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Research Article

Ulam Stability for Fractional Differential Equation in Complex Domain

Rabha W. Ibrahim

Institute of Mathematical Sciences, University Malaya, 50603 Kuala Lumpur, Malaysia

Correspondence should be addressed to Rabha W. Ibrahim, rabhaibrahim@yahoo.com

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The present paper deles with a fractional differential equation $z^{\alpha}D_z^{\alpha}u(z) + zu'(z) + (z^2 - a^2)u(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha}$, $1 < \alpha \leq 2$, where $z \in U := \{z : |z| < 1\}$ in sense of Srivastava-Owa fractional operators. The existence and uniqueness of holomorphic solutions are established. Ulam stability for the approximation and holomorphic solutions are suggested.

1. Introduction

Fractional calculus is a rapidly growing subject of interest for physicists and mathematicians. The reason for this is that problems may be discussed in a much more stringent and elegant way than using traditional methods. Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations [1–7].

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems, and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators, Erdlyi-Kober operators, Weyl-Riesz operators, Caputo operators, and Grünwald-Letnikov operators, have appeared during the past three decades with its applications in other field. Moreover, the existence and uniqueness of holomorphic solutions for nonlinear fractional differential equations such as Cauchy problems and diffusion problems in complex domain are established and posed [8–15].

The present article deals with a nonhomogeneous fractional differential equation. The nonhomogeneous fractional differential equations involving the Bessel differential equation which appears frequently in practical problems and applications. These equations have proved useful in many branches of physics and engineering. They have been used in problems of treating the boundary value problems exhibiting cylindrical symmetries.

In [1], Srivastava and Owa gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows.

Definition 1.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1, \tag{1.1}$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0, \tag{1.2}$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Remark 1.3. From Definitions 1.1 and 1.2, we have $D_z^0 f(z) = f(0)$, $\lim_{\alpha \to 0} I_z^\alpha f(z) = f(z)$ and $\lim_{\alpha \to 0} D_z^{1-\alpha} f(z) = f'(z)$. Moreover,

$$\begin{aligned} D_z^\alpha \{z^\mu\} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \{z^{\mu-\alpha}\}, \quad \mu > -1;\ 0 \le \alpha < 1, \\ I_z^\alpha \{z^\mu\} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} \{z^{\mu+\alpha}\}, \quad \mu > -1;\ \alpha > 0. \end{aligned} \tag{1.3}$$

Further properties of these operators can be found in [10, 11, 13].

2. Nonhomogeneous Fractional Differential Equation

In this section, we will express the solution for the nonhomogeneous problem

$$z^\alpha D_z^\alpha u(z) + z u'(z) + \left(z^2 - a^2\right) u(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha}, \quad 1 < \alpha \le 2, \tag{2.1}$$

where the radius of convergence of power series is at least ρ, for all $z \in U$ and $u : U \to \mathbb{C}$. Note that for $\alpha = 2$, (2.1) reduces to the Bessel differential equation.

Theorem 2.1. *Consider the problem* (2.1) *with* $a^2 < \alpha,\ 1 < \alpha \le 2$, *and there exists a constant* $\delta > 0$ *satisfying the condition*

$$|a_{n+2}| \le \frac{\Gamma(n+\alpha+1) + \left(n+\alpha-a^2\right)\Gamma(n+1)}{\delta\Gamma(n+1)}|c_n|, \tag{2.2}$$

for all sufficiently large integers n, *where*

$$c_n = \begin{cases} \dfrac{a_0}{\Gamma(\alpha+1)-a^2}, & n = 0 \\ \dfrac{a_n - c_{n-2}}{\Gamma(n+\alpha+1)/\Gamma(n+1) + (n+\alpha) - a^2}, & n \ge 1, \end{cases} \tag{2.3}$$

where $c_{-1} = 0$. *Then every solution* $u : U \to \mathbb{C}$ *of the fractional differential equation* (2.1) *can be expressed by*

$$u(z) = u_h(z) + \sum_{n=0}^{\infty} c_n z^{n+\alpha}, \tag{2.4}$$

where $u_h(z)$ *is a solution of the homogeneous*

$$z^\alpha D_z^\alpha u(z) + z u'(z) + \left(z^2 - a^2\right)u(z) = 0. \tag{2.5}$$

