Research Article

Composite Tracking Control for Generalized Practical Synchronization of Duffing-Holmes Systems with Parameter Mismatching, Unknown External Excitation, Plant Uncertainties, and Uncertain Deadzone Nonlinearities

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The generalized practical synchronization (GPS) of uncertain Duffing-Holmes chaotic systems with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities is investigated. Based on the composite control approach, a tracking control is derived to realize the GPS for the uncertain Duffing-Holmes chaotic systems with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities. Besides, the guaranteed exponential decay rate, convergence radius, and desired scaling factor can be prespecified. Finally, numerical simulations are provided to illustrate the feasibility and effectiveness of the proposed GPS scheme.

1. Introduction

Recently, synchronizations of various dynamic systems or chaotic systems have been intensively studied; see, for instance, [1–6] and the references therein. Occasionally, chaos in many systems is a source of the generation of oscillation. Chaos synchronizations exist in certain fields of application, such as, secure communication, ecological systems, and system identification.

It is well known that there inevitably exist nonlinearities in electric components, such as, operational amplifier, resistor, inductors, capacitor, and electromechanical actuator. Furthermore, the control schemes of controlled driver-response chaotic systems can be realized by various electric components. Input nonlinearities not only often appear in the controlled driver-response chaotic systems but also frequently cause undesirable behavior,

such as, instabilities or spurious limit cycles. Over the past decades, researchers have been concerned with several input nonlinearities common in dynamical systems, such as, deadzones, saturation, hysteresis, relays, and others; see, for instance, [7–10]. Generally speaking, the synchronization of chaotic system with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities is in general not as simple as that without any uncertainties and nonlinearities.

Over the past decades, Duffing systems have been received a great deal of interest due to theoretical interests and successful applications in numerous areas. In [11], the virtual stabilizability of Duffing-Holmes control systems has been studied and a tracking control has been proposed such that the states of Duffing-Holmes control system track the desired trajectories. In addition, a harmonic balance method in conjunction with the successive integration technique has been offered in [12] to solve the Duffing oscillator equation with damping and excitation. In this paper, the concept of the GPS for chaotic systems is presented and the GPS of uncertain Duffing-Holmes chaotic systems with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities is explored. Using the composite control strategy, a tracking control is offered to realize the GPS for uncertain Duffing-Holmes chaotic systems, with any prespecified exponential decay rate, convergence radius, and desired scaling factor.

2. Problem Formulation and Main Result

Before presenting the problem formulation, let us introduce a definition as follows.

Definition 2.1. The deadzone nonlinearities $D(u, \overline{d}, m)$, with $\overline{d} \geq 0$ and $m > 0$, is defined to be the collection of all functions $\Delta\phi(u) := \Re \rightarrow \Re$ satisfying

$$
\Delta\phi(u) := \begin{cases} m(u - \Delta\overline{r}), & \text{if } u \geq \Delta\overline{r}, \\ 0, & \text{if } -\Delta\underline{r} \leq u \leq \Delta\overline{r}, \\ m(u + \Delta\underline{r}) & \text{if } u \leq -\Delta\underline{r}, \end{cases} \tag{2.1}
$$

for any $\Delta\overline{r}$ and $\Delta\underline{r}$ with $0 \leq \Delta\underline{r},\ \Delta\overline{r} \leq \overline{d}$.

In this paper, we consider the following uncertain Duffing-Holmes chaotic systems with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities described as

Driver system:

$$
\ddot{r} + p_1 r + p_2 \dot{r} + r^3 + p_3 \cos(wt) + \Delta f_1(t, r, \dot{r}) = 0, \quad t \geq 0,
$$
$$
[r(0)\ \ \dot{r}(0)] = [r_{i1}\ \ r_{i2}], \tag{2.2}
$$

Response system:

$$
\ddot{q} + p_1 q + p_2 \dot{q} + \Delta f_2(t, q, \dot{q}) + \Delta\phi(u) = 0, \quad t \geq 0,
$$
$$
[q(0)\ \ \dot{q}(0)] = [q_{i1}\ \ q_{i2}], \tag{2.3}
$$

where r, $q \in \Re$, Δf_1 and Δf_2, for all $i \in \{1, 2\}$, represent the mixed uncertainties (parameter mismatching, unknown external excitation, and plant uncertainties), $[r_{i1} \quad r_{i2}]$ and $[q_{i1} \quad q_{i2}]$ represent the initial conditions, $u \in \Re$ is the control input, and $\Delta \phi(u) \in D(u, \bar{d}, m)$ is the uncertain deadzone nonlinearity. For the existence of the solutions of (2.2), we assume that the uncertainties Δf_1 and Δf_2 are continuous. It is noted that the system (2.2) without any uncertainties (i.e., $\Delta f_1 = \Delta f_2 = 0$) displays chaotic behavior for certain values of the parameters [13]. Letting

