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Research Article

A Generalized Nonuniform Contraction and Lyapunov Function

Fang-fang Liao,[1] Yongxin Jiang,[2] and Zhiting Xie[2]

[1] *Nanjing College of Information Technology, Nanjing 210046, China*
[2] *Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China*

Correspondence should be addressed to Fang-fang Liao, liaofangfang8178@sina.com

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For nonautonomous linear equations $x' = A(t)x$, we give a complete characterization of general nonuniform contractions in terms of Lyapunov functions. We consider the general case of nonuniform contractions, which corresponds to the existence of what we call nonuniform (D, μ)-contractions. As an application, we establish the robustness of the nonuniform contraction under sufficiently small linear perturbations. Moreover, we show that the stability of a nonuniform contraction persists under sufficiently small nonlinear perturbations.

1. Introduction

We consider nonautonomous linear equations

$$x' = A(t)x, \tag{1.1}$$

where $A : \mathbb{R}_0^+ \to \mathcal{B}(X)$ is a continuous function with values in the space of bounded linear operators in a Banach space X. Our main aim is to characterize the existence of a general nonuniform contraction for (1.1) in terms of Lyapunov functions.

We assume that each solution of (1.1) is global, and we denote the corresponding evolution operator by $T(t, s)$, which is the linear operator such that

$$T(t, s)x(s) = x(t), \quad t, s \in \mathbb{R}_0^+, \tag{1.2}$$

for any solution $x(t)$ of (1.1). Clearly, $T(t,t) = \mathrm{Id}$ and

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t,\tau,s \in \mathbb{R}_0^+. \tag{1.3}$$

We shall say that an increasing function $\mu : \mathbb{R}_0^+ \to [1,+\infty)$ is a *growth rate* if

$$\mu(0) = 1, \qquad \lim_{t\to+\infty} \mu(t) = +\infty. \tag{1.4}$$

Given two growth rates μ,ν, we say that (1.1) admits a *nonuniform* (μ,ν)*-contraction* if there exist constants $K,\alpha > 0$ and $\varepsilon \geq 0$ such that

$$\|T(t,s)\| \leq K\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{1.5}$$

We emphasize that the notion of nonuniform (μ,ν)-contraction often occurs under reasonably weak assumptions. We refer the reader to [1] for details.

In this work, we mainly consider more general nonuniform contractions (see (2.1) below) and we give a complete characterization of such contractions in terms of Lyapunov functions, especially in terms of quadratic Lyapunov functions, which are Lyapunov functions defined in terms of quadratic forms. The importance of Lyapunov functions is well established, particularly in the study of the stability of trajectories both under linear and nonlinear perturbations. This study goes back to the seminal work of Lyapunov in his 1892 thesis [2]. For more results, we refer the reader to [3–6] for the classical exponential contractions and dichotomies, [7–9] for the nonuniform exponential contractions and nonuniform exponential dichotomies.

The proof of this paper follows from the ideas in [9, 10]. As an application, we provide a very direct proof of the robustness of the nonuniform contraction, that is, of the persistence of the nonuniform contraction in the equation

$$x' = [A(t) + B(t)]x \tag{1.6}$$

for any sufficiently small linear perturbation $B(t)$. We remark that the so-called robustness problem also has a long history. In particular, the problem was discussed by Massera and Schäffer [11], Perron [12], Coppel [3] and in the case of Banach spaces by Daletskiĭ and Kreĭn [13]. For more recent work we refer to [14–16] and the references therein.

Furthermore, for a large class of nonlinear perturbations $f(t,x)$ with $f(t,0) = 0$ for every t, we show that if (1.1) admits a nonuniform contraction, then the zero solution of the equation

$$x' = A(t)x + f(t,x) \tag{1.7}$$

is stable. The proof uses the corresponding characterization between the nonuniform contractions and quadratic Lyapunov functions.

