u) + \Phi_p(v)\right), \quad \text{if } p \geqslant 2. \end{gathered} \tag{2.1}$$

Definition 2.2. Let R^2 be an 2-dimensional Euclidean space with an appropriate norm $|\cdot|$. A function $f : [0,T] \times R^2 \to R$ is called Φ_q-Carathéodory if and only if

(1) for each $x \in R^2$, $t \mapsto f(t,x)$ is measurable on $[0,T]$;

(2) for a.e. $t \in [0,T]$, $x \mapsto f(t,x)$ is continuous on R^2;

(3) for each $r > 0$, there exists a nonnegative function $\varphi_r \in L^1[0,T]$ with $\varphi_{r,q}(t) := \Phi_q(\int_t^T \varphi_r(\tau)d\tau) \in L^1[0,T]$ such that

$$|x| \leqslant r \text{ implies } \left|f(t,x)\right|\varphi_r(t), \quad \text{a.e. } t \in [0,T]. \tag{2.2}$$

Next we state Ge-Mawhin's continuation theorem [3, 4].

Definition 2.3. Let X, Z be two Banach spaces. A continuous opeartor $M : X \cap \operatorname{dom} M \to Z$ is called quasi-linear if and only if $\operatorname{Im} M$ is a closed subset of Z and $\operatorname{Ker} M$ is linearly homeomorphic to R^n, where n is an integer.

Let X_2 be the complement space of $\operatorname{Ker} M$ in X, that is, $X = \operatorname{Ker} M \oplus X_2$. $\Omega \subset X$ an open and bounded set with the origin $0 \in \Omega$.

Definition 2.4. A continuous operator $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is said to be M-compact in $\overline{\Omega}$ if there is a vector subspace $Z_1 \subset Z$ with $\dim Z_1 = \dim \operatorname{Ker} M$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda\left(\overline{\Omega}\right) \subset \operatorname{Im} M \subset (I-Q)Z, \tag{2.3}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Longleftrightarrow QNx = 0, \quad \forall x \in \Omega, \tag{2.4}$$

$$R(\cdot,0) \text{ is the zero operator}, \;\; R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}, \tag{2.5}$$

$$M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda, \tag{2.6}$$

where P, Q are projectors such that $\operatorname{Im} P = \operatorname{Ker} M$ and $\operatorname{Im} Q = Z_1$, $N = N_1$, $\Sigma_\lambda = \{x \in \overline{\Omega}, Mx = N_\lambda x\}$.

Theorem 2.5 (Ge-Mawhin's continuation theorem). *Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be two Banach spaces, $\Omega \subset X$ an open and bounded set. Suppose $M : X \cap \operatorname{dom} M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is M-compact. In addition, if*

(i) *$Mx \neq N_\lambda x$, for $x \in \operatorname{dom} M \cap \partial\Omega$, $\lambda \in (0,1)$,*

(ii) *$QNx \neq 0$, for $x \in \operatorname{Ker} M \cap \partial\Omega$,*

(iii) *$\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) \neq 0$,*

where $N = N_1$. Then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$.

According to the usual direct-sum spaces such as those in [3, 5, 7, 11–13], it is difficult (maybe impossible) to define the projector Q under the at most linearly increasing conditions. We have to weaken the conditions of Ge-Mawhin continuation theorem to resolve such problem.

Definition 2.6. Let Y_1 be finite dimensional subspace of Y. $Q : Y \to Y_1$ is called a semiprojector if and only if Q is semilinear and idempotent, where Q is called semilinear provided $Q(\lambda x) = \lambda Q(x)$ for all $\lambda \in R$ and $x \in Y$.

Remark 2.7. Using similar arguments to those in [3], we can prove that when Q is a semiprojector, Ge-Mawhin's continuation theorem still holds.

3. Existence Results for the Finite Case

Consider the Banach spaces $X = C^1[0,T]$ endowed with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{0 \leqslant t \leqslant T} |x(t)|$ and $Z = L^1[0,T]$ with the usual Lebesgue norm denoted by $\|\cdot\|_z$. Define the operator M by

$$M : \operatorname{dom} M \cap X \longrightarrow Z, \qquad (Mx)(t) = \left(\Phi_p(x'(t))\right)', \quad t \in [0,T], \tag{3.1}$$

where $\operatorname{dom} M = \{x \in C^1[0,T], \Phi_p(x') \in C^1[0,T], x(0) = x(\eta), x'(T) = 0\}$. Then by direct calculations, one has

$$\begin{gathered} \operatorname{Ker} M = \{x \in \operatorname{dom} M \cap X : x(t) = c \in R,\ t \in [0,T]\}, \\ \operatorname{Im} M = \left\{ y \in Z : \int_0^\eta \Phi_q\left(\int_s^T y(\tau)d\tau \right) ds = 0 \right\}. \end{gathered} \tag{3.2}$$

Obviously, $\operatorname{Ker} M \simeq R$ and $\operatorname{Im} M$ is close. So the following result holds.

