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Research Article

Existence and Uniqueness of Solutions for the System of Nonlinear Fractional Differential Equations with Nonlocal and Integral Boundary Conditions

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In the present study, the nonlocal and integral boundary value problems for the system of nonlinear fractional differential equations involving the Caputo fractional derivative are investigated. Theorems on existence and uniqueness of a solution are established under some sufficient conditions on nonlinear terms. A simple example of application of the main result of this paper is presented.

1. Introduction

Differential equations of fractional order have been proved to be valuable tools in the modeling of many phenomena of various fields of science and engineering. Indeed, we can obtain numerous applications in viscoelasticity [1–3], dynamical processes in self-similar structures [4], biosciences [5], signal processing [6], system control theory [7], electrochemistry [8], diffusion processes [9], and linear time-invariant systems of any order with internal point delays [10]. Furthermore, fractional calculus has been found many applications in classical mechanics [11], and the calculus of variations [12], and is a very useful means for obtaining solutions of nonhomogenous linear ordinary and partial differential equations. For more details, we refer the reader to [13].

There are several approaches to fractional derivatives such as Riemann-Liouville, Caputo, Weyl, Hadamar and Grunwald-Letnikov, and so forth. Applied problems require those definitions of a fractional derivative that allow the utilization of physically interpretable initial and boundary conditions. The Caputo fractional derivative satisfies these demands, while the Riemann-Liouville derivative is not suitable for mixed boundary conditions. The details can be found in [14–17].

The study of existence and uniqueness, periodicity, asymptotic behavior, stability, and methods of analytic and numerical solutions of fractional differential equations have been studied extensively in a large cycle works (see, e.g., [10, 18–37] and the references therein). However, many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering, and cellular systems. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multipoint, and nonlocal boundary value problems as special cases, see [38–41].

In the present paper, we study existence and uniqueness of the problem for the system of nonlinear fractional differential equations of form

$$ {}^{c}D_{0+}^{\alpha}x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1.1}$$

with the nonlocal and integral boundary condition

$$Ex(0) + Bx(T) = \int_0^T g(s, x(s))ds, \tag{1.2}$$

where $E \in R^{n\times n}$ is an identity matrix, $B \in R^{n\times n}$ is the given matrix, and

$$\|B\| < 1. \tag{1.3}$$

Here, $f(t, x(t))$ and $g(t, x(t)) \in R^n$ are smooth vector functions, ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α, $0 < \alpha \leq 1$.

The organization of this paper is as follows. In Section 2, we provide necessary background. In Section 3, theorems on existence and uniqueness of a solution are established under some sufficient conditions on nonlinear terms. Finally, in Section 4, a simple example of application of the main result of this paper is presented.

2. Preliminaries

In this section, we present some basic definitions and preliminary facts which are used throughout the paper. By $C([0, T], R^n)$, we denote the Banach space of all vector continuous functions $x(t)$ from $[0, T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0, T]\}. \tag{2.1}$$

Definition 2.1. If $g(t) \in C[a, b]$ and $\alpha > 0$, then the operator I_{a+}^{α}, defined by

$$I_{a+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g(s)}{(t-s)^{1-\alpha}}ds, \tag{2.2}$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α. Here $\Gamma(\cdot)$ is the Gamma function defined for any complex number z as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.3}$$

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (a, b) \to R$ is defined by

$${}^cD_{a+}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.4}$$

where $n = [\alpha] + 1$, (the notation $[\alpha]$ stands for the largest integer not greater than α).

Remark 2.3. Under natural conditions on $g(t)$, the Caputo fractional derivative becomes the conventional integer order derivative of the function $g(t)$ as $\alpha \to n$.

Remark 2.4. Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:

$${}^cD_{0+}^\alpha t^\beta = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-1}, \quad \beta > n, \qquad {}^cD_{0+}^\alpha t^k = 0, \quad k = 0, 1, \ldots, n-1. \tag{2.5}$$

Lemma 2.5 (see, [42]). *For $\alpha > 0$, $g(t) \in C(0,1) \cap L_1(0,1)$, the homogenous fractional differential equation*

$${}^cD_{0+}^\alpha g(t) = 0, \tag{2.6}$$

has a solution

$$g(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.7}$$

where, $c_i \in R$, $i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.