2. Lyapunov Functions and Nonuniform Contractions

Given a growth rate μ and a function $D : \mathbb{R}_0^+ \to (0, +\infty)$, we say that (1.1) admits a *nonuniform* (D, μ)*-contraction* if there exists a constant $\alpha > 0$ such that

$$\|T(t,s)\| \leq D(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}, \quad t \geq s \geq 0. \tag{2.1}$$

The nonuniform (μ, ν)-contraction is a special case of nonuniform (D, μ)-contraction with $D(s) = K\nu^{\varepsilon}(s)$.

Now we introduce the notion of Lyapunov functions. We say that a continuous function $V : (0, +\infty) \times X \to \mathbb{R}_0^-$ is a *strict Lyapunov function* to (1.1) if

(1) for every $t > 0$ and $x \in X$,

$$\|x\| \leq |V(t,x)| \leq D(t)\|x\|, \tag{2.2}$$

(2) for every $t \geq s > 0$ and $x \in X$,

$$V(s,x) \leq V(t, T(t,s)x), \tag{2.3}$$

(3) there exists a constant $\gamma > 0$ such that for every $t \geq s > 0$ and $x \in X$,

$$|V(t, T(t,s)x)| \leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} |V(s,x)|. \tag{2.4}$$

The following result gives an optimal characterization of nonuniform (D, μ)-contractions in terms of strict Lyapunov functions.

Theorem 2.1. (1.1) *admits a nonuniform* (D, μ)*-contraction if and only if there exists a strict Lyapunov function for* (1.1)*.*

Proof. We assume that there exists a strict Lyapunov function for (1.1). By (1) and (3), for every $t \geq s > 0$ and $x \in X$, we have

$$\begin{aligned} \|T(t,s)x\| &\leq |V(t, T(t,s)x)| \\ &\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} |V(s,x)| \\ &\leq D(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \|x\|. \end{aligned} \tag{2.5}$$

Therefore, (1.1) admits a nonuniform (D, μ)-contraction with $\alpha = \gamma$.

Next we assume that (1.1) admits a nonuniform (D,μ)-contraction. For $t > 0$ and $x \in X$, we set

$$V(t,x) = -\sup\left\{ \|T(\tau,t)x\| \left(\frac{\mu(\tau)}{\mu(t)}\right)^{\alpha} : \tau \geq t \right\}. \tag{2.6}$$

By (2.1), we have $|V(t,x)| \leq D(t)\|x\|$. Moreover, setting $\tau = t$, we obtain $|V(t,x)| \geq \|x\|$. This establishes (1). Furthermore, for $t \geq s$, we have

$$\begin{aligned} |V(t,T(t,s)x)| &= \sup\left\{ \|T(\tau,t)T(t,s)x\| \left(\frac{\mu(\tau)}{\mu(t)}\right)^{\alpha} : \tau \geq t \right\} \\ &= \left(\frac{\mu(s)}{\mu(t)}\right)^{\alpha} \sup\left\{ \|T(\tau,s)x\| \left(\frac{\mu(\tau)}{\mu(s)}\right)^{\alpha} : \tau \geq t \right\} \\ &\leq \left(\frac{\mu(s)}{\mu(t)}\right)^{\alpha} \sup\left\{ \|T(\tau,s)x\| \left(\frac{\mu(\tau)}{\mu(s)}\right)^{\alpha} : \tau \geq s \right\} \\ &= \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} |V(s,x)|. \end{aligned} \tag{2.7}$$

Therefore, V is a strict Lyapunov function for (1.1). □

Next we consider another class of Lyapunov functions, namely, those defined in terms of quadratic forms.

Let $S(t) \in \mathcal{B}(X)$ be a symmetric positive-definite operator for each $t > 0$. A *quadratic Lyapunov function* V is given as

$$H(t,x) = \langle S(t)x, x \rangle, \qquad V(t,x) = -\sqrt{H(t,x)}. \tag{2.8}$$

Given linear operators M, N, we write $M \leq N$ if $\langle Mx, x \rangle \leq \langle Nx, x \rangle$ for $x \in X$.