Lemma 3.1. *Let M be defined as* (3.1), *then M is a quasi-linear operator.*

Set the projector P and semiprojector Q by

$$P : X \longrightarrow X, \qquad (Px)(t) = x(0), \quad t \in [0,T], \tag{3.3}$$

$$Q : Z \longrightarrow Z, \qquad (Qy)(t) = \frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T y(\tau)d\tau\right)ds\right), \quad t \in [0,T], \tag{3.4}$$

where $\rho = ((1/q)(T^q - (T-\eta)^q))^{p-1}$. Define the operator $N_\lambda : X \to Z$, $\lambda \in [0,1]$ by

$$(N_\lambda x)(t) = -\lambda f(t, x(t), x\prime(t)), \quad t \in [0,T]. \tag{3.5}$$

Lemma 3.2. *Let $\Omega \subset X$ be an open and bounded set. If f is a Carathéodory function, N_λ is M-compact in $\overline{\Omega}$.*

Proof. Choose $Z_1 = \operatorname{Im} Q$ and define the operator $R : \overline{\Omega} \times [0,1] \to \operatorname{Ker} P$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_q\left(\int_s^T \lambda\left(f(\tau, x(\tau), x\prime(\tau)) - (Qf)(\tau)\right)d\tau\right)ds, \quad t \in [0,T]. \tag{3.6}$$

Obviously, $\dim Z_1 = \dim \operatorname{Ker} M = 1$. Since f is a Carathéodory function, we can prove that $R(\cdot,\lambda)$ is continuous and compact for any $\lambda \in [0,1]$ by the standard theories.

It is easy to verify that (2.3)–(2.5) in Definition 2.3 hold. Besides, for any $x \in \operatorname{dom} M \cap \overline{\Omega}$,

$$\begin{aligned} M[Px + R(x,\lambda)](t) &= \left(\Phi_p\left[x(0) + \int_0^t \Phi_q\left(\int_s^T \lambda\left(f(\tau, x(\tau), x'(\tau))d\tau - (Qf)(\tau)\right)d\tau\right)ds\right]'\right)' \\ &= ((I-Q)N_\lambda x)(t), \quad t \in [0,T]. \end{aligned} \tag{3.7}$$

So N_λ is M-compact in $\overline{\Omega}$. □

Theorem 3.3. *Let $f : [0,T] \times R^2 \to R$ be a Carathéodory function. Suppose that*

(H1) *there exist $e(t) \in L^1[0,T]$ and Carathéodory functions g_1, g_2 such that*

$$\begin{gathered} \left|f(t,u,v)\right| \leqslant g_1(t,u) + g_2(t,v) + e(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (u,v) \in R^2, \\ \lim_{x\to\infty} \frac{\int_0^T g_i(\tau,x)d\tau}{\Phi_p(|x|)} = r_i \in [0,+\infty), \quad i = 1,2; \end{gathered} \tag{3.8}$$

(H2) *there exists $B_1 > 0$ such that for all $t_\eta \in [0,\eta]$ and $x \in C^1[0,T]$ with $\|x\|_\infty > B_1$,*

$$\int_{t_\eta}^T f(\tau, x(\tau), x'(\tau))\, d\tau \neq 0; \tag{3.9}$$

(H3) *there exists* $B_2 > 0$ *such that for each* $t \in [0,T]$ *and* $u \in R$ *with* $|u| > B_2$ *either* $uf(t,u,0) \leqslant 0$ *or* $uf(t,u,0) \geqslant 0$. *Then BVP* (1.1) *has at least one solution provided*

$$\begin{aligned} \alpha_1 &:= 2^{q-2}\left(T^{p-1}r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p < 2, \\ \alpha_2 &:= \left(2^{p-2}T^{p-1}r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \end{aligned} \tag{3.10}$$

Proof. Let X, Z, M, N_λ, P, and Q be defined as above. Then the solutions of BVPs (1.1) coincide with those of $Mx = Nx$, where $N = N_1$. So it is enough to prove that $Mx = Nx$ has at least one solution.

Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x,\ \lambda \in (0,1)\}$. If $x \in \Omega_1$, then $QN_\lambda x = 0$. Thus,

$$\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau, x(\tau), x'(\tau))d\tau\right)ds\right) = 0. \tag{3.11}$$

The continuity of Φ_p and Φ_q together with condition (H2) implies that there exists $\xi \in [0,T]$ such that $|x(\xi)| \leqslant B_1$. So

$$|x(t)| \leqslant |x(\xi)| + \int_\xi^t \left|x'(s)\right|ds \leqslant B_1 + T\|x\prime\|_\infty, \quad t \in [0,T]. \tag{3.12}$$

Noting that $Mx = N_\lambda x$, we have

$$\begin{aligned} x'(t) &= \Phi_q\left(\int_t^T \lambda f(\tau, x(\tau), x'(\tau))d\tau\right), \\ x(t) &= x(0) + \int_0^t \Phi_q\left(\int_s^T \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)ds. \end{aligned} \tag{3.13}$$

If $p < 2$, choose $\epsilon > 0$ such that

$$\alpha_{1,\epsilon} := 2^{q-2}\left(T^{p-1}(r_1 + \epsilon) + (r_2 + \epsilon)\right)^{q-1} < 1. \tag{3.14}$$

For this $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^T g_i(\tau, x)d\tau \leqslant (r_i + \epsilon)\Phi_p(|x|) \quad \forall |x| > \delta,\ i = 1,2. \tag{3.15}$$

Set

$$g_{i,\delta} = \int_0^T \left(\max_{|x|\leqslant\delta} g_i(\tau, x)\right)d\tau, \quad i = 1,2. \tag{3.16}$$

Noting (3.12)-(3.13), we have

$$\begin{aligned}
|x'(t)| &= \left|\Phi_q\left(\int_t^T \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)\right| \leqslant \Phi_q\left(\int_0^T |f(\tau, x(\tau), x'(\tau))|d\tau\right) \\
&\leqslant \Phi_q\left(\int_0^T (g_1(\tau, x) + g_2(\tau, x') + e(\tau))d\tau\right) \\
&\leqslant \Phi_q((r_1+\epsilon)\Phi_p(|x|) + (r_2+\epsilon)\Phi_p(|x'|) + g_{1,\delta} + g_{2,\delta} + \|e\|_{L^1}) \\
&\leqslant \alpha_{1,\epsilon}\|x'\|_\infty + B_\delta,
\end{aligned} \tag{3.17}$$

where $B_\delta = 2^{q-2}((r_1+\epsilon)B_1^{p-1} + g_{1,\delta} + g_{2,\delta} + \|e\|_{L^1})^{q-1}$. So

$$\|x'\|_\infty \leqslant \frac{B_\delta}{1-\alpha_{1,\epsilon}} := B'. \tag{3.18}$$

And then $\|x\|_X \leqslant \max\{B_1 + TB', B'\} := B$.

Similarly, if $p \geqslant 2$, we can obtain $\|x\|_X \leqslant \max\{B_1 + T\widetilde{B}', \widetilde{B}'\} := \widetilde{B}$, where

$$\begin{gathered}
\widetilde{B}' = \frac{\left(2^{p-2}(r_1+\epsilon)B_1^{p-1} + g_{1,\delta} + g_{2,\delta} + \|e\|_{L^1}\right)^{q-1}}{1-\alpha_{2,\epsilon}}, \\
\alpha_{2,\epsilon} = \left(2^{p-2}T^{p-1}(r_1+\epsilon) + (r_2+\epsilon)\right)^{q-1}.
\end{gathered} \tag{3.19}$$

Above all, Ω_1 is bounded.

Set $\Omega_{2,i} := \{x \in \operatorname{Ker} M : (-1)^i \mu x + (1-\mu)JQNx = 0, \mu \in [0,1]\}$, $i = 1,2$, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism defined by $Ja = a$ for any $a \in R$. Next we show that $\Omega_{2,1}$ is bounded if the first part of condition (H3) holds. Let $x \in \Omega_{2,1}$, then $x = a$ for some $a \in R$ and

$$\mu a = (1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau, a, 0)d\tau\right)ds\right). \tag{3.20}$$

If $\mu = 0$, we can obtain that $|a| \leqslant B_1$. If $\mu \neq 0$, then $|a| \leqslant B_2$. Otherwise,

$$\begin{aligned}
\mu a^2 &= a(1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau, a, 0)d\tau\right)ds\right) \\
&= (1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T af(\tau, a, 0)d\tau\right)ds\right) \leqslant 0,
\end{aligned} \tag{3.21}$$

which is a contraction. So $\|x\|_X = |a| \leqslant \max\{B_1, B_2\}$ and $\Omega_{2,1}$ is bounded. Similarly, we can obtain that $\Omega_{2,2}$ is bounded if the other part of condition (H3) holds.

Let $\Omega = \{x \in X : \|x\|_X < \max\{B(\widetilde{B}), B_1, B_2\} + 1\}$. Then $\Omega_1 \cup \Omega_{2,1}$ $(\cup\, \Omega_{2,2}) \subset \Omega$. It is obvious that $Mx \neq N_\lambda x$ for each $(x, \lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0, 1)$.