$$
\begin{aligned}
x(t) &:= \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T := \begin{bmatrix} r(t) & \dot{r}(t) \end{bmatrix}^T, \\
z(t) &:= \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T := \begin{bmatrix} q(t) & \dot{q}(t) \end{bmatrix}^T,
\end{aligned}
\tag{2.4}
$$

the corresponding state-space equations of system (2.2) and system (2.3) are

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -p_1 x_1 - p_2 x_2 - x_1^3 - p_3 \cos(wt) - \Delta f_1(t, x_1, x_2), \quad \forall t \geq 0,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= -p_1 z_1 - p_2 z_2 - \Delta f_2(t, z_1, z_2) - \Delta \phi(u), \quad \forall t \geq 0.
\end{aligned}
\tag{2.6}
$$

Throughout this paper, an assumption is made as follows.

(A1) There exist two continuous functions $g_1(t, r, \dot{r}) \geq 0$ and $g_2(t, q, \dot{q}) \geq 0$ such that, for all arguments,

$$
\left| \Delta f_1(t, r, \dot{r}) \right| \leq g_1(t, r, \dot{r}), \qquad \left| \Delta f_2(t, q, \dot{q}) \right| \leq g_2(t, q, \dot{q}).
\tag{2.7}
$$

For brevity, let us define the synchronous error vector as

$$
e(t) := \begin{bmatrix} e_1(t) & e_2(t) \end{bmatrix}^T := z(t) - \beta x(t),
\tag{2.8}
$$

where $\beta \in \Re$ is the scaling factor.

The purpose of this paper is to search a composite control $u = u_1 + u_2$ such that the states z_1 and z_2 of the response system (2.6) track, respectively, the signals βx_1 and βx_2 of the driver system (2.5), with any desired scaling factor β.

The precise definition of the GPS, in terms of error response, is given as follows.

Definition 2.2. The driver-response dynamical systems (2.5) and (2.6) are said to realize the GPS, provided that, for any $\alpha > 0$, $\varepsilon > 0$, and $\beta \in \Re$, there exists a tracking control $u(t) := u(\alpha, \varepsilon, \beta)$ and positive number k such that the synchronous error satisfies

$$
\|e(t)\| \leq k \cdot e^{-\alpha t} + \varepsilon, \quad \forall t \geq 0.
\tag{2.9}
$$

In this case, the positive number ε is called the convergence radius, the positive number α is called the exponential decay rate, and β is called the desired scaling factor. In other words, the

GPS means that the states z_1 and z_2 of the response system can track, respectively, the signals βx_1 and βx_2 of the driver system, with any prespecified exponential decay rate, convergence radius, and desired scaling factor.

From (2.5)–(2.8), we have the following error dynamical system:

$$
\begin{aligned}
\dot{e} &= \begin{bmatrix} 0 & 1 \\ -p_1 & -p_2 \end{bmatrix} e + B \begin{bmatrix} 0 \\ -\Delta\phi - \Delta f_2 + \beta x_1^3 + \beta p_3 \cos(wt) + \beta \Delta f_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -p_1 & -p_2 \end{bmatrix} e - mBu + B \begin{bmatrix} 0 \\ \Delta f \end{bmatrix}, \quad \forall t \geq 0,
\end{aligned}
\tag{2.10}
$$

with $B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\Delta f := [-\Delta\phi(u) + mu] + \beta x_1^3 + \beta p_3 \cos(wt) + \beta \Delta f_1 - \Delta f_2$. First consider the case of the system (2.10) without any parameter mismatching, unknown external excitation, and plant uncertainties, that is, $\Delta f = 0$. Thus the nominal system of (2.10) can represented as

$$
\dot{e} = \begin{bmatrix} 0 & 1 \\ -p_1 & -p_2 \end{bmatrix} e - mBu.
\tag{2.11}
$$

According to the linear system theory, it can be easily obtained that given any positive constant α, the system (2.11) subjected to the linear control law:

$$
u(t) = Ke(x) = \begin{bmatrix} \dfrac{(\alpha+1)^2 - p_1}{m} & \dfrac{2(\alpha+1) - p_2}{m} \end{bmatrix} e(t),
\tag{2.12}
$$

is globally exponentially stable with guaranteed exponential decay rate α. Nevertheless, the nominal control law (2.12) may not suffice to render the uncertain error system (2.10) to be globally exponentially stable. Hence a corrective control term must be added to overcome the uncertain part of (2.10). In the following, a composite control, consisting of a nominal control and a corrective control, is proposed such that the GPS between systems (2.5) and (2.6) can be guaranteed.

