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Research Article

Upper and Lower Solution Method for Fourth-Order Four-Point Boundary Value Problem on Time Scales

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We consider a fourth-order four-point boundary value problem for dynamic equations on time scales. By the upper and lower solution method, some results on the existence of solutions of the fourth-order four-point boundary value problem on time scales are obtained. An example is also included to illustrate our results.

1. Introduction

Let $\mathbb{T}$ be a closed nonempty subset of $\mathbb{R}$, and let $\mathbb{T}$ have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. In some of the current literature, $\mathbb{T}$ is called a time scale (or measure chain). For notation, we shall use the convention that, for each interval J of $\mathbb{R}$, J will denote the time scales interval, that is, $J := J \cap \mathbb{T}$. Some preliminary definitions and theorems on time scales can be found in the books [1, 2], which are excellent references for calculus of time scales.

In this paper, let $\mathbb{T}$ be a time scale and $\sigma(t)$ the forward jump function in $\mathbb{T}$. We are concerned with the following fourth-order four-point boundary value problem on time scales $\mathbb{T}$:

$$\begin{gathered} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = f\left(t, y(\sigma(t)), y^{\Delta^2}(\sigma(t))\right), \\ y(a) = 0, \qquad y\left(\sigma^2(b)\right) + \lambda y^{\Delta}\left(\sigma^2(b)\right) = 0, \\ \zeta y^{\Delta^2}(\xi_1) - \eta y^{\Delta^3}(\xi_1) = 0, \qquad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) = 0, \end{gathered} \tag{1.1}$$

for $t \in [a,b] \subset \mathbb{T}$, $a \le \xi_1 < \xi_2 \le \sigma(b)$. We will assume that the following conditions are satisfied.

$(H1)$ $\zeta,\gamma,\eta,\delta \ge 0$, $\lambda \ge \sigma^3(b) - \sigma^2(b)$,

$(H2)$ $q(t) \ge 0$. If $q(t) \equiv 0$, then $\zeta + \gamma > 0$,

$(H3)$ $k = \zeta\delta + \eta\gamma + \zeta\gamma(\xi_2 - \xi_1) > 0$, $\eta - \zeta\xi_1 \ge 0$, $\delta - \gamma(\sigma(b) - \xi_2) \ge 0$.

The upper and lower solution method has been used to deal with the boundary value problems for dynamic equations in recent years. In most of these studies, two-point boundary value problem for second-order dynamic equations is considered [3–7].

Pang and Bai [8] studied the following fourth-order four-point BVP on time scales:

$$u^{\Delta\Delta\Delta\Delta}(t) = f\left(t, u(\sigma(t)), u^{\Delta\Delta}(t)\right), \quad t \in [0,1],$$
$$u(0) = u\left(\sigma^4(1)\right) = 0, \tag{1.2}$$
$$\alpha y^{\Delta^2}(\xi_1) - \beta y^{\Delta^3}(\xi_1) = 0, \qquad \gamma y^{\Delta^2}(\xi_2) + \eta y^{\Delta^3}(\xi_2) = 0$$

for $0 \le \xi_1 < \xi_2 \le \sigma(b)$, and $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and α,β,γ,η are nonnegative constants satisfying $\alpha\eta + \beta\gamma + \alpha\gamma(\xi_2 - \xi_1) > 0$. They establish criteria for the existence of a solution by developing the upper and lower solution method and the monotone iterative technique. Our problem is more general than the problems in [8], and our results are even new for the differential equations as well as for dynamic equations on general time scales.

2. Preliminaries

To prove the main results in this paper, we will employ several lemmas. We consider the linear boundary value problem

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = h(t), \quad t \in [\xi_1, \rho(\xi_2)],$$
$$\zeta y(\xi_1) - \eta y^{\Delta}(\xi_1) = 0, \qquad \gamma y(\xi_2) + \delta y^{\Delta}(\xi_2) = 0. \tag{2.1}$$

Denote by φ and ψ, the solutions of the corresponding homogeneous equation

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = 0, \quad t \in [\xi_1, \rho(\xi_2)] \tag{2.2}$$

under the initial conditions

$$\varphi(\xi_1) = \eta, \qquad \varphi^{\Delta}(\xi_1) = \zeta,$$
$$\psi(\xi_2) = \delta, \qquad \psi^{\Delta}(\xi_2) = -\gamma, \tag{2.3}$$

so that φ and ψ satisfy the first and second boundary conditions of (2.1), respectively. Let us set

$$D := \zeta\psi(\xi_1) - \eta\psi^{\Delta}(\xi_1) = \delta\varphi^{\Delta}(\xi_2) + \gamma\varphi(\xi_2). \tag{2.4}$$

Using the initial conditions (2.3), we can deduce from (2.2) for φ and ψ the following equations:

$$\varphi(t) = \eta + \zeta(t - \xi_1) + \int_{\xi_1}^{t}\int_{\xi_1}^{\tau} q(s)\varphi(\sigma(s))\Delta s\Delta\tau, \tag{2.5}$$

$$\psi(t) = \delta + \gamma(\xi_2 - t) + \int_{t}^{\xi_2}\int_{\tau}^{\xi_2} q(s)\psi(\sigma(s))\Delta s\Delta\tau. \tag{2.6}$$

Lemma 2.1. *Under the conditions (H1) and (H2), the following inequalities*

$$\begin{aligned} &\varphi(t) \geq 0, \quad t \in [\xi_1, \sigma(\xi_2)]; \qquad \psi(t) \geq 0, \quad t \in [\xi_1, \xi_2]; \\ &\varphi^{\Delta}(t) \geq 0, \quad t \in [\xi_1, \xi_2]; \qquad \psi^{\Delta}(t) \leq 0, \quad t \in [\xi_1, \xi_2]; \end{aligned} \tag{2.7}$$

yield.