Take the homotopy $H_i : (\operatorname{Ker} M \cap \overline{\Omega}) \times [0, 1] \to X$ by

$$H_i(x, \mu) = (-1)^i \mu x + (1-\mu) JQNx, \quad i = 1 \text{ or } 2. \tag{3.22}$$

Then for each $x \in \operatorname{Ker} M \cap \partial\Omega$ and $\mu \in [0, 1]$, $H_i(x, \mu) \neq 0$, so by the degree theory

$$\deg = \{JQN, \operatorname{Ker} M \cap \Omega, 0\} = \deg\left\{(-1)^i I, \operatorname{Ker} M \cap \Omega, 0\right\} \neq 0. \tag{3.23}$$

Applying Theorem 2.5 together with Remark 2.7, we obtain that $Mx = Nx$ has a solution in $\operatorname{dom} M \cap \overline{\Omega}$. So (1.1) is solvable. □

Corollary 3.4. *Let $f : [0, T] \times R^2 \to R$ be a Carathéodory function. Suppose that (H2), (H3) in Theorem 3.3 hold. Suppose further that*

(H1′) *there exist nonnegative functions $g_i \in L^1[0, T]$, $i = 0, 1, 2$ such that*

$$\left|f(t, u, v)\right| \leqslant g_1(t)|u|^{p-1} + g_2(t)|v|^{p-1} + g_0(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } (u, v) \in R^2. \tag{3.24}$$

Then BVP (1.1) *has at least one solution provided*

$$\begin{gathered} 2^{q-2}\left(T^{p-1}\|g_1\|_{L^1} + \|g_2\|_{L^1}\right)^{q-1} < 1, \quad \text{if } p < 2, \\ \left(2^{p-2}T^{p-1}\|g_1\|_{L^1} + \|g_2\|_{L^1}\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \end{gathered} \tag{3.25}$$

If f is a continuous function, we can establish the following existence result.

Theorem 3.5. *Let $f : [0, T] \times R^2 \to R$ be a continuous function. Suppose that (H1), (H3) in Theorem 3.3 hold. Suppose further that*

(H2′) *there exist B_3, $a > 0$, $b, c \geqslant 0$ such that for all $u \in R$ with $|u| > B_3$, it holds that*

$$\left|f(t, u, v)\right| \geqslant a|u| - b|v| - c \quad \forall t \in [0, T] \text{ and all } v \in R. \tag{3.26}$$

Then BVP (1.1) *has at least one solution provided*

$$\begin{gathered} 2^{q-2}\left(\left(\frac{b}{a} + T\right)^{p-1} r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p < 2, \\ \left(2^{p-2}\left(\frac{b}{a} + T\right)^{p-1} r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \end{gathered} \tag{3.27}$$

Proof. If $x \in \operatorname{dom} M$ such that $Mx = N_\lambda x$ for some $\lambda \in (0,1)$, we have $QN_\lambda x = 0$. The continuity of f and Φ_q imply that there exists $\xi \in [0,T]$ such that $f(\xi, x(\xi), x'(\xi)) = 0$. From (H2′), it holds

$$|x(\xi)| \leqslant \max\left\{B_3, \frac{b}{a}\|x'\|_\infty + \frac{c}{a}\right\}. \tag{3.28}$$

Therefore,

$$|x(t)| \leqslant |x(\xi)| + \int_\xi^t |x'(s)| ds \leqslant \left(\frac{b}{a} + T\right)\|x'\|_\infty + \frac{c}{a} + B_1, \quad t \in [0,T]. \tag{3.29}$$

With a similar way to those in Theorem 3.3, we can prove that (1.1) has at least one solution. □

Corollary 3.6. *Let $f : [0,T] \times R^2 \to R$ be a continuous function. Suppose that conditions in Corollary 3.4 hold except (H2) changed with* (H2′). *Then BVP* (1.1) *is also solvable.*

4. Existence Results for the Infinite Case

In this section, we consider the BVP (1.1) on a half line. Since the half line is noncompact, the discussions are more complicated than those on finite intervals.

Consider the spaces X and Z defined by

$$\begin{gathered} X = \left\{x \in C^1[0,+\infty), \lim_{t\to+\infty} x(t) \text{ exists}, \lim_{t\to+\infty} x'(t) \text{ exists}\right\}, \\ Z = \left\{y \in L^1[0,+\infty), \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} |y(\tau)| d\tau\right) ds < +\infty\right\}, \end{gathered} \tag{4.1}$$

with the norms $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $\|y\|_Z = \|y\|_{L^1}$, respectively, where $\|x\|_\infty = \sup_{0\leqslant t<+\infty}|x(t)|$. By the standard arguments, we can prove that $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are both Banach spaces.