Theorem 2.2. *Assume that there exist constants $c > 0$ and $d \geq 1$ such that*

$$\|T(t,s)\| \leq c \quad \textit{whenever } \mu(t) \leq d\mu(s), \ t \geq s > 0. \tag{2.9}$$

Then (1.1) admits a nonuniform (D,μ)-contraction (up to a multiplicative constant) if and only if there exist symmetric positive definite operators $S(t)$ and constants $C, K > 0$ such that $S(t)$ is of class C^1 in $t > 0$ and

$$\|S(t)\| \leq CD(t)^2, \tag{2.10}$$

$$S'(t) + A^*(t)S(t) + S(t)A(t) \leq -(\mathrm{Id} + KS(t))\frac{\mu'(t)}{\mu(t)}. \tag{2.11}$$

Proof. We first assume that (1.1) admits a nonuniform (D,μ)-contraction. Consider the linear operators

$$S(t) = \int_t^{\infty} T(\tau,t)^* T(\tau,t) \left(\frac{\mu(\tau)}{\mu(t)} \right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau, \tag{2.12}$$

for some constant $\rho \in (0,\alpha)$. Clearly, $S(t)$ is symmetric for each $t > 0$. Moreover, by (2.8), we have

$$\begin{aligned} \|H(t,x)\| &= \int_t^{\infty} \|T(\tau,t)x\|^2 \left(\frac{\mu(\tau)}{\mu(t)} \right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &\le D(t)^2 \|x\|^2 \int_t^{\infty} \left(\frac{\mu(\tau)}{\mu(t)} \right)^{-2\rho} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &= \frac{D(t)^2}{2\rho} \|x\|^2. \end{aligned} \tag{2.13}$$

Since $S(t)$ is symmetric, we obtain

$$\|S(t)\| = \sup_{x \neq 0} \frac{|H(t,x)|}{\|x\|^2} \le \frac{D(t)^2}{2\rho} \tag{2.14}$$

and therefore (2.10) holds. Since

$$\frac{\partial}{\partial t} T(\tau,t) = -T(\tau,t)A(t), \qquad \frac{\partial}{\partial t} T(\tau,t)^* = -A(t)^* T(\tau,t)^*, \tag{2.15}$$

we find that $S(t)$ is of class C^1 in t with derivative

$$\begin{aligned} S'(t) = &-\frac{\mu'(t)}{\mu(t)} - \int_t^{\infty} A(t)^* T(\tau,t)^* T(\tau,t) \left(\frac{\mu(\tau)}{\mu(t)} \right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &- \int_t^{\infty} T(\tau,t)^* T(\tau,t) A(t) \left(\frac{\mu(\tau)}{\mu(t)} \right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &- 2(\alpha-\rho) \frac{\mu'(t)}{\mu(t)} \int_t^{\infty} T(\tau,t)^* T(\tau,t) \left(\frac{\mu(\tau)}{\mu(t)} \right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau, \end{aligned} \tag{2.16}$$

which implies that

$$S'(t) = -\frac{\mu'(t)}{\mu(t)} - A(t)^* S(t) - S(t)A(t) - 2(\alpha-\rho) \frac{\mu'(t)}{\mu(t)} S(t). \tag{2.17}$$

Therefore,

$$S'(t) + A(t)^*S(t) + S(t)A(t) = -\frac{\mu'(t)}{\mu(t)}\left(\mathrm{Id} + 2(\alpha - \rho)S(t)\right), \tag{2.18}$$

which establishes (2.11) with $K = 2(\alpha - \rho)$.