Proof. We assume that $u : U \to \mathbb{C}$ is a function given in the form (2.4), and we define that $u_p(z) = u(z) - u_h(z) = \sum_{n=0}^{\infty} c_n z^{n+\alpha}$. Then, it follows from (2.2) and (2.3) that

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{c_{n+2}}{c_n}\right| &= \lim_{n\to\infty}\frac{1}{\Gamma((n+\alpha+1)+(n+\alpha-a^2)\Gamma(n+1))/\Gamma(n+1)}\left|\frac{a_{n+2}-c_n}{c_n}\right| \\ &= \lim_{n\to\infty}\frac{1}{\Gamma((n+\alpha+1)+(n+\alpha-a^2)\Gamma(n+1))/\Gamma(n+1)}\left|\frac{a_{n+2}}{c_n}-1\right| \\ &\le \frac{1}{\delta}, \quad \delta > 0. \end{aligned} \tag{2.6}$$

That is, the power series for $u_p(z)$ converges for all $z \in U$. Hence, we see that the domain of $u(z)$ is well defined. We now prove that the function $u_p(z)$ satisfies the nonhomogeneous equation (2.1). Indeed, it follows from (2.3) that

$$\begin{aligned} z^\alpha D_z^\alpha u_p(z) + z u'(z) + \left(z^2 - a^2\right)u_p(z) = {}& \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}c_n z^{n+\alpha} + \sum_{n=1}^{\infty}(n+\alpha)c_n z^{n+\alpha} \\ &+ \sum_{n=0}^{\infty} c_n z^{n+\alpha+2} - \sum_{n=0}^{\infty} c_n a^2 z^{n+\alpha} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} c_n z^{n+\alpha} + \sum_{n=1}^{\infty} (n+\alpha) c_n z^{n+\alpha} \\
&\quad + \sum_{n=2}^{\infty} c_{n-2} z^{n+\alpha} - \sum_{n=0}^{\infty} c_n a^2 z^{n+\alpha} \\
&= \left(\Gamma(\alpha+1) - a^2\right) c_0 z^{\alpha} + \left(\Gamma(\alpha+2) + (\alpha+1) - a^2\right) c_1 z^{\alpha+1} \\
&\quad + \sum_{n=2}^{\infty} \left[\left(\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} + (\alpha+n) - a^2\right) c_n + c_{n-2}\right] z^{\alpha+n} \\
&:= \sum_{n=0}^{\infty} a_n z^{n+\alpha}.
\end{aligned}
\tag{2.7}
$$

Hence $u_p(z)$ is a particular solution of the nonhomogeneous equation (2.1). On the other hand, since every solution to (2.1) can be expressed as a sum of a solution $u_h(z)$ of the homogeneous equation and a particular solution $u_p(z)$ of the nonhomogeneous equation, every solution of (2.1) is certainly of the form (2.4). □

3. Ulam Stability

A classical problem in the theory of functional equations is that if a function f approximately satisfies functional equation $\mathcal{E}$ when does, there exists an exact solution of $\mathcal{E}$ which f approximates. In 1960, Ulam [16] imposed the question of the stability of Cauchy equation, and, in 1941, Hyers solved it [17]. In 1978, Rassias [18] provided a generalization of Hyers, theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [19–21]). Recently, Li and Hua [22] have discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham et al. [23] have introduced the Hyers-Ulam stability of generalized finite polynomial equation.

In this section, we consider the Hyers-Ulam stability for fractional differential equation (2.1). Let $\mathcal{H}$ be the space of all analytic functions on U.

Definition 3.1. Let p be a real number. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{3.1}$$

has the generalized Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: for every $\epsilon > 0$, $w \in \overline{U} = U \bigcup \partial U$, if

$$\left| \sum_{n=0}^{\infty} a_n w^{n+\alpha} \right| \le \epsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n} \right), \tag{3.2}$$

then there exists some $z \in \overline{U}$ that satisfies (2.1) such that

$$|z^m - w^m| \le \epsilon K, \\ \left(z, w \ \in \overline{U},\ m \in \mathbb{N}\right). \tag{3.3}$$

We need the following results.