Let the operators M, N_λ, and P be defined as (3.1), (3.3), and (3.5), respectively, expect T replaced by $+\infty$. Set $\omega(t) = ((1 - e^{-(q-1)\eta})/(q-1))^{1-p} e^{-t}$, $t \in [0,+\infty)$ and define the semiprojector $Q : Y \to Y$ by

$$(Qy)(t) = w(t)\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^{+\infty} y(\tau) d\tau\right) ds\right), \quad t \in [0,+\infty). \tag{4.2}$$

Similarly, we can show that M is a quasi-linear operator. In order to prove that N_λ is M-compact in $\overline{\Omega}$, the following criterion is needed.

Theorem 4.1 (see [16]). *Let $M \subset C_\infty = \{x \in C[0,+\infty), \lim_{t\to+\infty} x(t)$ exists$\}$. Then M is relatively compact if the following conditions hold:*

(a) *all functions from M are uniformly bounded;*

(b) *all functions from M are equicontinuous on any compact interval of* $[0,+\infty)$;

(c) *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|f(t) - f(+\infty)| < \epsilon$, for all $t > T$, $f \in M$.*

Lemma 4.2. *Let $\Omega \subset X$ an open and bounded set with $0 \in \Omega$. If f is a Φ_q-Carathéodory function, N_λ is M-compact in $\overline{\Omega}$.*

Proof. Let $Z_1 = \operatorname{Im} Q$ and define the operator $R : \overline{\Omega} \times [0,1] \longrightarrow \operatorname{Ker} P$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_q\left(\int_s^{+\infty} \lambda\big(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\big)d\tau\right)ds, \quad t \in [0,+\infty). \tag{4.3}$$

We just prove that $R(\cdot,\lambda) : \overline{\Omega} \times [0,1] \longrightarrow X$ is what we need. The others are similar and are omitted here.

Firstly, we show that R is well defined. Let $x \in \Omega$, $\lambda \in [0,1]$. Because Ω is bounded, there exists $r > 0$ such that for any $x \in \Omega$, $\|x\|_X \leqslant r$. Noting that f is a Φ_q-Carathéodory function, there exists $\varphi_r \in L^1[0,+\infty)$ with $\varphi_{r,q} \in L^1[0,+\infty)$ such that

$$\big|f(t, x(t), x'(t)) \leqslant \big|\varphi_r(t), \quad \text{a.e. } t \in [0,+\infty). \tag{4.4}$$

Therefore

$$\begin{aligned} |R(x,\lambda)(t)| &= \left|\int_0^t \Phi_q\left(\int_s^{+\infty} \lambda\big(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\big)d\tau\right)ds\right| \\ &\leqslant \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} (\varphi_r(\tau) + \Upsilon_r\omega(\tau))d\tau\right)ds < +\infty, \quad \forall t \in [0,+\infty), \end{aligned} \tag{4.5}$$

where $\Upsilon_r = \Phi_p(\int_0^\eta \Phi_q(\int_s^{+\infty} \varphi_r(\tau)d\tau)ds)$. Meanwhile, for any $t_1, t_2 \in [0,+\infty)$, we have

$$\begin{aligned} |R(x,\lambda)(t_1) - R(x,\lambda)(t_2)| &\leqslant \int_{t_1}^{t_2} \Phi_q\left(\int_s^{+\infty} \lambda\big|f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\big|d\tau\right)ds \\ &\leqslant \int_{t_1}^{t_2} \Phi_q\left(\int_s^{+\infty} (\varphi_r(\tau) + \Upsilon_r\omega(\tau))d\tau\right)ds \\ &\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2, \end{aligned} \tag{4.6}$$

$$\left|\int_{t_1}^{t_2} \lambda\big(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\big)d\tau\right| \leqslant \int_{t_1}^{t_2} (\varphi_r(\tau) + \Upsilon_r\omega(\tau))d\tau \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \tag{4.7}$$

The continuity of Φ_q concludes that

$$\big|R(x,\lambda)'(t_1) - R(x,\lambda)'(t_2)\big| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \tag{4.8}$$

It is easy to see that $\lim_{t\to+\infty} R(x,\lambda)(t)$ exists and $\lim_{t\to+\infty} R(x,\lambda)\prime(t) = 0$. So $R(x,\lambda) \in X$.