Proof. We apply the induction principle for time scales to the statement

$$A(t) : \varphi(t) \geq 0, \qquad \varphi^{\Delta}(t) \geq 0, \tag{2.8}$$

where $t \in [\xi_1, \xi_2]$.

(I) The statement $A(\xi_1)$ is true, since $\varphi(\xi_1) = \eta$ and $\varphi^{\Delta}(\xi_1) = \zeta$.

(II) Let t be right-scattered and let $A(t)$ be true, that is, $\varphi(t) \geq 0$ and $\varphi^{\Delta}(t) \geq 0$. We need to show that $\varphi(\sigma(t)) \geq 0$ and $\varphi^{\Delta}(\sigma(t)) \geq 0$. By the definition of Δ-derivative, we have

$$\varphi(\sigma(t)) = \varphi(t) + [\sigma(t) - t]\varphi^{\Delta}(t). \tag{2.9}$$

Further, by the definition of Δ-derivative and (2.2) for $\varphi(t)$, we have

$$\varphi^{\Delta}(\sigma(t)) = \varphi^{\Delta}(t) + [\sigma(t) - t]\varphi^{\Delta^2}(t) = \varphi^{\Delta}(t) + [\sigma(t) - t]q(t)\varphi(\sigma(t)). \tag{2.10}$$

From (2.9), we get $\varphi(\sigma(t)) \geq 0$, and then from (2.10), we get $\varphi^{\Delta}(\sigma(t)) \geq 0$.

(III) Let t_0 be right-dense, $A(t_0)$ be true and $t_1 \in [\xi_1, \xi_2]$ such that $t_1 > t_0$ and is sufficiently close to t_0. We need to prove that $A(t)$ is true for $t \in [t_0, t_1]$.

From (2.2) with $y(t) = \varphi(t)$, the equations

$$\varphi^{\Delta}(t) = \varphi^{\Delta}(t_0) + \int_{t_0}^{t} q(s)\varphi(\sigma(s))\Delta s, \tag{2.11}$$

$$\varphi(t) = \varphi(t_0) + \varphi^{\Delta}(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} q(s)\varphi(\sigma(s))\Delta s\Delta\tau \tag{2.12}$$

follow. To investigate the function $\varphi(t)$ appearing in (2.12), we consider the equation

$$y(t) = \varphi(t_0) + \varphi^{\Delta}(t_0)(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{\tau} q(s)y(\sigma(s))\Delta s\Delta\tau, \tag{2.13}$$

where $y(t)$ is the desired solution. Our aim is to show that with t_1 sufficiently close to t_0, (2.13) has a unique continuous solution $y(t)$ satisfying inequality

$$y(t) \geq \varphi(t_0) + \varphi^{\Delta}(t_0)(t - t_0), \quad t \in [t_0, t_1]. \tag{2.14}$$

We solve (2.13) by the method of successive approximations, setting

$$\begin{aligned} y_0(t) &= \varphi(t_0) + \varphi^{\Delta}(t_0)(t - t_0), \\ y_j(t) &= \int_{t_0}^{t}\int_{t_0}^{\tau} q(s)y_{j-1}(\sigma(s))\Delta s\Delta\tau, \quad j = 1, 2, 3, \ldots. \end{aligned} \tag{2.15}$$

If the series $\sum_{j=0}^{\infty} y_j(t)$ converges uniformly with respect to $t \in [t_0, t_1]$, then its sum will be, obviously, a continuous solution of (2.13). To prove the uniform convergence of this series, we let

$$M_0 = \varphi(t_0) + \varphi^{\Delta}(t_0)(t_1 - t_0), \qquad M_1 = \int_{t_0}^{t_1}\int_{t_0}^{\tau} q(s)\Delta s\Delta\tau. \tag{2.16}$$

Then the estimate

$$0 \leq y_j(t) \leq M_0 M_1^j, \quad t \in [t_0, t_1], \ j = 0, 1, 2, \ldots \tag{2.17}$$

can easily be obtained. Indeed, (2.17) evidently holds for $j = 0$. Let it also hold for $j = n$. Then from (2.15), we get for $t \in [t_0, t_1]$,

$$\begin{aligned} 0 \leq y_{n+1}(t) &\leq \int_{t_0}^{t_1}\int_{t_0}^{\tau} q(s)y_n(\sigma(s))\Delta s\Delta\tau \\ &\leq \int_{t_0}^{t_1} \max_{t_0 \leq s \leq \rho(\tau)} y_n(\sigma(s)) \int_{t_0}^{\tau} q(s)\Delta s\Delta\tau \\ &\leq \max_{t_0 \leq s \leq \rho^2(t_1)} y_n(\sigma(s)) \int_{t_0}^{t_1}\int_{t_0}^{\tau} q(s)\Delta s\Delta\tau \\ &\leq M_0 M_1^n M_1 = M_0 M_1^{n+1}. \end{aligned} \tag{2.18}$$

Therefore, by the usual mathematical induction principle, (2.17) holds for all $j = 0, 1, 2, \ldots$.