Lemma 2.6 (see, [42]). *Assume that $g(t) \in C(0,1) \cap L_1(0,1)$, with derivative of order n that belongs to $C(0,1) \cap L_1(0,1)$, then*

$$I_{0+}^\alpha\, {}^cD_{0+}^\alpha g(t) = g(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.8}$$

where $c_i \in R$, $i = 0, 1, \ldots, n-1$, and $n = [\alpha] + 1$.

Lemma 2.7 (see, [42]). *Let $p, q \geq 0$, $f(t) \in L_1[0,T]$. Then*

$$I_{0+}^p I_{0+}^q f(t) = I_{0+}^{p+q} f(t) = I_{0+}^q I_{0+}^p f(t) \tag{2.9}$$

is satisfied almost everywhere on $[0,T]$. Moreover, if $f(t) \in C[0,T]$, then identity (2.9) is true for all $t \in [0,T]$.

Lemma 2.8 (see, [42]). *If $\alpha > 0, f(t) \in C[0,T]$, then ${}^cD^{\alpha}_{0+}I^{\alpha}_{0+}f(t) = f(t)$ for all $t \in [0,T]$.*

3. Main Results

Lemma 3.1. *Let $0 < \alpha \leq 1, y(t)$ and $g(t) \in C([0,T], R^n)$. Then, nonlocal boundary value problem*

$$ {}^cD^{\alpha}_{0+}x(t) = y(t), \quad t \in [0,T], \tag{3.1} $$

$$ Ex(0) + Bx(T) = \int_0^T g(s)ds \tag{3.2} $$

has a unique solution $x(t) \in C([0,T], R^n)$ given by

$$ x(t) = \int_0^T G(t,s)y(s)ds + C, \tag{3.3} $$

where

$$ G(t,s) = \begin{cases} E\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - (E+B)^{-1}B\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t, \\ \\ -(E+B)^{-1}B\dfrac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}, & t \leq s \leq T, \end{cases} \tag{3.4} $$

$$ C = (E+B)^{-1}\int_0^T g(s)ds. \tag{3.5} $$

Proof. Assume that $x(t)$ is a solution of nonlocal boundary value problem (3.1) and (3.2), then using Lemma 2.6, we get

$$ x(t) = I^{\alpha}_{0+}y(t) + c_1, \quad c_1 \in R^n. \tag{3.6} $$

Applying condition (3.2) and identity (3.6), we get

$$ c_1 + B\left(I^{\alpha}_{0+}y(T) + c_1\right) = \int_0^T g(s)ds. \tag{3.7} $$

From condition (1.3) it follows that the inverse of the matrix $E + B$ exists. Therefore, we can write

$$ c_1 = (E+B)^{-1}\int_0^T g(s)ds - (E+B)^{-1}BI^{\alpha}_{0+}y(T). \tag{3.8} $$

Using formulas (3.6) and (3.8), we obtain

$$x(t) = I_{0+}^{\alpha} y(t) + (E+B)^{-1} \int_0^T g(s)ds - (E+B)^{-1} B I_{0+}^{\alpha} y(T), \tag{3.9}$$

which can be written as (3.3). Lemma 3.1 is proved. □

Lemma 3.2. *Assume that f and $g \in C\ ([0,T] \times R^n, R^n)$. Then, the vector function $x(t) \in C\ ([0,T], R^n)$ is a solution of the boundary value problem* (1.1) *and* (1.2) *if and only if it is a solution of the integral equation*

$$x(t) = \int_0^T G(t,s) f(s, x(s))ds + (E+B)^{-1} \int_0^T g(s, x(s))ds. \tag{3.10}$$

Proof. If $x(t)$ solves boundary value problem (1.1) and (1.2). Then, by the same manner as in Lemma 3.1, we can prove that $x(t)$ is solution of integral equation (3.10). Conversely, let $x(t)$ is solution of integral equation (3.10). We denote that

$$v(t) = \int_0^T G(t,s) f(s, x(s))ds + (E+B)^{-1} \int_0^T g(s, x(s))ds. \tag{3.11}$$

Then, by Lemmas 2.7 and 2.8, we obtain

$$\begin{aligned} v(t) = {} & I_{0+}^{\alpha} f(t, x(t)) - (E+B)^{-1} B \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds \right) \\ & + (E+B)^{-1} \int_0^T g(s, x(s))ds. \end{aligned} \tag{3.12}$$

