Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 603748, 8 pages
doi:10.1155/2012/603748

Research Article

A New Method for Riccati Differential Equations Based on Reproducing Kernel and Quasilinearization Methods

F. Z. Geng and X. M. Li

Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, China

Correspondence should be addressed to F. Z. Geng, gengfazhan@126.com

Received 1 December 2011; Accepted 17 December 2011

Academic Editor: Shaher Momani

We introduce a new method for solving Riccati differential equations, which is based on reproducing kernel method and quasilinearization technique. The quasilinearization technique is used to reduce the Riccati differential equation to a sequence of linear problems. The resulting sets of differential equations are treated by using reproducing kernel method. The solutions of Riccati differential equations obtained using many existing methods give good approximations only in the neighborhood of the initial position. However, the solutions obtained using the present method give good approximations in a larger interval, rather than a local vicinity of the initial position. Numerical results compared with other methods show that the method is simple and effective.

1. Introduction

In this paper, we consider the following Riccati differential equation:

$$\begin{gathered} u'(x) = p(x) + q(x)u(x) + r(x)u^2(x), \quad 0 \le x \le X, \\ u(0) = 0. \end{gathered} \tag{1.1}$$

Without loss of generality, we only consider initial condition $u(0) = 0$, for $u(0) = \alpha$ can be easily reduced to $u(0) = 0$. Riccati differential equations play a significant role in many fields of applied science [1]. For example, as is well known, a one-dimensional static Schrödinger equation is closely related to a Riccati differential equation. Solitary wave solution of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [2]. Such type of problem also arises in the optimal control literature. Therefore, the problem has attracted much attention and has been studied by many authors. However, deriving its analytical solution in an explicit form seems to be unlikely except for certain special situations. For example, some Riccati equations with constant coefficients can be solved analytically by various methods [3]. Therefore, one has to go for the numerical techniques or approximate approaches for getting its solution. Recently, Adomian's decomposition method and multistage Adomian's decomposition method have been proposed for solving Riccati differential equations in [4]. Abbasbandy [5–7] solved a special Riccati differential equation, quadratic Riccati differential equation using He's VIM, homotopy perturbation method (HPM) and iterated He's HPM and compared the accuracy of the obtained solution with that derived by Adomians decomposition method. Dehghan and Lakestani [8] solved Riccati differential equations by using the cubic B-spline scaling functions and Chebyshev cardinal functions and obtained good approximate solutions. Geng et al. [9] introduced a piecewise variational iteration method for Riccati differential equations, which is a modified variational iteration method. Tang and Li [10] introduced a new method for determining the solution of Riccati differential equations. Ghorbani and Momani [11] proposed an effective variational iteration algorithm for solving Riccati differential equations. In [12–14], the authors presented some methods for solving fractional Riccati differential equations. Mohammadi and Hosseini [15] introduced a comparison of some numerical methods for solving quadratic Riccati differential equations.

Reproducing kernel theory has important application in numerical analysis, differential equation, probability, statistics, and so on [16, 17]. Recently, Cui et al. present reproducing kernel method for solving linear and nonlinear differential equations [18–22].

In this paper, based on reproducing kernel method (RKM) and quasilinearization technique, we present a new method for (1.1) and obtain a highly accurate numerical solution. The advantage of the present method over existing methods for solving this problem is that the solution of (1.1) obtained using the present method is efficient not only for a smaller value of x but also for a larger value.

The rest of the paper is organized as follows. In the next section, the RKM for first order linear ordinary differential equations (ODEs) is introduced. The method for solving (1.1) is presented in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Analysis of RKM for First-Order Linear ODEs

In this section, we illustrate how to solve the following linear first order ODEs using RKM:

$$
\begin{gathered}
Lu(x) = u'(x) + a(x)u(x) = f(x), \quad 0 < x < 1, \\
u(0) = 0,
\end{gathered}
\tag{2.1}
$$

where $a(x)$ and $f(x)$ are continuous.