Now choosing t_1 appropriately, we obtain $M_1 < 1$. Then (2.13) will have a continuous solution

$$y(t) = \sum_{j=0}^{\infty} y_j(t) \quad \text{for } t \in [t_0, t_1]. \tag{2.19}$$

Since $y_j(t) \geq 0$, it follows that $y(t) \geq y_0(t)$ thereby proving the validity of inequality (2.14). To prove uniqueness of solution of (2.13) for $t \in [t_0, t_1]$, suppose it has two solutions y_1 and y_2, and passing on to the modulus, we get

$$\begin{aligned} \left|y_1(t) - y_2(t)\right| &\leq \int_{t_0}^{t} \int_{t_0}^{\tau} q(s) \left|y_1(\sigma(s)) - y_2(\sigma(s))\right| \Delta s \Delta \tau \\ &\leq \max_{t_0 \leq s \leq \rho^2(t_1)} \left|y_1(\sigma(s)) - y_2(\sigma(s))\right| \int_{t_0}^{t_1} \int_{t_0}^{\tau} q(s) \Delta s \Delta \tau. \end{aligned} \tag{2.20}$$

Thus,

$$\left|y_1(t) - y_2(t)\right| \leq M_1 \max_{t_0 \leq s \leq \rho^2(t_1)} \left|y_1(\sigma(s)) - y_2(\sigma(s))\right|, \quad t \in [t_0, t_1]. \tag{2.21}$$

Since $M_1 < 1$, hence it follows that $y_1(t) = y_2(t)$ for $t \in [t_0, t_1]$.

From (2.12) and (2.13) in view of the uniqueness of solution, we get that $\varphi(t) = y(t)$, $t \in [t_0, t_1]$. Therefore,

$$\varphi(t) \geq \varphi(t_0) + \varphi^{\Delta}(t_0)(t - t_0), \quad t \in [t_0, t_1]. \tag{2.22}$$

Hence by making use of the induction hypothesis $A(t_0)$ being true, we obtain $\varphi(t) \geq 0$ for $t \in [t_0, t_1]$. Taking this into account, from (2.11), we also get $\varphi^{\Delta}(t) \geq 0$ for $t \in [t_0, t_1]$. Thus, $A(t)$ is true for all $t \in [t_0, t_1]$.

(IV) Let $t \in (\xi_1, \xi_2]$ and assume t is left-dense and such that $A(s)$ is true for all $s < t$, that is,

$$\varphi(s) \geq 0, \quad \varphi^{\Delta}(s) \geq 0, \quad \forall s \in [\xi_1, t). \tag{2.23}$$

Passing on here to the limit as $s \to t$, we get by the continuity of $\varphi(s)$ and $\varphi^{\Delta}(s)$ that $\varphi(t) \geq 0$ and $\varphi^{\Delta}(t) \geq 0$, thereby verifying the validity of $A(t)$.

Consequently, by the induction principle on time scales, (2.8) holds for all $t \in [\xi_1, \xi_2]$.

From (2.9) and (2.8) for $t = \xi_2$, we also get $\varphi(\sigma(\xi_2)) \geq 0$. So the statements (2.7) for φ are proved.

We can prove the statements of the lemma for ψ similarly applying the backward induction principle on time scales. The lemma is proved. □

Lemma 2.2. *Under the conditions (H1) and (H2), the inequality $D > 0$ holds.*

Proof. By (2.4) and (2.5), we have

$$D = \zeta\delta + \eta\gamma + \zeta\gamma(\xi_2 - \xi_1) + \delta\int_{\xi_1}^{\xi_2} q(s)\varphi(\sigma(s))\Delta s + \gamma\int_{\xi_1}^{\xi_2}\int_{\xi_1}^{\tau} q(s)\varphi(\sigma(s))\Delta s\Delta\tau. \tag{2.24}$$

Since $\varphi(t) \geq 0$ for $t \in [\xi_1, \sigma(\xi_2)]$, from (2.8), we have

$$D \geq \zeta\delta + \eta\gamma + \zeta\gamma(\xi_2 - \xi_1). \tag{2.25}$$

If $q(t) \equiv 0$, then in (2.25) the equality holds. From the condition $(H2)$, we get $D > 0$. This proof is completed. □

Lemma 2.3. *Assume that the conditions (H1) and (H2) are satisfied. If* $h \in C[\xi_1, \rho(\xi_2)]$, *then the boundary value problem*

$$\begin{gathered} y^{\Delta^2}(t) - q(t)y(\sigma(t)) = h(t), \quad t \in [\xi_1, \rho(\xi_2)], \\ \zeta y(\xi_1) - \eta y^{\Delta}(\xi_1) = 0, \\ \gamma y(\xi_2) + \delta y^{\Delta}(\xi_2) = 0 \end{gathered} \tag{2.26}$$

has a unique solution

$$y(t) = -\int_{\xi_1}^{\xi_2} G(t,s)h(s)\Delta s, \tag{2.27}$$

where

$$G(t,s) = \frac{1}{D}\begin{cases} \psi(\sigma(s))\varphi(t), & t \leq s, \\ \psi(t)\varphi(\sigma(s)), & t \geq \sigma(s). \end{cases} \tag{2.28}$$