The application of the fractional differential operator ${}^cD_{0+}^{\alpha}$ to both sides of (3.12) yields

$$\begin{aligned} {}^cD_{0+}^{\alpha} v(t) = {} & {}^cD_{0+}^{\alpha} I_{0+}^{\alpha} f(t, x(t)) - (E+B)^{-1} B {}^cD_{o+}^{\alpha} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds \right) \\ & + {}^cD_{o+}^{\alpha} C = f(t, x(t)). \end{aligned} \tag{3.13}$$

Hence, $x(t)$ solves fractional differential equation (1.1). Also, it is easy to see that $x(t)$ satisfies nonlocal boundary condition (1.2). □

The first main statement of this paper is an existence and uniqueness of boundary value problem (1.1) and (1.2) result that it is based on a Banach fixed point theorem.

Theorem 3.3. *Assume that:*

(H1) *There exists a constant $L > 0$ such that*

$$\left|f(t,x) - f(t,y)\right| \leq L\left|x - y\right|, \tag{3.14}$$

for each $t \in [0,T]$ and all $x, y \in R^n$.

(H2) *There exists a constant $M > 0$ such that*

$$\left|g(t,x) - g(t,y)\right| \leq M\left|x - y\right|, \tag{3.15}$$

for each $t \in [0,T]$ and all $x, y \in R^n$.

If

$$\frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1 + (1 - \|B\|)^{-1}\|B\|\right)\right] + (1 - \|B\|)^{-1}MT < 1, \tag{3.16}$$

then boundary value problem (1.1) *and* (1.2) *has unique solution on* $[0,T]$.

Proof . Transform problem (1.1) and (1.2) into a fixed point problem. Consider the operator

$$P : C([0,T], R^n) \longrightarrow C([0,T], R^n), \tag{3.17}$$

defined by

$$P(x)(t) = \int_0^T G(t,s)f(s,x(s))ds + (E+B)^{-1}\int_0^T g(s,x(s))ds. \tag{3.18}$$

Clearly, the fixed points of the operator P are solution of problem (1.1) and (1.2). We will use the Banach contraction principle to prove that under assumption (3.16) operator P has a fixed point. It is clear that the operator P maps into itself and that

$$\begin{aligned}
&\left|P(x)(t) - P(y)(t)\right| \\
&\quad\leq \int_0^T |G(t,s)|\left|f(s,x(s)) - f(s,y(s))\right|ds + \left\|(E+B)^{-1}\right\|\int_0^T \left|g(s,x(s)) - g(s,y(s))\right|ds \\
&\quad\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1}\left|f(s,x(s)) - f(s,y(s))\right|ds + \left\|(E+B)^{-1}B\right\|\right. \\
&\qquad\qquad \left.\times\int_0^T (T-s)^{\alpha-1}\left|f(s,x(s)) - f(s,y(s))\right|ds\right] + \left\|(E+B)^{-1}\right\|MT\|x-y\| \\
&\quad\leq \left\{\frac{1}{\Gamma(\alpha+1)}\left[LT^{\alpha}\left(1 + \left\|(E+B)^{-1}B\right\|\right)\right] + \left\|(E+B)^{-1}\right\|MT\right\}\|x-y\|,
\end{aligned} \tag{3.19}$$

for any $x, y \in C([0,T], R^n)$ and $t \in [0,T]$. From condition (1.3) it follows that

$$\left\| (E+B)^{-1} \right\| \leq \frac{1}{1-\|B\|}. \tag{3.20}$$

Then, using estimates (3.19) and (3.20), we get

$$\begin{aligned} &\|P(x)(\cdot) - P(y)(\cdot)\| \\ &\quad \leq \left\{ \frac{1}{\Gamma(\alpha+1)} \left[LT^{\alpha} \left(1 + (1-\|B\|)^{-1}\|B\| \right) \right] + (1-\|B\|)^{-1} MT \right\} \|x-y\|. \end{aligned} \tag{3.21}$$

Consequently, by assumption (3.16) operator P is a contraction. As a consequence of Banach's fixed point theorem, we deduce that operator P has a fixed point which is a solution of problem (1.1) and (1.2). Theorem 3.3 is proved. □

The second main statement of this paper is an existence of boundary value problem (1.1) and (1.2) result that it is based on Schaefer's fixed point theorem.