In order to solve (2.1) using RKM, we first construct a reproducing kernel Hilbert space $W_2^2[0,1]$, in which every function satisfies the initial condition of (2.1).

Definition 2.1 (Reproducing kernel). Let E be a nonempty abstract set. A function $K : E \times E \to C$ is a reproducing kernel of the Hilbert space H if and only if

(a)

$$\forall\, t \in E, \quad K(\cdot, t) \in H, \tag{2.2}$$

(b)

$$\forall\, t \in E, \forall \varphi \in H,\ \left(\varphi(\cdot),\ K(\cdot, t)\right) = \varphi(t). \tag{2.3}$$

The last condition is called "the reproducing property": the value of the function φ at the point t is reproduced by the inner product of $\varphi(\cdot)$ with $K(\cdot, t)$.

A Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space (RKHS).

2.1. The Reproducing Kernel Hilbert Space $W_2^2[0, X]$

The inner product space $W_2^2[0, X]$ is defined as $W_2^2[0, X] = \{u(x) \mid u, u'$ are absolutely continuous real value functions, $u'' \in L^2[0, X], u(0) = 0\}$. The inner product in $W_2^2[0, X]$ is given by

$$(u(y), v(y))_{W_2^2} = u(0)v(0) + u(X)v(X) + \int_0^X u''v''dy, \tag{2.4}$$

and the norm $\|u\|_{W_2^2}$ is denoted by $\|u\|_{W_2^2} = \sqrt{(u, u)_{W_2^2}}$, where $u, v \in W_2^2[0, X]$.

Theorem 2.2. *The space $W_2^2[0, X]$ is a reproducing kernel Hilbert space. That is, there exists $R_x(y) \in W_2^2[0, X]$, for any $u(y) \in W_2^2[0, X]$, and each fixed $x \in [0, X]$, $y \in [0, X]$, such that $(u(y), R_x(y))_{W_2^2} = u(x)$. The reproducing kernel $R_x(y)$ can be denoted by*

$$R_x(y) = \begin{cases} R1(x, y), & y \le x, \\ R1(y, x), & y > x, \end{cases} \tag{2.5}$$

where $R1(x, y) = y((x - X)Xy^2 + x(2X^3 - 3xX^2 + x^2X + 6))/(6X^2)$.

The method for obtaining unknown coefficients of (2.5) can be found in [16].

In [22], Li and Cui defined a reproducing kernel Hilbert space $W_2^1[0, X]$ and gave its reproducing kernel $\overline{R}_x(y)$.

2.2. The Solution of (2.1)

In (2.1), it is clear that $L : W_2^2[0, X] \to W_2^1[0, X]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{R}_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. The orthonormal system

$\{\overline{\varphi}_i(x)\}_{i=1}^{\infty}$ of $W_2^2[0,X]$ can be derived from Gram-Schmidt orthogonalization process of $\{\varphi_i(x)\}_{i=1}^{\infty}$:

$$\overline{\varphi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\varphi_k(x), \quad (\beta_{ii} > 0,\ i = 1,2,\ldots). \tag{2.6}$$

Theorem 2.3. *For* (2.1), *if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,X]$, *then* $\{\varphi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^2[0,1]$ *and* $\varphi_i(x) = L_y R_x(y)|_{y=x_i}$. *The subscript* y *by the operator* L *indicates that the operator* L *applies to the function of* y.

Theorem 2.4. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,X]$ *and the solution of* (2.1) *is unique, then the solution of* (2.1) *is*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\varphi}_i(x). \tag{2.7}$$

Proof. Applying Theorem 2.3, it is easy to see that $\{\overline{\varphi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^2[0,X]$. Note that $(v(x), \varphi_i(x)) = v(x_i)$ for each $v(x) \in W_2^1[0,1]$. Hence we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \left(u(x), \overline{\varphi}_i(x)\right)\overline{\varphi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left(u(x), L^*\varphi_k(x)\right)\overline{\varphi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left(Lu(x), \varphi_k(x)\right)\overline{\varphi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left(f(x), \varphi_k(x)\right)\overline{\varphi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\varphi}_i(x), \end{aligned} \tag{2.8}$$

and the proof of the theorem is complete. □

Now, an approximate solution $u_N(x)$ can be obtained by the N-term intercept of the exact solution $u(x)$ and

$$u_N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\varphi}_i(x). \tag{2.9}$$