Here D, φ, ψ *are as in* (2.4), (2.5), *and* (2.6), *respectively.*

Proof. Taking

$$z(t) = -\frac{1}{D}\int_{\xi_1}^{t}\left[\varphi(\sigma(s))\psi(t) - \varphi(t)\psi(\sigma(s))\right]h(s)\Delta s, \tag{2.29}$$

we have

$$\begin{aligned} z^{\Delta^2}(t) = &-\frac{1}{D}\left[\varphi(\sigma(t))\psi^{\Delta}(\sigma(t)) - \varphi^{\Delta}(\sigma(t))\psi(\sigma(t))\right]h(t) \\ &- \frac{1}{D}\int_{\xi_1}^{t}\left[\varphi(\sigma(s))\psi^{\Delta^2}(t) - \varphi^{\Delta^2}(t)\psi(\sigma(s))\right]h(s)\Delta s. \end{aligned} \tag{2.30}$$

By Corollary 3.14 in [1], since the Wronskian of any two solutions of (2.2) is independent of t, we get

$$D = W(\psi,\varphi) = \psi(\sigma(t))\varphi^{\Delta}(\sigma(t)) - \varphi(\sigma(t))\psi^{\Delta}(\sigma(t)). \tag{2.31}$$

Hence we get

$$\begin{aligned}
z^{\Delta^2}(t) &= -\frac{1}{D}\left[-Dh(t) + q(t)\int_{\xi_1}^{t}[\varphi(\sigma(s))\psi(\sigma(t)) - \varphi(\sigma(t))\psi(\sigma(s))]h(s)\Delta s\right] \\
&= h(t) + q(t)\left[-\frac{1}{D}\int_{\xi_1}^{t}[\varphi(\sigma(s))\psi(\sigma(t)) - \varphi(\sigma(t))\psi(\sigma(s))]h(s)\Delta s\right] \\
&= h(t) + q(t)\left[-\frac{1}{D}\int_{\xi_1}^{\sigma(t)}[\varphi(\sigma(s))\psi(\sigma(t)) - \varphi(\sigma(t))\psi(\sigma(s))]h(s)\Delta s\right] \\
&= h(t) + q(t)z(\sigma(t)).
\end{aligned} \tag{2.32}$$

So the general solution of equation

$$y^{\Delta^2}(t) - q(t)y(\sigma(t)) = h(t), \quad t \in [\xi_1, \rho(\xi_2)], \tag{2.33}$$

has the form

$$y(t) = c_1\varphi(t) + c_2\psi(t) - \frac{1}{D}\int_{\xi_1}^{t}[\varphi(\sigma(s))\psi(t) - \varphi(t)\psi(\sigma(s))]h(s)\Delta s, \tag{2.34}$$

where c_1 and c_2 are arbitrary constants. Substituting this expression for $y(t)$ in the boundary conditions of BVP (2.26), we can evaluate c_1 and c_2. After some easy calculations, we can get (2.27) and (2.28). □

Lemma 2.4. *Under the conditions (H1) and (H2), the Green's function of BVP* (2.26) *possesses the following property:*

$$G(t,s) \geq 0, \quad (t,s) \in [\xi_1, \xi_2] \times [\xi_1, \rho(\xi_2)]. \tag{2.35}$$

Proof. The lemma follows from (2.28), Lemmas 2.1 and 2.2 immediately. □

Lemma 2.5. *Assume that the conditions (H1) and (H2) are satisfied. If $h \in C[a,b]$, then the boundary value problem*

$$\begin{gathered}
y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = h(t), \quad t \in [a,b], \\
y(a) = 0, \qquad y\left(\sigma^2(b)\right) + \lambda y^{\Delta}\left(\sigma^2(b)\right) = 0, \\
\zeta y^{\Delta^2}(\xi_1) - \eta y^{\Delta^3}(\xi_1) = 0, \qquad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) = 0
\end{gathered} \tag{2.36}$$

has a unique solution

$$y(t) = \int_a^{\sigma^2(b)} G_1(t,\xi) \int_{\xi_1}^{\xi_2} G_2(\xi,s)h(s)\Delta s \Delta\xi, \tag{2.37}$$

where

$$G_1(t,s) = \frac{1}{\sigma^2(b) - a + \lambda} \begin{cases} \left(\sigma^2(b) - \sigma(s) + \lambda\right)(t-a), & t \le s, \\ \left(\sigma^2(b) - t + \lambda\right)(\sigma(s) - a), & t \ge \sigma(s), \end{cases} \tag{2.38}$$

$$G_2(t,s) = \frac{1}{D} \begin{cases} \psi(\sigma(s))\varphi(t), & t \le s, \\ \psi(t)\varphi(\sigma(s)), & t \ge \sigma(s). \end{cases} \tag{2.39}$$

Here D, φ, ψ *are as in* (2.4), (2.5), *and* (2.6), *respectively.*

Proof. Let us consider the following BVP:

$$\begin{gathered} y^{\Delta^2}(t) = -\int_{\xi_1}^{\xi_2} G_2(t,s)h(s)\Delta s, \quad t \in \left[a, \sigma^2(b)\right], \\ y(a) = 0, \qquad y\left(\sigma^2(b)\right) + \lambda y^{\Delta}\left(\sigma^2(b)\right) = 0. \end{gathered} \tag{2.40}$$