Theorem 3.4. *Assume that:*

(H3) *The function* $f : [0,T] \times R^n \longrightarrow R^n$ *is continuous.*

(H4) *There exists a constant* $N_1 > 0$ *such that* $|f(t,x)| \leq N_1$ *for each* $t \in [0,T]$ *and all* $x \in R^n$.

(H5) *The function* $g : [0,T] \times R^n \longrightarrow R^n$ *is continuous.*

(H6) *There exists a constant* $N_2 > 0$ *such that* $|g(t,x)| \leq N_2$ *for each* $t \in [0,T]$ *and all* $x \in R^n$.

Then, boundary value problem (1.1) *and* (1.2) *has at least one solution on* $[0,T]$.

Proof. We will divide the proof into four main steps in which we will show that under the assumptions of theorem operator P has a fixed point. □

Step 1. Operator P under the assumptions of theorem is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T],R^n)$. Then, for each $t \in [0,T]$

$$\begin{aligned}
&|P(x_n)(t) - P(x)(t)| \\
&\quad\le \int_0^T |G(t,s)| \left| f(s,x_n(s)) - f(s,x(s)) \right| ds + \left\| (E+B)^{-1} \right\| \int_0^T \left| g(s,x_n(s)) - g(s,x(s)) \right| ds \\
&\quad\le \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} \max \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \right. \\
&\qquad\qquad \left. +(1-\|B\|)^{-1}\|B\| \int_0^T (T-s)^{\alpha-1} \max \left| f(s,x_n(s)) - f(s,x(s)) \right| ds \right] \\
&\qquad + (1-\|B\|)^{-1} MT \max \left| g(s,x_n(s)) - g(s,x(s)) \right| \\
&\quad\le \frac{1}{\Gamma(\alpha+1)} \left[LT^\alpha \left(1 + (1-\|B\|)^{-1}\|B\| \right) \right] \max \left| f(s,x_n(s)) - f(s,x_n(s)) \right| \\
&\qquad + (1-\|B\|)^{-1} MT \max \left| g(s,x_n(s)) - g(s,x(s)) \right|.
\end{aligned} \tag{3.22}$$

Since f and g are continuous functions, we have

$$\begin{aligned}
&\|P(x_n)(\cdot) - P(x)(\cdot)\| \\
&\quad\le \frac{1}{\Gamma(\alpha+1)} \left[LT^\alpha \left(1 + (1-\|B\|)^{-1}\|B\| \right) \right] \max \left| f(s,x_n(s)) - f(s,x_n(s)) \right| \\
&\qquad + (1-\|B\|)^{-1} MT \max \left| g(s,x_n(s)) - g(s,x(s)) \right| \longrightarrow 0
\end{aligned} \tag{3.23}$$

as $n \to \infty$.

Step 2. Operator P maps bounded sets in bounded sets in $C([0,T],R^n)$. Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l such that for each $x \in B_\eta = \{x \in C([0,T],R^n) : \|x\| \le \eta\}$, we have $\|P(x(\cdot))\| \le l$. By assumptions (H4) and (H6), we have for each $t \in [0,T]$,

$$|P(x)(t)| \le \int_0^T |G(t,s)| \left| f(s,x(s)) \right| ds + \left\| (E+B)^{-1} \right\| \int_0^T \left| g(s,x(s)) \right| ds. \tag{3.24}$$

Hence,

$$|P(x)(t)| \le \frac{N_1 T^\alpha}{\Gamma(\alpha+1)} \left[1 + (1-\|B\|)^{-1}\|B\| \right] + N_2 (1-\|B\|)^{-1} T. \tag{3.25}$$

Thus,

$$\|P(x)(\cdot)\| \le \frac{N_1 T^\alpha}{\Gamma(\alpha+1)}\left[1+(1-\|B\|)^{-1}\|B\|\right]+N_2(1-\|B\|)^{-1}T=l. \tag{3.26}$$

Step 3. Operator P maps bounded sets into equicontinuous sets of $([0,T],R^n)$.