Next, we verify that $R(\cdot,\lambda)$ is continuous. Obviously $R(x,\lambda)$ is continuous in λ for any $x\in\Omega$. Let $\lambda\in[0,1]$, $x_n \to x$ in Ω as $n\to+\infty$. In fact,

$$\left|\int_0^{+\infty}\left(f(\tau,x_n,x_n')-f(\tau,x,x')\right)d\tau\right| \leqslant 2\|\varphi_r\|_{L^1},$$
$$\left|\int_0^t\left[\Phi_q\left(\int_s^{+\infty}f(\tau,x_n,x_n')d\tau\right)-\Phi_q\left(\int_s^{+\infty}f(\tau,x,x')d\tau\right)\right]ds\right| \leqslant 2\|\varphi_{r,q}\|_{L^1}. \tag{4.9}$$

So by Lebesgue Dominated Convergence theorem and the continuity of Φ_q, we can obtain

$$\|R(x_n,\lambda)-R(x,\lambda)\|_X \longrightarrow 0, \quad \text{as } n\longrightarrow+\infty. \tag{4.10}$$

Finally, $R(\cdot,\lambda)$ is compact for any $\lambda\in[0,1]$. Let $U\subset X$ be a bounded set and $\lambda\in[0,1]$, then there exists $r_0>0$ such that $\|x\|_X\leqslant r_0$ for any $x\in U$. Thus we have

$$\begin{aligned}
\|R(x,\lambda)\|_X &= \max\left\{\|R(x,\lambda)\|_\infty, \left\|R'(x,\lambda)\right\|_\infty\right\}\\
&\leqslant \max\left\{\int_0^{+\infty}\Phi_q\left(\int_s^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon_{r_0}\omega(\tau)\right)d\tau\right)ds,\right.\\
&\qquad\left.\Phi_q\left(\int_0^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon\omega(\tau)\right)d\tau\right)\right\},\\
|R(x,\lambda)(t)-R(x,\lambda)(+\infty)| &= \left|\int_t^{+\infty}\Phi_q\left(\int_s^{+\infty}\lambda\left(f(\tau,x(\tau),x'(\tau))-(Qf)(\tau)\right)d\tau\right)ds\right|\\
&\leqslant \int_t^{+\infty}\Phi_q\left(\int_s^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon_{r_0}\omega(\tau)\right)d\tau\right)ds \longrightarrow 0,\\
&\qquad \text{uniformly as } t\longrightarrow+\infty,\\
\left|R(x,\lambda)'(t)-R(x,\lambda)'(+\infty)\right| &= \left|\Phi_q\left(\int_t^{+\infty}\lambda\left(f(\tau,x(\tau),x'(\tau))-(Qf)(\tau)\right)d\tau\right)\right|\\
&\leqslant \Phi_q\left(\int_t^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon\omega(\tau)\right)d\tau\right) \longrightarrow 0,\\
&\qquad \text{uniformly as } t\to+\infty.
\end{aligned} \tag{4.11}$$

Those mean that $R(\cdot,\lambda)$ is uniformly bounded and equiconvergent at infinity. Similarly to the proof of (4.3) and (4.6), we can show that $R(\cdot,\lambda)$ is equicontinuous. Through Lemma 4.2, $R(\cdot,\lambda)U$ is relatively compact. The proof is complete. □

Theorem 4.3. *Let $f:\ [0,+\infty)\times R^2 \to R$ be a continuous and Φ_q-Carathéodory function. Suppose that*

(H4) *there exist functions* $g_0, g_1, g_2 \in L^1[0,+\infty)$ *such that*

$$\left|f(t,u,v)\right| \leqslant g_1(t)|u|^{p-1} + g_2(t)|v|^{p-1} + g_0(t) \quad \text{for a.e. } t \in [0,+\infty) \ \text{ and all } \ (u,v) \in R^2,$$
$$\left\|g_{i,q}\right\|_{L^1} := \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} \left|g_i(\tau)\right| d\tau\right) ds < +\infty, \quad i = 0,1,2,$$
$$\left\|g_1\right\|_1 := \int_0^{+\infty} t^{p-1}\left|g_1(\tau)\right| d\tau < +\infty; \tag{4.12}$$

(H5) *there exists* $\gamma > 0$ *such that for all* ζ *satisfying*

$$f(\zeta,u,v) = 0, \qquad f(t,u,v) \neq 0, \quad t \in [0,\zeta), \ (u,v) \in R^2, \tag{4.13}$$

it holds $\zeta \leqslant \gamma$;

(H6) *there exist* B_4, $a > 0$, $b, c \geqslant 0$ *such that for all* $u \in R$ *with* $|u| > B_4$, *it holds*

$$\left|f(t,u,v)\right| \geqslant a|u| - b|v| - c \quad \forall t \in [0,\gamma], \ v \in R; \tag{4.14}$$