The Green's function associated with the BVP (2.40) is $G_1(t,s)$. This completes the proof. □

Lemma 2.6. *Assume that the conditions (H1)–(H3) are satisfied. If* y *satisfies*

$$\begin{gathered} y^{\Delta^4} - q(t)y^{\Delta^2}(\sigma(t)) \ge 0, \quad t \in [a,b], \\ y(a) \ge 0, \qquad y\left(\sigma^2(b)\right) + \lambda y^{\Delta}\left(\sigma^2(b)\right) \ge 0, \\ \zeta y^{\Delta^2}(\xi_1) - \eta y^{\Delta^3}(\xi_1) \le 0, \qquad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) \le 0, \end{gathered} \tag{2.41}$$

then $y(t) \ge 0,\ t \in [a, \sigma^2(b)]$ *and* $y^{\Delta^2}(t) \le 0,\ t \in [a, \sigma(b)]$.

Proof. Let

$$\begin{gathered} y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = h(t), \quad t \in [a,b], \\ y(a) = t_0, \qquad y\left(\sigma^2(b)\right) + \lambda y^{\Delta}\left(\sigma^2(b)\right) = t_1, \\ \zeta y^{\Delta^2}(\xi_1) - \eta y^{\Delta^3}(\xi_1) = t_2, \qquad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) = t_3, \end{gathered} \tag{2.42}$$

where $t_0 \ge 0,\ t_1 \ge 0,\ t_2 \le 0,\ t_3 \le 0,\ h \ge 0$.

It is easy to check that y and y^{Δ^2} can be given by the expression

$$y(t) = S(t) - \int_a^{\sigma^2(b)} G_1(t,\xi)R(\xi)\Delta\xi + \int_a^{\sigma^2(b)} G_1(t,\xi)\int_{\xi_1}^{\xi_2} G_2(\xi,s)h(s)\Delta s\Delta\xi,$$
$$y^{\Delta^2}(t) = R(t) - \int_{\xi_1}^{\xi_2} G_2(t,s)h(s)\Delta s, \tag{2.43}$$

where

$$S(t) = \frac{1}{\sigma^2(b) - a + \lambda}\left[\left(\sigma^2(b) + \lambda - t\right)t_0 + (t-a)t_1\right],$$
$$R(t) = \frac{1}{k}\left[(\zeta(t-\xi_1) + \eta)t_3 + (\gamma(\xi_2 - t) + \delta)t_2\right], \tag{2.44}$$

and $G_1(t,s)$, $G_2(t,s)$are as in (2.38) and (2.39), respectively. The hypothesis of the lemma implies that $S(t) \geq 0$ for $t \in [a,\sigma^2(b)]$, $R(t) \leq 0$ for $t \in [a,\sigma(b)]$, $G_1(t,s) \geq 0$ for $(t,s) \in [a,\sigma^2(b)] \times [a,\sigma(b)]$, and $G_2(t,s) > 0$ for $(t,s) \in [a,\sigma(b)] \times [a,b]$. Therefore, we get $y(t) \geq 0$ for $t \in [a,\sigma^2(b)]$ and $y^{\Delta^2}(t) \leq 0$ for $t \in [a,\sigma(b)]$. The proof is completed. □

3. Upper and Lower Solution Method

In this section, we present existence results for the BVP (1.1) by using the method of upper and lower solutions. We define the set

$$D := \left\{y : y^{\Delta^n} \in C\left[a,\sigma^4(b)\right]^{k^n},\ k = 0,1,2,3,4\right\}. \tag{3.1}$$

Definition 3.1. Letting $\alpha(t) \in D$ on $[a,\sigma^4(b)]$, we say α is a lower solution for the problem (1.1) if α satisfies

$$\alpha^{\Delta^4}(t) - q(t)\alpha^{\Delta^2}(\sigma(t)) \leq f\left(t,\alpha(\sigma(t)),\alpha^{\Delta^2}(\sigma(t))\right),$$
$$\alpha(a) \leq 0, \qquad \alpha\left(\sigma^2(b)\right) + \lambda\alpha^{\Delta}\left(\sigma^2(b)\right) \leq 0, \tag{3.2}$$
$$\zeta\alpha^{\Delta^2}(\xi_1) - \eta\alpha^{\Delta^3}(\xi_1) \geq 0, \qquad \gamma\alpha^{\Delta^2}(\xi_2) + \delta\alpha^{\Delta^3}(\xi_2) \geq 0.$$

Definition 3.2. Letting $\beta(t) \in D$, on $[a,\sigma^4(b)]$, we say β is an upper solution for the problem (1.1) if β satisfies

$$\beta^{\Delta^4}(t) - q(t)\beta^{\Delta^2}(\sigma(t)) \geq f\left(t,\beta(\sigma(t)),\beta^{\Delta^2}(\sigma(t))\right),$$
$$\beta(a) \geq 0, \qquad \beta\left(\sigma^2(b)\right) + \lambda\beta^{\Delta}\left(\sigma^2(b)\right) \geq 0, \tag{3.3}$$
$$\zeta\beta^{\Delta^2}(\xi_1) - \eta\beta^{\Delta^3}(\xi_1) \leq 0, \qquad \gamma\beta^{\Delta^2}(\xi_2) + \delta\beta^{\Delta^3}(\xi_2) \leq 0.$$

We assume that the function $f(t, y, y^{\Delta^2})$ satisfies the following condition.