Now we assume that conditions (2.9) and (2.10)-(2.11) hold. Set $x(t) = T(t,\tau)x(\tau)$. By (2.10), we have

$$\|H(t, x(t))\| \le \|S(t)\| \cdot \|x(t)\|^2 \le CD(t)^2\|x(t)\|^2. \tag{2.19}$$

Lemma 2.3. *There exists a constant $\eta > 0$ such that*

$$H(t, x(t)) \ge \eta\|x(t)\|^2. \tag{2.20}$$

Proof of Lemma 2.3. Note that

$$\begin{aligned}\frac{d}{dt}H(t, x(t)) &= \langle S'(t)x(t), x(t)\rangle + \langle S(t)A(t)x(t), x(t)\rangle + \langle S(t)x(t), A(t)x(t)\rangle \\ &= \langle (S'(t) + S(t)A(t) + A(t)^*S(t))x(t), x(t)\rangle.\end{aligned} \tag{2.21}$$

Hence, by condition (2.11), and the fact that $K > 0$ we obtain

$$\frac{d}{dt}H(t, x(t)) \le -\frac{\mu'(t)}{\mu(t)}\|x(t)\|^2. \tag{2.22}$$

Now given $\tau > 0$, take $t > \tau$ such that $\mu(t) = d\mu(\tau)$ with d as in (2.9). Then

$$\begin{aligned}H(t, x(t)) - H(\tau, x(\tau)) &= \int_\tau^t \frac{d}{dv}H(v, x(v))dv \\ &\le -\int_\tau^t \frac{\mu'(v)}{\mu(v)}\|x(v)\|^2 dv \\ &= -\int_\tau^t \frac{\mu'(v)}{\mu(v)}\|T(v,\tau)x(\tau)\|^2 dv \\ &\le -\|x(\tau)\|^2 \int_\tau^t \frac{\mu'(v)}{\mu(v)}\frac{1}{\|T(\tau, v)\|^2}dv.\end{aligned} \tag{2.23}$$

It follows from (2.9) that

$$
\begin{aligned}
H(t,x(t)) - H(\tau,x(\tau)) &\le -\frac{1}{c^2}\|x(\tau)\|^2 \int_\tau^t \frac{\mu'(v)}{\mu(v)}dv \\
&= -\frac{\log d}{c^2}\|x(\tau)\|^2.
\end{aligned}
\tag{2.24}
$$

Since $H(t,x(t)) \ge 0$, we have

$$
H(\tau,x(\tau)) \ge H(\tau,x(\tau)) - H(t,x(t)) \ge \frac{\log d}{c^2}\|x(\tau)\|^2 \tag{2.25}
$$

which yields (2.20) with $\eta = (\log d)/c^2 > 0$.

Lemma 2.4. *For $t \ge \tau$, one has*

$$
H(t,x(t)) \le \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} H(\tau,x(\tau)). \tag{2.26}
$$

Proof of Lemma 2.4. By conditions (2.11) and (2.21), we have

$$
\frac{d}{dt}H(t,x(t)) \le -K\frac{\mu'(t)}{\mu(t)}H(t,x(t)). \tag{2.27}
$$

Therefore,

$$
\begin{aligned}
H(t,x(t)) - H(\tau,x(\tau)) &= \int_\tau^t \frac{d}{dv}H(v,x(v))dv \\
&\le -K\int_\tau^t \frac{\mu'(v)}{\mu(v)}H(v,x(v))dv.
\end{aligned}
\tag{2.28}
$$

It follows from Gronwall's lemma that

$$
H(t,x(t)) \le \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} H(\tau,x(\tau)), \tag{2.29}
$$

which yields the desired result.