(H7) *there exists* $B_5 > 0$ *such that for all* $t \in [0,+\infty)$ *and* $u \in R$ *with* $|u| > B_5$ *either* $uf(t,u,0) \leqslant 0$ *or* $uf(t,u,0) \geqslant 0$. *Then BVP* (1.1) *has at least one solution provided*

$$\max\left\{2^{q-2}\left\|g_{1,q}\right\|_{L^1}, \beta_1\right\} < 1, \quad \text{if } p < 2,$$
$$\max\left\{\left\|g_{1,q}\right\|_{L^1}, \beta_2\right\} < 1, \quad \text{if } p \geqslant 2, \tag{4.15}$$

where

$$\beta_1 := 2^{q-2}\left(\left(\frac{b}{a}+\gamma\right)^{p-1}\left\|g_1\right\|_{L^1} + \left\|g_1\right\|_1 + \left\|g_2\right\|_{L^1}\right)^{q-1},$$
$$\beta_2 := \left(2^{2(p-2)}\left(\frac{b}{a}+\gamma\right)^{p-1}\left\|g_1\right\|_{L^1} + 2^{2(q-2)}\left\|g_1\right\|_1 + \left\|g_2\right\|_{L^1}\right)^{q-1}. \tag{4.16}$$

Proof. Let X, Z, M, N_λ, P, and Q be defined as above. Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x, \lambda \in (0,1)\}$. We will prove that Ω_1 is bounded. In fact, for any $x \in \Omega_1$, $QN_\lambda x = 0$, that is,

$$\omega(t)\ \Phi_p\left(\int_0^{\eta} \Phi_q\left(\int_s^{+\infty} \lambda f(\tau, x(\tau), x'(\tau)) d\tau\right) ds\right) = 0. \tag{4.17}$$

The continuity of Φ_p and Φ_q together with conditions (H5) and (H6) implies that there exists $\xi \leqslant \gamma$ such that

$$|x(\xi)| \leqslant \max\left\{B_4, \frac{b}{a}\|x\prime\|_\infty + \frac{c}{a}\right\}. \tag{4.18}$$

So, we have

$$|x(t)| \leqslant |x(\xi)| + \left|\int_\xi^t x'(s)ds\right| \leqslant \max\left\{B_4, \frac{b}{a}\|x\prime\|_\infty + \frac{c}{a}\right\} + (t+\gamma)\left\|x'\right\|_\infty, \quad t \in [0,+\infty). \tag{4.19}$$

If $p < 2$, it holds

$$|x(t)|^{p-1} \leqslant \left(\left(\frac{b}{a} + \gamma\right)^{p-1} + t^{p-1}\right)\left\|x'\right\|_\infty^{p-1} + \left(\frac{c}{a} + B_4\right)^{p-1}, \quad t \in [0,+\infty). \tag{4.20}$$

Therefore

$$\begin{aligned} \left|x'(t)\right| &= \left|\Phi_q\left(\int_t^{+\infty} \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)\right| \\ &\leqslant \Phi_q\left(\int_0^{+\infty}\left(g_1(\tau)|x(\tau)|^{p-1} + g_2(\tau)\left|x'(\tau)\right|^{p-1} + g_0(\tau)\right)d\tau\right) \\ &\leqslant \beta_1\left\|x'\right\|_\infty + 2^{q-2}\left((c/a + B_4)^{p-1}\left\|g_1\right\|_{L^1} + \left\|g_0\right\|_{L^1}\right)^{q-1}, \quad t \in [0,+\infty) \end{aligned} \tag{4.21}$$

concludes that

$$\left\|x'\right\|_\infty \leqslant \frac{2^{q-2}\left((c/a + B_4)^{p-1}\left\|g_1\right\|_{L^1} + \left\|g_0\right\|_{L^1}\right)^{q-1}}{1-\beta_1} := C. \tag{4.22}$$

Meanwhile

$$\begin{aligned} |x(t)| &= \left|x(0) + \int_0^t \Phi_q\left(\int_s^{+\infty} \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)ds\right| \\ &\leqslant |x(0)| + \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty}\left(g_1|x|^{p-1} + g_2\left|x'\right|^{p-1} + g_0\right)d\tau\right)ds \\ &\leqslant 2^{q-2}\left\|g_{1,q}\right\|_{L^1}\|x\|_\infty + C_0 \end{aligned} \tag{4.23}$$

implies that

$$\|x\|_\infty \leqslant \frac{C_0}{1 - 2^{q-2}\left\|g_{1,q}\right\|_{L^1}}, \tag{4.24}$$

where $C_0 = (b/a + \gamma + 2^{2(q-2)}\|g_{2,q}\|_{L^1})C + B_4 + c/a + 2^{2(q-2)}\|g_{0,q}\|_{L^1}$.