Let $t_1,t_2\in(0,T]$, $t_1<t_2$, B_η be a bounded set of $C([0,T],R^n)$ as in Step 2, and let $x\in B_\eta$. Then,

$$\begin{aligned}|P(x)(t_2)-P(x)(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]f(s,x(s))ds\right.\\ &\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s))ds\right| \le \frac{N_1}{\Gamma(\alpha+1)}\left[2(t_2-t_1)^\alpha+\left(t_2^\alpha-t_1^\alpha\right)\right].\end{aligned} \tag{3.27}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that the operator $P:C([0,T],R^n)\to C([0,T],R^n)$ is completely continuous.

Step 4. A priori bounds. Now, it remains to show that the set

$$\Delta=\{x\in C([0,T],R^n): x=\lambda P(x)\ \text{for some}\ 0<\lambda<1\} \tag{3.28}$$

is bounded.

Let $x=\lambda(Px)$ for some $0<\lambda<1$. Then, for each $t\in[0,T]$ we have

$$x(t)=\lambda\left[\int_0^T G(t,s)f(s,x(s))ds+(E+B)^{-1}\int_0^T g(s,x(s))ds\right]. \tag{3.29}$$

This implies by assumptions (H4) and (H6) (as in Step 2) that for each $t\in[0,T]$ we have

$$|P(x)(t)| \le \frac{N_1 T^\alpha}{\Gamma(\alpha+1)}\left[1+(1-\|B\|)^{-1}\|B\|\right]+N_2(1-\|B\|)^{-1}T. \tag{3.30}$$

Thus, for every $t\in[0,T]$, we have

$$|x(t)| \le \frac{N_1 T^\alpha}{\Gamma(\alpha+1)}\left[1+(1-\|B\|)^{-1}\|B\|\right]+N_2(1-\|B\|)^{-1}T=R. \tag{3.31}$$

Therefore,

$$\|x\|\le R. \tag{3.32}$$

This shows that the set Δ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of problem (1.1) and (1.2). Theorem 3.4 is proved.

Moreover, we will give an existence result for problem (1.1) and (1.2) by means of an application of a Leray-Schauder type nonlinear alternative, where conditions (H4) and (H6) are weakened.

Theorem 3.5. *Assume that (H3), (H5) and the following conditions hold.*

(H7) *There exist $\theta_f \in L_1([0,T], R^+)$ and continuous and nondecreasing*

$$\psi_f : [0,\infty) \longrightarrow [0,\infty), \tag{3.33}$$

such that $|f(t,x)| \le \theta_f(t)\psi_f(|x|)$ for each $t \in [0,T]$ and all $x \in R^n$.

(H8) *There exist $\theta_g \in L_1([0,T], R^+)$ and continuous nondecreasing*

$$\psi_g : [0,\infty) \longrightarrow [0,\infty), \tag{3.34}$$

such that $|g(t,x)| \le \theta_g(t)\psi_g(|x|)$ for each $t \in [0,T]$ and all $x \in R^n$.

(H9) *There exists a number $K > 0$ such that*

$$\frac{K}{\psi_f(K)\left[\left\|I^\alpha\theta_f\right\|_{L_1} + (1-\|B\|)^{-1}\|B\|\left(I^\alpha\theta_f\right)(T)\right] + \psi_g(K)(1-\|B\|)^{-1}\left\|\theta_g\right\|_{L_1}} > 1. \tag{3.35}$$

Then, boundary value problem (1.1) *and* (1.2) *has at least one solution on* $[0,T]$.

Proof. Consider the operator P defined in Theorems 3.3 and 3.4. It can be easily shown that operator P is continuous and completely continuous. Let $x(t) = (Px)(t)$ for each $t \in [0,T]$. Then, from assumptions (H7) and (H8) if follows that for each $t \in [0,T]$