3. Solution of Riccati Differential Equation (1.1)

To solve Riccati differential equation (1.1), quasilinearization technique is used to reduce (1.1) to a sequence of linear problems. Let $f(x,u) = p(x) + r(x)u^2$. By choosing a reasonable initial approximation $u_0(x)$ for the function $u(x)$ in $f(x,u)$ and expanding $f(x,u)$ around the function $u_0(x)$, one obtains

$$f(x,u_1) = (x,u_0) + (u_1 - u_0)\frac{\partial f}{\partial u}\bigg|_{u=u_0} + \dots . \tag{3.1}$$

In general, one can write for $k = 1,2,\dots$, (k = iteration index):

$$f(x,u_k) = f(x,u_{k-1}) + (u_k - u_{k-1})\frac{\partial f}{\partial u}\bigg|_{u=u_{k-1}} + \dots . \tag{3.2}$$

Hence, we can obtain the following iteration formula for Riccati differential equation: (1.1)

$$\begin{gathered} u_k'(x) - a_k(x)u_k(x) = f_k(x), \quad k = 1,2,\dots, \\ u_k(0) = 0, \end{gathered} \tag{3.3}$$

where $a_k(x) = q(x) + \partial f/\partial u|_{u=u_{k-1}} = q(x) + 2r(x)u_{k-1}(x)$, $f_k(x) = f(x,u_{k-1}) - \partial f/\partial u|_{u=u_{k-1}} = p(x) - r(x)u_{k-1}^2(x)$, $u_0(x)$ is the initial approximation.

Therefore, to solve Riccati differential equation (1.1), it is suffice for us to solve the series of linear problem (3.3).

By using RKM presented in Section 2, one can obtain the solution of problem (3.3)

$$u_k(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x), \tag{3.4}$$

where $A_j = \sum_{l=1}^{j} \beta_{jl} f_k(x_l)$.

Therefore, N-term approximations $u_{k,N}(x)$ to $u_k(x)$ are obtained

$$u_{k,N}(x) = \sum_{j=1}^{N} A_j \overline{\psi}_j(x). \tag{3.5}$$

4. Numerical Examples

In this section, we apply the method presented in Section 3 to some Riccati differential equations. Numerical results show that the MVIM is very effective.

Example 4.1. Consider the following Riccati differential equation [4–7, 9]:

$$\begin{gathered} u'(x) = 1 + 2u(x) - u^2(x), \quad 0 \le x \le 4, \\ u(0) = 0. \end{gathered} \tag{4.1}$$

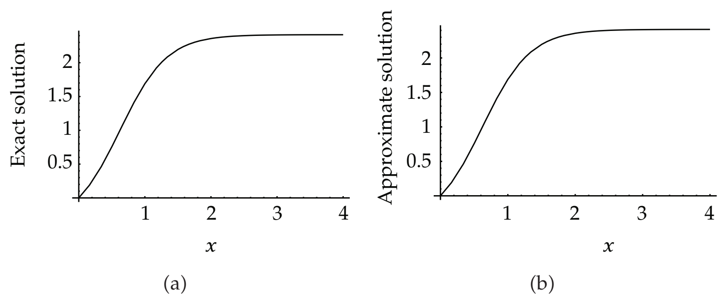

Figure 1: Comparison of approximate solutions with the exact solutions for Example 4.1. ((a): exact solution; (b): approximate solution.).