By Lemmas 2.3 and 2.4 together with (2.19), we obtain

$$\begin{aligned}\|T(t,\tau)x(\tau)\|^2 &= \|x(t)\|^2\\ &\le \eta^{-1}H(t,x(t))\\ &\le \eta^{-1}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K}H(\tau,x(\tau))\\ &\le \eta^{-1}CD(\tau)^2\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K}\|x(\tau)\|^2,\end{aligned} \tag{2.30}$$

and therefore,

$$\|T(t,\tau)\|^2 \le \eta^{-1}CD(\tau)^2\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K}, \tag{2.31}$$

which implies that (1.1) admits a nonuniform (D,μ)-contraction. □

As an application of Theorem 2.2, we establish the robustness of nonuniform (D,μ)-contractions. Roughly speaking, a nonuniform contraction for (1.1) is said to be *robust* if (1.6) still admits a nonuniform contraction for any sufficiently small perturbation $B(t)$.

Theorem 2.5. *Let* $A, B : \mathbb{R}_0^+ \to \mathcal{B}(X)$ *be continuous functions such that* (1.1) *admits a nonuniform* (D,μ)*-contraction with condition* (2.9)*. Suppose further that* $D(t) \ge 1$ *for every* $t > 0$ *and*

$$\|B(t)\| \le \delta D^{-2}(t)\frac{\mu'(t)}{\mu(t)}, \quad t > 0 \tag{2.32}$$

for some $\delta > 0$ *sufficiently small. Then* (1.6) *admits a nonuniform* (D,μ)*-contraction.*

Proof. Let $U(t,s)$ be the evolution operator associated to (1.6). It is easy to verify that

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)d\tau. \tag{2.33}$$

For every $t \ge s > 0$ with $\mu(t) \le d\mu(s)$, we have

$$\begin{aligned}\|U(t,s)\| &\le c + \int_s^t c\delta D^{-2}(\tau)\frac{\mu'(\tau)}{\mu(\tau)}\|U(\tau,s)\|d\tau\\ &\le c + c\delta\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}\|U(\tau,s)\|d\tau.\end{aligned} \tag{2.34}$$

Using Gronwall's inequality, we obtain

$$\|U(t,s)\| \le c\exp\left(c\delta\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}d\tau\right) \le c\exp(c\delta\log d) \tag{2.35}$$

for every $t \ge s > 0$ with $\mu(t) \le d\mu(s)$. Therefore condition (2.9) also holds for the perturbed equation (1.6).

Now we consider the matrices $S(t)$ in (2.12). Condition (2.10) can be obtained as in the proof of Theorem 2.2. For condition (2.11), it is sufficient to show that

$$S(t)B(t) + B(t)^*S(t) \le \vartheta\frac{\mu'(t)}{\mu(t)}\mathrm{Id} \tag{2.36}$$

for some constant $\vartheta < 1$. Using (2.10) and (2.32), we have

$$\begin{aligned} S(t)B(t) + B(t)^*S(t) &\le 2\|S(t)\|\cdot\|B(t)\| \\ &\le 2CD(t)^2\delta D(t)^{-2}\frac{\mu'(t)}{\mu(t)} \\ &= 2C\delta\frac{\mu'(t)}{\mu(t)}, \end{aligned} \tag{2.37}$$

and taking δ sufficiently small, we find that (2.36) holds with some $\vartheta < 1$. □

3. Stability of Nonlinear Perturbations

Before stating the result, we fist prove an equivalent characterization of property (3). Given matrices $S(t) \in \mathcal{B}(X)$ for each $t \in \mathbb{R}_0^+$, we consider the functions

$$\begin{aligned} \dot{H}(t,x) &= \frac{d}{dh}H(t+h,T(t+h,h)x)\big|_{h=0}, \\ \dot{V}(t,x) &= \frac{d}{dh}V(t+h,T(t+h,h)x)\big|_{h=0}, \end{aligned} \tag{3.1}$$

whenever the derivatives are well defined and H, V are given as (2.8).