$$\begin{aligned}
|x(t)| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta_f(s)\psi_f(|x(s)|)ds + \frac{1}{\Gamma(\alpha)}\left\|(E+B)^{-1}B\right\|\int_0^T (T-s)^{\alpha-1}\theta_f(s)\psi_f(|x(s)|)ds\\
&+ \left\|(E+B)^{-1}\right\|\int_0^T \theta_g(s)\psi_g(|x(s)|)ds \le \psi_f(\|x\|)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\theta_f(s)ds\\
&+ \psi_f(\|x\|)\frac{1}{\Gamma(\alpha)}\left\|(E+B)^{-1}B\right\|\int_0^T (T-s)^{\alpha-1}\theta_f(s)ds + \psi_g(\|x\|)\left\|(E+B)^{-1}\right\|\\
&\times\int_0^T \theta_g(s)ds \le \psi_f(\|x\|)\left[\left\|I^\alpha\theta_f\right\|_{L_1} + (1-\|B\|)^{-1}\|B\|\left(I^\alpha\theta_f\right)(T)\right]\\
&+ \psi_g(\|x\|)(1-\|B\|)^{-1}\left\|\theta_g\right\|_{L_1}.
\end{aligned} \tag{3.36}$$

Thus,

$$\frac{\|x\|}{\psi_f(\|x\|)\left[\left\|I^\alpha\theta_f\right\|_{L_1}+(1-\|B\|)^{-1}\|B\|\left(I^\alpha\theta_f\right)(T)\right]+\psi_g(\|x\|)(1-\|B\|)^{-1}\left\|\theta_g\right\|_{L_1}}\leq 1. \tag{3.37}$$

Then, by condition (H9), there exists K such that $\|x\| \neq K$.

Let

$$U=\{x\in C([0,T],R):\|x\|<K\}. \tag{3.38}$$

The operator $P:\overline{U}\to C([0,T],R)$ is continuous and completely continuous. By the choice of U, there exists $x\in\partial U$ such that $x=\lambda P(x)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [43], we deduce that P has a fixed point x in $\overline{U}$, which is a solution of problem (1.1) and (1.2). This completes of proof of Theorem 3.5. □

4. An Example

In this section, we give an example to illustrate the usefulness of our main results. Let us consider the following nonlocal boundary value problem for the system of fractional differential equation

$$\begin{gathered}
{}^cD^\alpha x_1(t)=\frac{1}{10}\sin x_2,\quad t\in[0,1],\ 0<\alpha<1,\\
{}^cD^\alpha x_2(t)=\frac{|x_1|}{(9+e^t)(1+|x_1|)},\\
x_1=\int_0^1\sin 0.1x_2(t)\,dt,\qquad x_2(0)+0.5x_1(1)=1.
\end{gathered} \tag{4.1}$$

Evidently,

$$E+B=\begin{pmatrix}1&0\\0.5&1\end{pmatrix},\qquad B=\begin{pmatrix}0&0\\0.5&0\end{pmatrix},\qquad \|B\|=0.5,\qquad (1-\|B\|)^{-1}=2. \tag{4.2}$$

Hence, conditions (H1) and (H2) hold with $L=M=0,1$. We will check that condition (3.16) is satisfied for appropriate values of $0<\alpha\leq 1$ with $T=1$. Indeed,

$$\frac{0.2}{\Gamma(\alpha+1)}+0.2<1. \tag{4.3}$$

Then, by Theorem 3.3 boundary value problem (4.1) has a unique solution on $[0,1]$ for values of α satisfying condition (4.3). For example, if $\alpha=0.2$ then

$$\Gamma(\alpha+1)=\Gamma(1.2)=0.92,\qquad \frac{0.2}{\Gamma(\alpha+1)}+0.2=0.418<1. \tag{4.4}$$

5. Conclusion

In this work, some existence and uniqueness of a solution results have been established for the system of nonlinear fractional differential equations under the some sufficient conditions on nonlinear terms. Of course, such type existence and uniqueness results hold under the same sufficient conditions on nonlinear terms for the system of nonlinear fractional differential equations (1.1), subject to multipoint nonlocal and integral boundary conditions

$$Ex(0) + \sum_{j=1}^{J} B_j x(t_j) = \int_0^T g(s, x(s))ds, \tag{5.1}$$

where $B_j \in R^{n\times n}$ are given matrices and $\sum_{j=1}^{J} \|B_j\| < 1$. Here, $0 < t_1 < \cdots < t_J \le T$.

Moreover, applying the result of the paper [44] the first order of accuracy difference scheme for the numerical solution of nonlocal boundary value problem (1.1) and (5.1) can be presented. Of course, such type existence and uniqueness results hold under the some sufficient conditions on nonlinear terms for the solution system of this difference scheme.

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