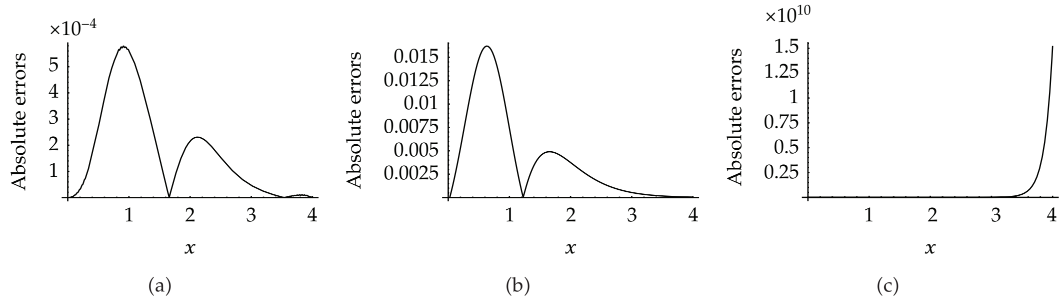

Figure 2: Comparison of absolute errors using the present method, MVIM [9] and VIM [6] for Example 4.1. ((a): The present method; (b): MVIM [9]; (c): VIM [6]).

The exact solution can be easily determined to be

$$u(x) = 1 + \sqrt{2}\ \tanh\left(\sqrt{2}\ x + \frac{\log\left(\left(-1+\sqrt{2}\right)/\left(1+\sqrt{2}\right)\right)}{2}\right). \tag{4.2}$$

According to (3.3), (3.4), and (3.5), taking $k = 3$ and $N = 100$, we can obtain the approximations of (4.1) on $[0, 4]$. The numerical results are shown in Figures 1 and 2. Figure 1 shows a comparison of approximations obtained using the present method with the exact solution. From Figure 1, it is easily found that the present approximations are effective for a larger interval, rather than a local vicinity of the initial position. The comparison of absolute errors using the present method with conventional VIM [6] and piecewise VIM [9] is shown in Figure 2. From Figure 2, we find that the solution derived by VIM [6] gives a good approximation only in the neighborhood of the initial position.

Remark 4.2. The solutions of Example 4.1 derived by ADM [4], HPM [5], and VIM [6] give good approximations only in the neighborhood of the initial position. The approximations derived by the present piecewise VIM [9] and iterated HPM [7] are both efficient for the whole interval. However, the present method is more accurate than piecewise VIM [9] and iterated HPM [7].

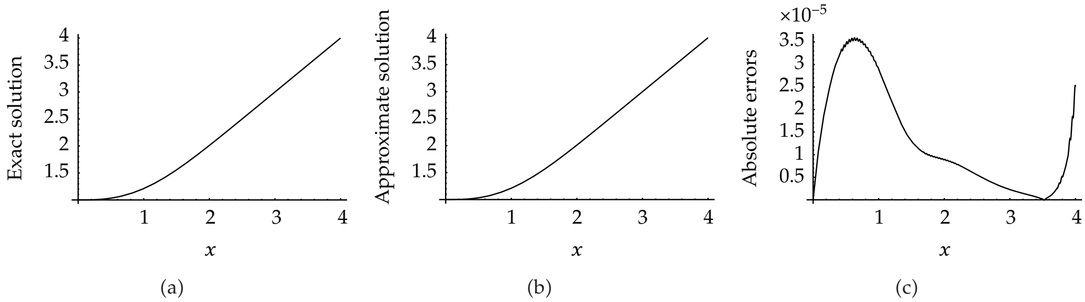

Figure 3: The numerical results for Example 4.3. ((a): exact solution; (b): approximate solution; (c): absolute error.).

Example 4.3. Consider the following Riccati differential equation [9]:

$$u'(x) = 1 + x^2 - u^2(x), \quad 0 \le x \le 4,$$
$$u(0) = 1. \tag{4.3}$$

The exact solution can be easily determined to be

$$u(x) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2} dt}. \tag{4.4}$$

According to (3.3), (3.4), and (3.5), taking $k = 3$ and $N = 100$, we can obtain the approximations of (4.1) on $[0, 4]$. The numerical results are shown in Figure 3.