(H4) $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$f(t, y_2, z) - f(t, y_1, z) \geq 0, \quad \text{for } \alpha(\sigma(t)) \leq y_1 \leq y_2 \leq \beta(\sigma(t)),\ z \in \mathbb{R},\ t \in [a, b],$$
$$f(t, y, z_2) - f(t, y, z_1) \leq 0, \quad \text{for } \beta^{\Delta^2}(\sigma(t)) \leq z_1 \leq z_2 \leq \alpha^{\Delta^2}(\sigma(t)),\ y \in \mathbb{R},\ t \in [a, b], \tag{3.4}$$

where α, β are lower and upper solutions, respectively, for the BVP (1.1), and satisfy $\alpha \leq \beta$, $\alpha^{\Delta^2} \geq \beta^{\Delta^2}$.

Theorem 3.3. *Assume that the conditions (H1)–(H4) are satisfied. Then the problem* (1.1) *has a solution $y(t)$ with*

$$\alpha(t) \leq y(t) \leq \beta(t), \qquad \alpha^{\Delta^2}(t) \geq y^{\Delta^2}(t) \geq \beta^{\Delta^2}(t) \tag{3.5}$$

for $t \in [a, \sigma^3(b)]$ and $t \in [a, \sigma(b)]$, respectively.

Proof. Consider the BVP,

$$y^{\Delta^4}(t) - q(t)y^{\Delta^2}(\sigma(t)) = F\Big(t, y(\sigma(t)), y^{\Delta^2}(\sigma(t))\Big), \quad t \in [a, b],$$
$$y(a) = 0, \qquad y\Big(\sigma^2(b)\Big) + \lambda y^{\Delta}\Big(\sigma^2(b)\Big) = 0, \tag{3.6}$$
$$\zeta y^{\Delta^2}(\xi_1) - \eta y^{\Delta^3}(\xi_1) = 0, \qquad \gamma y^{\Delta^2}(\xi_2) + \delta y^{\Delta^3}(\xi_2) = 0,$$

where

$$F(t, x, y) = \begin{cases} f^*(t, \alpha(\sigma(t)), y), & x < \alpha(\sigma(t)), \\ f^*(t, x, y), & \alpha(\sigma(t)) \leq x \leq \beta(\sigma(t)), \\ f^*(t, \beta(\sigma(t)), y), & x > \beta(\sigma(t)), \end{cases}$$
$$f^*(t, x, y) = \begin{cases} f\Big(t, x, \beta^{\Delta^2}(\sigma(t))\Big), & y < \beta^{\Delta^2}(\sigma(t)), \\ f(t, x, y), & \beta^{\Delta^2}(\sigma(t)) \leq y \leq \alpha^{\Delta^2}(\sigma(t)), \\ f\Big(t, x, \alpha^{\Delta^2}(\sigma(t))\Big), & y > \alpha^{\Delta^2}(\sigma(t)). \end{cases} \tag{3.7}$$

By Lemma 2.5, it is clear that the solutions of the BVP (3.6) are the fixed points of the operator

$$Ay(t) = \int_a^{\sigma^2(b)} G_1(t, \xi) \int_{\xi_1}^{\xi_2} G_2(\xi, s) F\Big(s, y(\sigma(s)), y^{\Delta^2}(\sigma(s))\Big) \Delta s \Delta \xi, \quad t \in \Big[a, \sigma^4(b)\Big], \tag{3.8}$$

where $G_1(t,s)$ and $G_2(t,s)$ are as in (2.38) and (2.39), respectively. It is clear that A is continuous. Since the function $f(t,y,y^{\Delta^2})$ satisfies the conditions (3.4),

$$f\Big(t,\alpha(\sigma(t)),\alpha^{\Delta^2}(\sigma(t))\Big) \le F\Big(t,y,y^{\Delta^2}\Big) \le f\Big(t,\beta(\sigma(t)),\beta^{\Delta^2}(\sigma(t))\Big) \quad \text{for } t\in[a,b]. \tag{3.9}$$

Thus, there exists a positive constant M such that $|F(t,y,y^{\Delta^2})| \le M$, which implies that the operator A is uniformly bounded. Moreover, the operator A is equicontinuous. Therefore, from the Arzela-Ascoli theorem, the operator A is a compact operator. Thus, by Schauder's fixed point theorem, there exists a solution y of the BVP (3.6).