Now we present the main result for the GPS between system (2.5) and system (2.6).

Theorem 2.3. *The GPS between system (2.5) and system (2.6) can be achieved under the composite control $u(t) := u(\alpha, \varepsilon, \beta)$ defined by*

$$
u(t) = u_1(t) + u_2(t),
\tag{2.13}
$$

where

$$
u_1(t) = Ke(x) = \begin{bmatrix} \dfrac{(\alpha+1)^2 - p_1}{m} & \dfrac{2(\alpha+1) - p_2}{m} \end{bmatrix} e(t),
\tag{2.14}
$$

$$
u_2(t) = \frac{h^2(t)}{m \cdot h(t) \cdot |p_{21}e_1 + p_{22}e_2| + \alpha \cdot m \cdot \varepsilon^2 \cdot \lambda_{\min}(P)} (p_{21}e_1 + p_{22}e_2),
\tag{2.15}
$$

where

$$h(t) = m\overline{d} + |\beta| \cdot g_1 + g_2 + \left|\beta x_1^3\right| + |\beta p_3|, \tag{2.16}$$

and $P := |p_{ij}|_{2 \times 2} > 0$ is the unique solution to the following Lyapunov equation:

$$(A + \alpha I)^T P + P(A + \alpha I) = -2I, \tag{2.17}$$

with $A := \begin{bmatrix} 0 & 1 \\ -(\alpha+1)^2 & -2(\alpha+1) \end{bmatrix}$.

Proof. From (2.10), (2.13), and (2.14), the dynamical error system can be performed

$$\begin{aligned}
\dot{e} &= \begin{bmatrix} 0 & 1 \\ -p_1 & -p_2 \end{bmatrix} e - mB(u_1 + u_2) + B\begin{bmatrix} 0 \\ \Delta f \end{bmatrix} \\
&= Ae - mBu_2 + B\begin{bmatrix} 0 \\ \Delta f \end{bmatrix}, \quad \forall\, t \geq 0,
\end{aligned} \tag{2.18}$$

with $B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\Delta f := [mu - \Delta\phi(u)] + \beta x_1^3 + \beta p_3 \cos(wt) + \beta \Delta f_1 - \Delta f_2$.

Obviously, one has $\sigma(A) = \{-\alpha - 1\}$, $\sigma(A + \alpha I) = \{-1\}$, and $|\Delta f| \leq h(t)$. Let

$$W(e(t)) = e^T(t)Pe(t). \tag{2.19}$$

The time derivative of $W(e(t))$ along the trajectories of dynamical error system, with (2.11)–(2.19) and (A1), is given by

$$\begin{aligned}
\dot{W}(e(t)) &= \dot{e}^T Pe + e^T P\dot{e} \\
&= [Ae + B(-mu_2 + \Delta f)]^T Pe + e^T P[Ae + B(-mu_2 + \Delta f)] \\
&= e^T\left(A^T P + PA\right)e - 2m \cdot e^T PBu_2 + 2e^T PB \cdot \Delta f \\
&= e^T(-2\alpha P - 2I)e - 2m \cdot e^T PBu_2 + 2e^T PB \cdot \Delta f \\
&\leq -2\alpha W - 2m \cdot e^T PBu_2 + 2\left|e^T PB\right| \cdot |\Delta f| \\
&\leq -2\alpha W - 2\frac{h^2(t)}{h \cdot |B^T Pe(t)| + \alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P)}\left|e^T PB\right|^2 + 2\left|e^T PB\right| \cdot h \\
&= -2\alpha W + 2\frac{\alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P) \cdot h \cdot |e^T PB|}{h \cdot |B^T Pe(t)| + \alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P)} \\
&= -2\alpha W + 2\alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P), \quad \forall t \geq 0,
\end{aligned} \tag{2.20}$$

in view of $|\Delta f(t)| \leq h(t)$ and $p_{21}e_1 + p_{22}e_2 = B^T Pe = e^T PB$. Thus, one has

$$e^{2\alpha_1 t} \cdot \dot{W} + e^{2\alpha t} \cdot 2\alpha W = \frac{d}{dt}\left[e^{2\alpha t} \cdot W\right] \leq e^{2\alpha t} \cdot 2\alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P), \quad \forall t \geq 0 \tag{2.21}$$