5. Conclusion

In this paper, based on reproducing kernel method and quasilinearization technique, a new method is presented to solve Riccati differential equations. Compared with other methods, the results of numerical examples demonstrate that the present method is more accurate than existing methods. Therefore, our conclusion is that the present method is quite effective for solving Riccati differential equations.

Acknowledgments

The authors would like to express thanks to unknown referees for their careful reading and helpful comments. The paper was supported by the National Natural Science Foundation of China (Grant no. 11026200) and the Special Funds of the National Natural Science Foundation of China (Grant No. 11141003).

References

[1] W. T. Reid, *Riccati Differential Equations*, Academic Press, New York, NY, USA, 1972.

[2] J. F. Carinena, G. Marmo, A. M. Perelomov, and M. F. Z. Rañada, "Related operators and exact solutions of Schrödinger equations," *International Journal of Modern Physics A*, vol. 13, no. 28, pp. 4913–4929, 1998.

[3] M. R. Scott, *Invariant Imbedding and Its Applications to Ordinary Differential Equations: an Introduction*, Addison-Wesley, London, UK, 1973.

[4] M. A. El-Tawil, A. A. Bahnasawi, and A. Abdel-Naby, "Solving Riccati differential equation using Adomian's decomposition method," *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 503–514, 2004.

[5] S. Abbasbandy, "Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method," *Applied Mathematics and Computation*, vol. 172, no. 1, pp. 485–490, 2006.

[6] S. Abbasbandy, "A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 59–63, 2007.

[7] S. Abbasbandy, "Iterated He's homotopy perturbation method for quadratic Riccati differential equation," *Applied Mathematics and Computation*, vol. 175, no. 1, pp. 581–589, 2006.

[8] M. Lakestani and M. Dehghan, "Numerical solution of Riccati equation using the cubic B-spline scaling functions and Chebyshev cardinal functions," *Computer Physics Communications*, vol. 181, no. 5, pp. 957–966, 2010.

[9] F. Z. Geng, Y. Z. Lin, and M. G. Cui, "A piecewise variational iteration method for Riccati differential equations," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2518–2522, 2009.

[10] B. Q. Tang and X. F. Li, "A new method for determining the solution of Riccati differential equations," *Applied Mathematics and Computation*, vol. 194, no. 2, pp. 431–440, 2007.

[11] A. Ghorbani and S. Momani, "An effective variational iteration algorithm for solving Riccati differential equations," *Applied Mathematics Letters*, vol. 23, no. 8, pp. 922–927, 2010.

[12] S. Momani and N. Shawagfeh, "Decomposition method for solving fractional Riccati differential equations," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1083–1092, 2006.

[13] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos, Solitons & Fractals*, vol. 36, no. 1, pp. 167–174, 2008.

[14] S. H. Hosseinnia, A. Ranjbar, and S. Momani, "Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3138–3149, 2008.

[15] F. Mohammadi and M. M. Hosseini, "A comparative study of numerical methods for solving quadratic Riccati differential equations," *Journal of the Franklin Institute*, vol. 348, no. 2, pp. 156–164, 2011.

[16] M. Cui and Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science Publishers Inc., New York, NY, USA, 2009.

[17] A. Berlinet and C. Thomas-Agnan, *Reproducing Kernel Hilbert Spaces in Probability and Statistics*, Kluwer Academic Publishers, Boston, Mass, USA, 2004.

[18] F. Z. Geng, "New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 165–172, 2009.

[19] F. Z. Geng, "Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2095–2102, 2009.

[20] F. Z. Geng and M. Cui, "Solving a nonlinear system of second order boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1167–1181, 2007.

[21] H. M. Yao and Y. Z. Lin, "Solving singular boundary-value problems of higher even-order," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 703–713, 2009.

[22] C. L. Li and M. G. Cui, "How to solve the equation AuBu + Cu = f," *Applied Mathematics and Computation*, vol. 133, no. 2-3, pp. 643–653, 2002.