Suppose y^* is a solution of the BVP (3.6). Since $f(t,y,y^{\Delta^2})$ satisfies the conditions (3.4), we know that

$$f\Big(t,\alpha(\sigma(t)),\alpha^{\Delta^2}(\sigma(t))\Big) \le F\Big(t,y^*,y^{*\Delta^2}\Big) \le f\Big(t,\beta(\sigma(t)),\beta^{\Delta^2}(\sigma(t))\Big) \quad \text{for } t\in[a,b]. \tag{3.10}$$

Thus,

$$\begin{gathered}
\begin{aligned}
(\beta-y^*)^{\Delta^4}(t) &- q(t)(\beta-y^*)^{\Delta^2}(\sigma(t)) \\
&\ge f(t,\beta(\sigma(t)),\beta^2(\sigma(t))) \\
&\quad -F\Big(t,y^*(\sigma(t)),y^{\Delta^2}(\sigma(t))\Big) \ge 0, \quad t\in[a,b],
\end{aligned}\\
(\beta-y^*)(a)\ge 0, \qquad (\beta-y^*)\Big(\sigma^2(b)\Big)+\lambda(\beta-y^*)^{\Delta}\Big(\sigma^2(b)\Big)\ge 0,\\
\zeta(\beta-y^*)^{\Delta^2}(\xi_1)-\eta(\beta-y^*)^{\Delta^3}(\xi_1)\le 0, \qquad \gamma(\beta-y^*)^{\Delta^2}(\xi_2)+\delta(\beta-y^*)^{\Delta^3}(\xi_2)\le 0.
\end{gathered} \tag{3.11}$$

By virtue of Lemma 2.6, $y^*(t) \le \beta(t)$ for $t\in[a,\sigma^2(b)]$ and $y^{*\Delta^2}(t) \ge \beta^{\Delta^2}(t)$ for $t\in[a,\sigma(b)]$. If $\sigma^2(b)$ is right-scattered, by using the inequality $(\beta-y^*)(\sigma^2(b))+\lambda(\beta-y^*)^{\Delta}(\sigma^2(b))\ge 0$, we get the inequality $(\lambda/(\sigma^3(b)-\sigma^2(b)))(\beta-y^*)(\sigma^3(b)) \ge ((\lambda/(\sigma^3(b)-\sigma(b)))-1)(\beta-y^*)(\sigma^2(b))$. So we get $y^*(t)\le\beta(t)$ on $[a,\sigma^3(b)]$. If $\sigma^2(b)$ is right-dense, it is trivial that the inequality $y^*(t)\le\beta(t)$ holds on $[a,\sigma^3(b)]$. Similarly, one can show that $\alpha(t)\le y^*(t)$ for $t\in[a,\sigma^3(b)]$ and $\alpha^{\Delta^2}(t)\ge y^{*\Delta^2}(t)$ for $t\in[a,\sigma(b)]$. This completes the proof. □

Theorem 3.4. *Assume that the conditions (H1)–(H4) are satisfied. Then there exist two monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$, *nonincreasing and nondecreasing, respectively, with* $\alpha_0=\alpha$ *and* $\beta_0=\beta$, *which converge to the extremal solutions in* $[\beta,\alpha]$ *of the problem* (1.1).

Proof. For any function ϑ which satisfies $\alpha(t)\le\vartheta(t)\le\beta(t)$ for $t\in[a,\sigma^2(b)]$, consider the following problem:

$$\begin{gathered}
y^{\Delta^4}(t)-q(t)y^{\Delta^2}(\sigma(t)) = f\Big(t,\vartheta(\sigma(t)),\vartheta^{\Delta^2}(\sigma(t))\Big), \quad t\in[a,b],\\
y(a)=0, \qquad y\Big(\sigma^2(b)\Big)+\lambda y^{\Delta}\Big(\sigma^2(b)\Big)=0,\\
\zeta y^{\Delta^2}(\xi_1)-\eta y^{\Delta^3}(\xi_1)=0, \qquad \gamma y^{\Delta^2}(\xi_2)+\delta y^{\Delta^3}(\xi_2)=0.
\end{gathered} \tag{3.12}$$

Clearly, this problem is type (2.12). Obviously, it has a unique solution given by the expression

$$y(t)=\int_a^{\sigma^2(b)} G_2(t,\xi)\int_{\xi_1}^{\xi_2} G_1(\xi,s) f\Big(s,\vartheta(\sigma(s)),\vartheta^{\Delta^2}(\sigma(s))\Big)\Delta s\Delta\xi \equiv A\vartheta(t),\quad t\in\Big[a,\sigma^4(b)\Big]. \tag{3.13}$$

Step 1. $\alpha \leq A\alpha,\ A\beta \leq \beta$ on $[a,\sigma^3(b)]$ and $\alpha^{\Delta^2} \geq (A\alpha)^{\Delta^2},\ (A\beta)^{\Delta^2} \geq \beta^{\Delta^2}$ on $[a,\sigma(b)]$.

Let $A\alpha(t) = w(t)$. Thus

$$\begin{gathered}
\begin{aligned}
&(w-\alpha)^{\Delta^4}(t) - q(t)(w-\alpha)^{\Delta^2}(\sigma(t))\\
&\quad \geq f\Big(t,\alpha(\sigma(t)),\alpha^{\Delta^2}(\sigma(t))\Big)\\
&\qquad - f\Big(t,\alpha(\sigma(t)),\alpha^{\Delta^2}(\sigma(t))\Big) = 0,\quad t\in[a,b],
\end{aligned}\\
(w-\alpha)(a)\geq 0,\qquad (w-\alpha)\Big(\sigma^2(b)\Big)+\lambda(w-\alpha)^{\Delta}\Big(\sigma^2(b)\Big)\geq 0,\\
\zeta(w-\alpha)^{\Delta^2}(\xi_1)-\eta(w-\alpha)^{\Delta^3}(\xi_1)\leq 0,\qquad \gamma(w-\alpha)^{\Delta^2}(\xi_2)+\delta(w-\alpha)^{\Delta^3}(\xi_2)\leq 0.
\end{gathered} \tag{3.14}$$