It follows that

$$
\int_0^t \frac{d}{dt}\Big[e^{2\alpha t} \cdot W(e(t))\Big] dt = e^{2\alpha t} \cdot W(e(t)) - W(e(0))
$$

$$
\leq \int_0^t e^{2\alpha t} \cdot 2\alpha \cdot \varepsilon^2 \cdot \lambda_{\min}(P) dt \tag{2.22}
$$

$$
= \varepsilon^2 \cdot \lambda_{\min}(P)\Big(e^{2\alpha t} - 1\Big), \quad \forall t \geq 0.
$$

This implies

$$
W(e(t)) \leq \Big[W(e(0)) - \varepsilon^2 \cdot \lambda_{\min}(P)\Big] \cdot e^{-2\alpha t} + \varepsilon^2 \cdot \lambda_{\min}(P), \quad \forall t \geq 0. \tag{2.23}
$$

Thus, from (2.19) and (2.23), it can be readily obtained that

$$
\lambda_{\min}(P)\|e(t)\|^2 \leq W(e(t)) \leq e^{-2\alpha t}\Big[W(e(0)) - \varepsilon^2 \lambda_{\min}(P)\Big] + \varepsilon^2 \lambda_{\min}(P), \quad \forall t \geq 0. \tag{2.24}
$$

Consequently, we conclude that

$$
\|e(t)\| \leq \sqrt{e^{-2\alpha t}\left[\frac{W(e(0)) + \varepsilon^2 \lambda_{\min}(P)}{\lambda_{\min}(P)}\right] + \varepsilon^2}
$$

$$
\leq \sqrt{e^{-2\alpha t}\left[\frac{W(e(0)) + \varepsilon^2 \lambda_{\min}(P)}{\lambda_{\min}(P)}\right]} + \sqrt{\varepsilon^2} \tag{2.25}
$$

$$
= \sqrt{\left[\frac{W(e(0)) + \varepsilon^2 \lambda_{\min}(P)}{\lambda_{\min}(P)}\right]} \cdot e^{-\alpha t} + \varepsilon, \quad \forall t \geq 0.
$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.4. In what follows, we present an algorithm to find a tracking control law of (2.13) stated in Theorem 2.3.

  INPUT: the driver-response systems (2.5) and (2.6), the prespecified exponential decay
         rate $\alpha > 0$, and the prespecified convergence radius $\varepsilon > 0$.
 OUPUT: tracking control of (2.13).

Step 1. Choose $g_1(t, r, \dot{r})$ and $g_2(t, q, \dot{q})$ such that (A1) is satisfied.

Step 2. Calculate m and $\overline{d}$, from Definition 2.1.

Step 3. Calculate P, $\lambda_{\min}(P)$, p_{21}, and p_{22}, from (2.17).
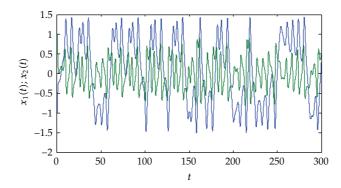
Step 4. Form e_1 and e_2 from (2.8).

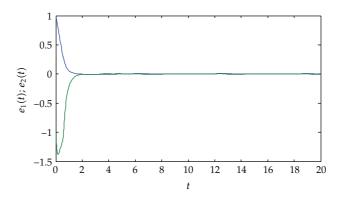**Figure 1:** Typical state trajectories of the uncertain Duffing-Holmes system for (3.1).

Figure 2: Synchronization errors of systems (3.1)–(3.7) with $\beta = 1$.

Step 5. Form u_1 from (2.14).

Step 6. Form $h(t)$ from (2.16).

Step 7. Form u_2 from (2.15).

Step 8. OUPUT $u(t) = u_1(t) + u_2(t)$.

3. Illustrative Example

In what follows, we provide an example to illustrate the main result.

Driver system:

$$\dot{x}_1 = x_2,$$
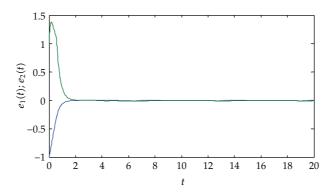$$\dot{x}_2 = -p_1 x_1 - p_2 x_2 - x_1^3 - p_3 \cos(wt) - \Delta a(t) x_1 x_2, \quad \forall t \geq 0. \tag{3.1}$$

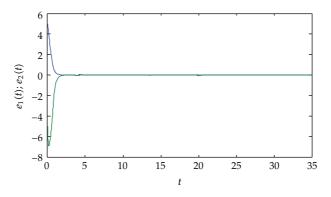**Figure 3:** Synchronization errors of systems (3.1)–(3.7) with $\beta = -1$.