Lemma 3.1. *Let V,μ be C^1 functions. Then property* (3) *is equivalent to*

$$\dot{V}(t,T(t,\tau)x) \ge -\gamma V(t,T(t,\tau)x)\frac{\mu'(t)}{\mu(t)}, \quad t > \tau. \tag{3.2}$$

Proof. Now we assume that property (3) holds. If $t > \tau$ and $h > 0$, then

$$\begin{aligned}
V(t+h, T(t+h,\tau)x) &= V(t+h, T(t+h,t)T(t,\tau)x) \\
&\geq \left(\frac{\mu(t+h)}{\mu(t)}\right)^{-\gamma} V(t, T(t,\tau)x), \\
\lim_{h\to 0^+} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} &\geq V(t, T(t,\tau)x) \lim_{h\to 0^+} \frac{(\mu(t+h)/\mu(t))^{-\gamma} - 1}{h} \\
&= -\gamma V(t, T(t,\tau)x) \frac{\mu'(t)}{\mu(t)}.
\end{aligned} \tag{3.3}$$

Similarly, if $h < 0$ is such that $t + h > \tau$, then

$$\begin{aligned}
&V(t+h, T(t+h,\tau)x) \leq \left(\frac{\mu(t+h)}{\mu(t)}\right)^{-\gamma} V(t, T(t,\tau)x), \\
&\lim_{h\to 0^-} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} \\
&\quad \geq V(t, T(t,\tau)x) \lim_{h\to 0^-} \frac{(\mu(t+h)/\mu(t))^{-\gamma} - 1}{h} \\
&\quad = -\gamma V(t, T(t,\tau)x) \frac{\mu'(t)}{\mu(t)}.
\end{aligned} \tag{3.4}$$

This establishes (3.2).

Next we assume that (3.2) holds. We rewrite (3.2) in the form

$$\frac{\dot{V}(t, T(t,\tau)x)}{V(t, T(t,\tau)x)} \geq -\gamma \frac{\mu'(t)}{\mu(t)}, \quad t > \tau, \tag{3.5}$$

which implies that

$$\begin{aligned}
\log\left(\frac{V(t, T(t,\tau)x)}{V(\tau, x)}\right) &= \int_\tau^t \frac{\dot{V}(v, T(v,\tau)x)}{V(v, T(v,\tau)x)} dv \\
&\geq -\gamma \int_\tau^t \frac{\mu'(v)}{\mu(v)} dv \\
&= \log\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\gamma},
\end{aligned} \tag{3.6}$$

and hence property (3) holds. □

Theorem 3.2. *Assume that* (1.1) *admits a nonuniform* (D,μ)*-contraction satisfying* (2.9). *Suppose further that there exists a constant* $l > 0$ *such that* $l < \alpha$ *and*

$$\|f(t,x)\| \leq l\frac{\mu'(t)}{\mu(t)}\|x\|, \quad t > 0,\ x \in X. \tag{3.7}$$

Then for each $k > -\alpha + l$, *there exists* $C > 0$ *such that*

$$\|y(t)\| \leq CD(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{k}\|y(s)\|, \quad t \geq s \tag{3.8}$$

for every solution $y(t)$ *of* (1.7).

Proof. For $S(t)$ as in (2.12) and $H(t,x(t))$ as in (2.8), we have, for every $t \geq s$,

$$\begin{aligned}
H(t,T(t,s)x(s)) &= \int_t^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&= \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}\int_t^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}\int_s^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&= \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}H(s,x(s)).
\end{aligned} \tag{3.9}$$

Since $V(t,x) = -\sqrt{H(t,x)}$, we have

$$V(t,T(t,s)x(s)) \geq \left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho)}V(s,x(s)), \quad t \geq s. \tag{3.10}$$

Applying Lemma 3.1, we obtain

$$\dot{V}(t,T(t,s)x(s)) \geq -(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}V(t,T(t,s)x(s)), \quad t \geq s. \tag{3.11}$$

In particular, for $t = s$,

$$\dot{V}(s,x(s)) \geq -(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}V(s,x(s)). \tag{3.12}$$

From the identity $\dot{H} = 2V\dot{V}$ that for every $s > 0$ and $x \in X$, we have

$$\dot{H}(s,x) \leq -2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}H(s,x). \tag{3.13}$$