If $p \geqslant 2$, we can prove that

$$\|x'\|_\infty \leqslant \frac{\left(2^{p-2}(B_4 + c/a)^{p-1}\|g_1\|_{L^1} + \|g_0\|_{L^1}\right)^{q-1}}{1-\beta_2} := \widetilde{C},$$
$$\|x\|_\infty \leqslant \frac{\left(b/a + \gamma + \|g_{2,q}\|_{L^1}\right)\widetilde{C} + B_4 + c/a + \|g_{0,q}\|_{L^1}}{1 - \|g_{1,q}\|_{L^1}}. \tag{4.25}$$

So Ω_1 is bounded. With the similar arguments to those in Theorem 3.3, we can complete the proof. □

5. Examples

Example 5.1. Consider the three-point BVPs for second-order differential equations

$$\left(x'(t)\left|x'(t)\right|\right)' = a_2(t)x'(t) + a_1(t)x^2(t)\operatorname{sgn} x(t) + a_0(t), \quad 0 < t < 1,$$
$$x(0) = x(\eta), \qquad x'(1) = 0, \tag{5.1}$$

where $a_i(t) \in C^1[0,1]$, $i = 0,1,2$ with $a_1 = \min|a_1(t)| > 0$.

Take

$$f(t,u,v) = a_1(t)u^2 \operatorname{sgn} u + a_2(t)v + a_0(t),$$
$$g_1(t,u) = |a_1(t)|u^2,$$
$$g_2(t,v) = |a_2(t)||v|, \tag{5.2}$$

and $e(t) = |a_0(t)|$. Then, we have

$$\left|f(t,u,v)\right| \leqslant g_1(t,u) + g_2(t,v) + e(t), \quad \text{for } (t,u,v) \in [0,1] \times R^2$$
$$\max_{0\leqslant t\leqslant 1} \frac{g_1(t,x)}{|x|} = \|a_1\|_{L^1} \in [0,+\infty),$$
$$\max_{0\leqslant t\leqslant 1} \frac{g_1(t,x)}{|x|} = 0, \tag{5.3}$$
$$\left|f(t,u,v)\right| \geqslant a_1|u| - \|a_2\|_\infty|v| - \|a_0\|_\infty, \quad \text{for } (t,|u|,v) \in [0,T] \times [1,+\infty) \times R,$$
$$uf(t,u,0) = a_1(t)|u|^3 + a_0(t)u \geqslant 0, \quad \text{for } (t,|u|) \in [0,1] \times \left[\sqrt{\frac{\|a_0\|_\infty}{a_1}}, +\infty\right).$$

By using Theorem 3.5, we can concluded that BVP (5.1) has at least one solution if

$$\left(\frac{\|a_2\|_\infty}{a_1} + 1\right)^2 \|a_1\|_\infty < \frac{1}{2}. \tag{5.4}$$

Example 5.2. Consider the three-point BVPs for second-order differential equations on a half line

$$\begin{gathered} x''(t) + e^{-\alpha t} p(t) x(t) + q(t) = 0, \quad 0 < t < +\infty, \\ x(0) = x(\eta), \qquad \lim_{t \to +\infty} x'(t) = 0, \end{gathered} \tag{5.5}$$

where $\alpha > (1+\sqrt{5})/2$, $p(t) = \max\{\sin \beta t, 1/2\}$ and $q(t)$ continuous on $[0, +\infty)$ with $q(t) > 0$ (or $q(t) < 0$) on $[0, 1)$ and $q \equiv 0$ on $[1, +\infty)$.

Denote $f(t, u) = e^{-\alpha t} p(t) u + q(t)$. Set $g_1(t) = e^{-\alpha t}$, $g_0(t) = q(t)$. By direct calculations, we obtain that $\|g_1\|_{L^1} = 1/\alpha$, $\|g_{1,q}\|_{L^1} = \|g_1\|_1 = 1/\alpha^2$ and $\|g_{0,q}\|_{L^1} \leqslant \|g_0\|_{L^1} \leqslant \|q\|_\infty$. Furthermore,

$$\begin{gathered} |f(t, u)| \leqslant |g_1(t)||u| + |g_0(t)|, \\ |f(t, u)| \geqslant \frac{1}{2} e^{-\alpha} |u| - \|q\|_\infty. \end{gathered} \tag{5.6}$$

If there exists $\xi \in [0, +\infty)$ such that $f(\xi, u) = 0$, then $\xi \leqslant 1$. Otherwise

$$u f(\xi, u) = e^{-\alpha \xi} p(\xi) u^2 \geqslant \frac{1}{2} e^{-\alpha \xi} u^2 > 0, \quad \forall u \in R \setminus \{0\} \tag{5.7}$$

which is a contraction.

Obviously $\max\{1/\alpha, 1/\alpha + 1/\alpha^2\} < 1$. Meanwhile, it is easy to verify that condition (H7) holds. So Theorem 4.3 guarantees that (5.5) has at least one solution.

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