Figure 4: Synchronization errors of systems (3.1)–(3.7) with $\beta = 5$.

Response system:

$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = -p_1 z_1 - p_2 z_2 - \Delta b(t)\sin(t) - \Delta \phi(u), \quad \forall t \geq 0, \tag{3.2}$$

where

$$p_1 = -1, \quad p_2 = 0.25 \quad p_3 = 0.3, \quad w = 1,$$
$$-0.1 \leq \Delta a(t) \leq 0.1, \quad -0.1 \leq \Delta b(t) \leq 0.1, \quad \Delta \phi(u) \in D(u, 1, 2). \tag{3.3}$$

Our objective, in this example, is to design a tracking control such that the systems (3.1) and (3.2) can realize the GPS, with the exponential decay rate $\alpha = 2$ and convergence radius $\varepsilon = 0.1$. Comparison of (3.1)-(3.2) with (2.5)-(2.6), it can be obtained that

$$\overline{d} = 1, \quad m = 2,$$
$$\Delta f_1(t, x_1, x_2) = \Delta a(t) \cdot x_1 x_2, \quad \Delta f_2(t, z_1, z_2) = \Delta b(t)\sin(t). \tag{3.4}$$
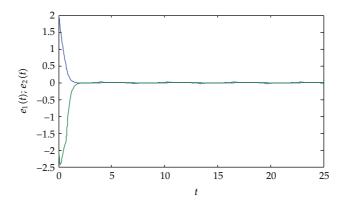
Figure 5: Synchronization errors of systems (3.1)–(3.7) with $\beta = 0$.

The condition (A1) is evidently satisfied if we let

$$g_1(t, x_1, x_2) = 0.1 \cdot |x_1 x_2|, \qquad g_2(t, z_1, z_2) = 0.1 \cdot |\sin(t)|. \tag{3.5}$$

Solving the Lyapunov equation (2.17) yields

$$P = \begin{bmatrix} 49 & 11 \\ 11 & 3 \end{bmatrix}, \qquad \lambda_{\min}(P) = 0.5049,$$
$$p_{21} = 11, \qquad p_{22} = 3. \tag{3.6}$$

Finally, owing to (2.13)–(2.16), we obtain the design controller

$$u(t) = 5e_1 + 2.875e_2 + \frac{h^2(t)}{2 \cdot h(t) \cdot |11e_1 + 3e_2| + 0.02}(11e_1 + 3e_2), \tag{3.7}$$

with

$$h(t) = 2 + 0.1 \cdot \left[|\beta| \cdot |x_1 x_2| + |\sin(t)| \right] + \left| \beta x_1^3 \right| + |0.3\beta|. \tag{3.8}$$

Consequently, by Theorem 2.3, we conclude that the GPS between the driver system (3.1) and the response system (3.2) can be achieved under the control law (3.7), with the guaranteed exponential decay rate $\alpha = 2$ and convergence radius $\varepsilon = 0.1$.

With, for instance,

$$\Delta a(t) = \Delta b(t) = 0.1, \tag{3.9}$$

the typical state trajectories of uncertain Duffing-Holmes chaotic system (3.1) are depicted in Figure 1. In addition, the synchronization errors of systems (3.1)–(3.7) with $\beta = 1, -1, 5$, and 0, are depicted in Figures 2, 3, 4 and 5, respectively. From the foregoing simulations results, it is seen that the controlled drive-response systems (3.1) and (3.2) achieve the GPS under the control law (3.7).

4. Conclusion

In this paper, the GPS of uncertain Duffing-Holmes chaotic systems with parameter mismatching, unknown external excitation, plant uncertainties, and uncertain deadzone nonlinearities has been investigated. Based on the composite control methodology, a tracking control has been proposed to realize the GPS for such uncertain Duffing-Holmes chaotic systems, with prespecified exponential decay rate, convergence radius, and desired scaling factor. Finally, numerical simulations have also been given to verify the feasibility and effectiveness of the proposed GPS scheme.

Nomenclature

$\Re^n$:   The n-dimensional real space
$|a|$:   The modulus of a real number a
$\|x\|$:   The Euclidean norm of the vector $x \in \Re^n$
I :   The unit matrix
A^T :   The transport of the matrix A
$\lambda_{\min}(P)$: The minimum eigenvalue of the matrix
                 P with real eigenvalues
$\sigma(A)$:   The spectrum of the matrix A
$P > 0$:   The matrix P is a symmetric positive
                 definite matrix.

Acknowledgments

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