Lemma 3.2 (see [24]). *If $f : D \to X$ (X is a complex Banach space) is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in D unless $\|f\|$ is of constant value throughout D.*

Theorem 3.3. *Let $f(z) := z^\alpha D_z^\alpha u(z) + zu'(z) + (z^2 - a^2)u(z)$ be holomorphic in the unit disk U and then (2.1) has the generalized Hyers-Ulam stability.*

Proof. In virtue of Theorem 2.1, (2.2) has a holomorphic solution in the U. According to Lemma 3.2, we have

$$\left|f(z)\right| < 1, \quad z \in U. \tag{3.4}$$

Let $\epsilon > 0$ and $w \in \overline{U}$ be such that

$$\left|\sum_{n=0}^{\infty} a_n w^{n+\alpha}\right| \le \epsilon\left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{2^n}\right). \tag{3.5}$$

We will show that there exists a constant K independent of ϵ such that

$$|w^m - u^m| \le \epsilon K, \quad w \in \overline{U},\ u \in U \tag{3.6}$$

and satisfies (3.1). We put the function

$$f(w) = \frac{-1}{\lambda a_m} \sum_{n=0,\, n\neq m}^{\infty} a_n w^{n+\alpha}, \quad a_m \neq 0,\ 0 < \lambda < 1, \tag{3.7}$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned} |w^m - u^m| &= \left|w^m - \lambda f(w) + \lambda f(w) - u^m\right| \\ &\le \left|w^m - \lambda f(w)\right| + \lambda\left|f(w) - u^m\right| \\ &< \left|w^m - \lambda f(w)\right| + \lambda|w^m - u^m| \\ &= \left|w^m + \frac{1}{a_m}\sum_{n=0,\, n\neq m}^{\infty} a_n w^{n+\alpha}\right| + \lambda|w^m - u^m| \\ &= \frac{1}{|a_m|}\left|\sum_{n=0}^{\infty} a_n w^{n+\alpha}\right| + \lambda|w^m - u^m|. \end{aligned} \tag{3.8}$$

Without loss of the generality, we consider $|a_m| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned}
|w^m - u^m| &\leq \frac{1}{|a_m|(1-\lambda)}\left|\sum_{n=0}^{\infty} a_n w^{n+\alpha}\right| \\
&\leq \frac{\epsilon}{|a_m|(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{2^n}\right) \\
&\leq \frac{\epsilon|a_m|^{p-1}}{(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right) \\
&\leq \frac{2\epsilon|a_m|^{p-1}}{(1-\lambda)} \\
&:= K\epsilon.
\end{aligned} \tag{3.9}$$

This completes the proof. □

Theorem 3.4. *If*

$$\sum_{n=0,\, n\neq 1}^{\infty} |a_n| \leq |a_1|, \quad a_1 \neq 0, \tag{3.10}$$

then (2.1) *has a unique solution in the unit disk.*

Proof. By setting

$$f(z) = \frac{-1}{a_1}\sum_{n=0,\, n\neq 1}^{\infty} a_n z^{n+\alpha}, \tag{3.11}$$

for $|z| \leq 1$, we pose

$$\begin{aligned}
\left|f(z)\right| &= \left|\frac{-1}{a_1}\sum_{n=0,\, n\neq 1}^{\infty} a_n z^{n+\alpha}\right| \\
&= \left|\frac{1}{a_1}\right|\left|\sum_{n=0,\, n\neq 1}^{\infty} a_n z^{n+\alpha}\right| \\
&\leq \left|\frac{1}{a_1}\right|\sum_{n=0,\, n\neq 1}^{\infty} |a_n| \\
&< 1.
\end{aligned} \tag{3.12}$$

Since $|f(z)| < 1$ for $|z| = 1$, hence for $|f(z)| = |-z|$ and by Rouche's theorem, we observe that $f(z) - z$ has exactly one zero in U, which yields that f has a unique fixed point in U. □

4. Conclusion

From above, we conclude that fractional differential equations of Bessel type have holomorphic solutions in the unit disk. The uniqueness imposed by employing the Rouche's theorem. Furthermore, this solution satisfied the generalized Ulam stability for infinite series of fractional power.

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