Using Lemma 2.6, we obtain that $\alpha \leq A\alpha$, $\alpha^{\Delta^2} \geq (A\alpha)^{\Delta^2}$ on $[a,\sigma^2(b)]$ and $[a,\sigma(b)]$, respectively. In case that $\sigma^2(b)$ is right scattered, the assumptions $\lambda \geq \sigma^3(b)-\sigma^2(b)$ and $(w-\alpha)(\sigma^2(b)) + (w-\alpha)^{\Delta}(\sigma^2(b)) \geq 0$ imply that $\alpha(\sigma^3(b)) \leq (A\alpha)(\sigma^3(b))$. Similarly, we show that $(A\beta)(t) \leq \beta(t),\ (A\beta)^{\Delta^2}(t) \geq \beta^{\Delta^2}(t)$ on $[a,\sigma^3(b)]$ and $[a,\sigma(b)]$, respectively.

Step 2. If $y_1,y_2 \in [\alpha,\beta]$; $y_1^{\Delta^2}, y_2^{\Delta^2} \in [\beta^{\Delta^2},\alpha^{\Delta^2}]$, $y_1 \leq y_2$ and $y_2^{\Delta^2} \leq y_1^{\Delta^2}$, then we have $Ay_1 \leq Ay_2$ and $(Ay_1)^{\Delta^2} \geq (Ay_2)^{\Delta^2}$.

Let $Ay_1 = w_1$ and $Ay_2 = w_2$. Thus

$$\begin{gathered}
\begin{aligned}
&(w_2-w_1)^{\Delta^4}(t) - q(t)(w_2-w_1)^{\Delta^2}(\sigma(t))\\
&\quad = f\Big(t,y_2(\sigma(t)),y_2^{\Delta^2}(\sigma(t))\Big)\\
&\qquad - f\Big(t,y_1(\sigma(t)),y_1^{\Delta^2}(\sigma(t))\Big) \geq 0,\quad t\in[a,b],
\end{aligned}\\
(w_2-w_1)(a)=0,\qquad (w_2-w_1)\Big(\sigma^3(b)\Big)+\lambda(w_2-w_1)^{\Delta}\Big(\sigma^3(b)\Big)=0,\\
\zeta(w_2-w_1)^{\Delta^2}(\xi_1)-\eta(w_2-w_1)^{\Delta^3}(\xi_1)=0,\qquad \gamma(w_2-w_1)^{\Delta^2}(\xi_2)+\delta(w_2-w_1)^{\Delta^3}(\xi_2)=0.
\end{gathered} \tag{3.15}$$

Hence we get that $Ay_1(t) \leq Ay_2(t)$ and $(Ay_1)^{\Delta^2}(t) \geq (Ay_2)^{\Delta^2}(t)$ on $[a,\sigma^3(b)]$ and $[a,\sigma(b)]$, respectively.

Now, we define the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ by

$$\alpha_0(t)=\alpha(t),\quad \alpha_{n+1}(t)=A\alpha_n(t),\quad \beta_0(t)=\beta(t),\quad \beta_{n+1}(t)=A\beta_n(t),\quad \text{for } n\geq 0. \tag{3.16}$$

From the properties of A, we have

$$\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0 = \beta,$$
$$\alpha^{\Delta^2} = \alpha_0^{\Delta^2} \leq \alpha_1^{\Delta^2} \leq \cdots \leq \beta_n^{\Delta^2} \leq \cdots \leq \beta_1^{\Delta^2} \leq \beta_0^{\Delta^2} = \beta^{\Delta^2}. \tag{3.17}$$

But then α^* and β^* defined by

$$\alpha^* = \lim_{n\to\infty} \alpha_n, \qquad \beta^* = \lim_{n\to\infty} \beta_n, \qquad {\alpha^*}^{\Delta^2} = \lim_{n\to\infty} \alpha_n^{\Delta^2}, \qquad {\beta^*}^{\Delta^2} = \lim_{n\to\infty} \beta_n \tag{3.18}$$

are extremal solutions of (1.1).

□

Example 3.5. Consider the BVP

$$y^{\Delta^4}(t) - t^2 y^{\Delta^2}(\sigma(t)) = -y^{\Delta^2}(\sigma(t)), \quad t \in \left[\frac{3}{2}, 4\right],$$
$$y\left(\frac{3}{2}\right) = 0, \qquad y\left(\sigma^2(4)\right) + y^{\Delta}\left(\sigma^2(4)\right) = 0, \tag{3.19}$$
$$y^{\Delta^2}(2) - 3y^{\Delta^3}(2) = 0, \qquad 7y^{\Delta^2}(3) + 8y^{\Delta^3}(3) = 0,$$

where $\sigma(4) \leq 29/7$ and $\sigma^3(4) - \sigma^2(4) \leq 1$. It is easy to check that $\alpha(t) = 0$, $\beta(t) = t$ are lower and upper solutions of the BVP (3.19), respectively, and that all assumptions of Theorem 3.3 are fulfilled. So the BVP (3.19) has a solution $y(t)$ satisfying $0 \leq y(t) \leq t$ for $t \in [3/2, \sigma^3(4)]$, $y^{\Delta^2}(t) = 0$ for $t \in [3/2, \sigma(4)]$.

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