On the other hand,

$$\dot{H}(s,x) = \langle (S'(s) + S(s)A(s) + A(s)^* S(s))x, x\rangle. \tag{3.14}$$

Therefore,

$$\begin{aligned} 0 &\geq \dot{H}(s,x) + 2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}H(s,x) \\ &= \left\langle \left(S'(s) + S(s)A(s) + A(s)^* S(s) + 2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}S(s)\right)x, x\right\rangle, \end{aligned} \tag{3.15}$$

and hence

$$S'(t) + S(t)A(t) + A(t)^* S(t) + 2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}S(t) \leq 0. \tag{3.16}$$

Therefore, if $y(t)$ is a solution of (1.7), then

$$\begin{aligned} \frac{d}{dt}H(t,y(t)) &= \langle S'(t)y(t), y(t)\rangle + \langle S(t)A(t)y(t), y(t)\rangle + \langle S(t)y(t), A(t)y(t)\rangle \\ &\quad + \langle S(t)f(t,y(t)), y(t)\rangle + \langle S(t)y(t), f(t,y(t))\rangle \\ &= \langle (S'(t) + S(t)A(t) + A(t)^* S(t))y(t), y(t)\rangle \\ &\quad + \langle (S(t) + S(t)^*)y(t), f(t,y(t))\rangle \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + \langle (S(t) + S(t)^*)y(t), f(t,y(t))\rangle \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + 2\|S(t)\| \cdot \|f(t,y(t))\| \cdot \|y(t)\| \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + 2l\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 \\ &= -2(\alpha-\rho-l)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2. \end{aligned} \tag{3.17}$$

If ρ is small enough such that $\alpha - \rho - l > 0$, then

$$\frac{d}{dt}H(t,y(t)) \leq -2(\alpha-\rho-l)\frac{\mu'(t)}{\mu(t)}H(t,y(t)), \tag{3.18}$$

and hence

$$H(t,y(t)) - H(s,y(s)) \leq -2(\alpha-\rho-l)\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}H(\tau,y(\tau))d\tau. \tag{3.19}$$

It follows from Gronwall's inequality that

$$H(t,y(t)) \leq H(s,y(s))\left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho-l)}, \quad t \geq s. \tag{3.20}$$

Now given $s > 0$, take $t > s$ such that $\mu(t) = d\mu(s)$ with d as in (2.9). Then

$$\begin{aligned}
H(s,y(s)) &= \int_s^\infty \|T(\tau,s)y(s)\|^2\left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\geq \int_s^t \|T(\tau,s)y(s)\|^2\left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\geq \frac{1}{c^2}\|y(s)\|^2\int_s^t \left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&= \frac{1}{2c^2(\alpha-\rho)}\|y(s)\|^2\left\{\left(\frac{\mu(t)}{\mu(s)}\right)^{2(\alpha-\rho)}-1\right\} \\
&= \frac{1}{2c^2(\alpha-\rho)}\|y(s)\|^2\left\{d^{2(\alpha-\rho)}-1\right\}.
\end{aligned} \tag{3.21}$$

Taking

$$\kappa = \frac{1}{2c^2(\alpha-\rho)}\left\{d^{2(\alpha-\rho)}-1\right\} > 0, \tag{3.22}$$

then

$$H(s,y(s)) \geq \kappa\|y(s)\|^2. \tag{3.23}$$

It follows from (2.13) and (3.20) that

$$\begin{aligned}
\|y(t)\| &\leq \kappa^{1/2}\sqrt{H(t,y(t))} \\
&\leq \kappa^{1/2}\sqrt{H(s,y(s))}\left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho-l)} \\
&\leq \kappa^{1/2}\sqrt{\frac{1}{2\rho}D(s)}\left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho-l)}\|y(s)\|.
\end{aligned} \tag{3.24}$$

Now the proof is finished. □

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