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Research Article

Normality Criteria of Meromorphic Functions That Share a Holomorphic Function

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Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $P(f)$, $P(g)$ share ψ, then $\mathcal{F}$ is normal in D.

1. Introduction and Main Results

Let $\mathbb{C}$ be complex plane. Let D be a domain in $\mathbb{C}$. Let $\mathcal{F}$ be a family meromorphic functions defined in the domain D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞.

Let $f(z)$ and $g(z)$ be two meromorphic functions, let a be a finite complex number. If $f(z) - a$ and $g(z) - a$ have the same zeros, then we say they share a or share a IM (ignoring multiplicity) (see [1–3]).

Definition 1.1. Let $a_i(z)$, $(i = 1, 2, \ldots, q-1)$, $b_j(z)$, $(j = 1, 2, \ldots, n)$ be analytic in D, let $n_0, n_1, \ldots, n_k$ be nonnegative integers, set

$$\begin{aligned} P(\omega) &= \omega^q + a_{q-1}(z)\omega^{q-1} + \cdots + a_1(z)\omega, \\ M\left(f, f', \ldots, f^{(k)}\right) &= f^{n_0}\left(f'\right)^{n_1} \cdots \left(f^{(k)}\right)^{n_k}, \\ \gamma_M &= n_0 + n_1 + \cdots + n_k, \\ \Gamma_M &= n_0 + 2n_1 + \cdots + (k+1)n_k, \end{aligned} \tag{1.1}$$

where $M(f, f', \ldots, f^{(k)})$ is called a differential monomial of f, γ_M the degree of $M(f, f', \ldots, f^{(k)})$, and Γ_M the weight of $M(f, f', \ldots, f^{(k)})$.

From Definition 1.1, we give Definition 1.2.

Definition 1.2. Let $M_j(f, f', \ldots, f^{(k)})$, $(j = 1, 2, \ldots, n)$ be differential monomials of f. Set

$$H\left(f, f', \ldots, f^{(k)}\right) = b_1(z) M_1\left(f, f', \ldots, f^{(k)}\right) + \cdots + b_n(z) M_n\left(f, f', \ldots, f^{(k)}\right),$$
$$\gamma_H = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_n}\},$$
$$\Gamma_H = \max\{\Gamma_{M_1}, \Gamma_{M_2}, \ldots, \Gamma_{M_n}\}, \tag{1.2}$$

where $H(f, f', \ldots, f^{(k)})$ is called the differential polynomial of f, γ_H the degree of $H(f, f', \ldots, f^{(k)})$, and Γ_H the weight of $H(f, f', \ldots, f^{(k)})$,

$$\left.\frac{\Gamma}{\gamma}\right|_H = \max\left\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \ldots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\right\},$$
$$G(f) = P\left(f^{(k)}\right) + H\left(f, f', \ldots, f^{(k)}\right). \tag{1.3}$$

In 1979, Gu [4] proved the following result.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let a be a nonzero constant. If, for each function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq a$ in D, then $\mathcal{F}$ is normal in D.*

Yang [5] and Schwick [6] proved that Theorem A still holds if a is replaced by a holomorphic function $\psi(\not\equiv 0)$ in Theorem A.

Xu [7] improved Theorem A by the ideas of shared values and obtained the following result.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$ be a holomorphic functions and with only simple zeros in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, f has all multiple poles and $f \neq 0$. If, for every pair of functions f and g, $f^{(k)}$ and $g^{(k)}$ share ψ in D, then $\mathcal{F}$ is normal in D.*

Recently, Xu [7] did not know whether the condition ψ has only simple zero in D and f has all multiple poles are necessary or not in Theorem B.

In 2007, Fang and Chang considered the case $a = 0$ in Theorem A. In this note, Fang and Chang [8] proved the following result.

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, and let k be a positive integer, and let b be a nonzero complex number. If, for each $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and the zeros of $f^{(k)} - b$ have multiplicity at least $(k+2)/k$, then $\mathcal{F}$ is normal in D.*

Remark 1.3. The number $(k+2)/k$ is sharp, as is shown by the examples in [8].

In 2009, Xia and Xu [9] replaced the constant 1 by a function $\psi(z) \not\equiv 0$ in Theorem C. They obtained the following result.

Theorem D. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and all zeros of $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f - \psi(z)$ have multiplicity at least $(k+2)/k$. If, for $k = 1$, ψ has only zeros with multiplicities at most 2 and, for $k \geq 2$, ψ has only simple zeros, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Theorem D can be improved by the ideas of shared values. In this paper, we investigate the problem and obtain the following results.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $P(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $P(f)$ and $P(g)$ share ψ, then $\mathcal{F}$ is normal in D.*

By Theorem 1.4, we immediately deduce.

Corollary 1.5. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\not\equiv 0)$, $a_0, a_1, , \ldots, a_{k-1}$ be holomorphic functions in D, and k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)}$ and $g^{(k)}$ share ψ, then $\mathcal{F}$ is normal in D.*

Remark 1.6. By the ideas of sharing values, Theorem 1.4 and Corollary 1.5 yield the number $(k+2)/k$ can be omitted.

Remark 1.7. Obviously, Corollary 1.5 omitted the conditions ψ with only simple zeros, and, for every function $f \in \mathcal{F}$, f has all multiple poles in Theorem D. But the condition for every function $f \in \mathcal{F}$, $f^{(k)} \neq 0$ is additional. Hence, Corollary 1.5 improves Theorem B in some sense.

The condition $\psi \not\equiv 0$ in Theorem 1.4 is necessary. For example, we consider the following families.

Example 1.8. $\mathcal{F} = \{f_m(z) = e^{mz},\ m = 1, 2, \ldots\}$, obviously, any $f \in F$ satisfies $f \neq 0$, $f^{(k)} \neq 0$. For distinct positive integers m, l, $f_m^{(k)}$, and $f_l^{(k)}$ share 0 IM. However, the families $\mathcal{F}$ are not normal at $z = 0$.

Remark 1.9. Some ideas of this paper are based on [7, 9, 10].

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

The well-known Zalcman's lemma is a very important tool in the study of normal families. It has also undergone various extensions and improvements. The following is one up-to-date local version, which is due to Pang and Zaclman [11].

Lemma 2.1 (see [11, 12]). *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc Δ with the property that, for each $f \in \mathcal{F}$, all zeros are of multiplicity at least k. Suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f = 0$. If $\mathcal{F}$ is not normal in Δ, then, for $0 \leq \alpha \leq k$, there exist*

(1) *a number $r \in (0, 1)$;*

(2) *a sequence of complex numbers z_n, $|z_n| < r$;*

(3) *a sequence of functions* $f_n \in \mathcal{F}$;

(4) *a sequence of positive numbers* $\rho_n \to 0^+$;

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ *converge locally uniformly (with respect to the spherical metric) to a nonconstant meromorphic function* $g(\xi)$ *on* $\mathbb{C}$*, and, moreover, the zeros of* $g(\xi)$ *are of multiplicity at least* k, $g^\sharp(\xi) \le g^\sharp(0) = kA + 1$. *In particular,* g *has order at most 2.*

Here, as usual, $g^\sharp(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

Lemma 2.2 (see [1]). *Let* $f(z)$ *be a transcendental meromorphic function in* $\mathbb{C}$*, let* $k(\ge 1)$ *be a integer, and let* b *be a nonzero finite value, then* f *or* $f^{(k)} - b$ *has infinite zeros.*

Lemma 2.3 (see [7]). *Let* $f(z)$ *be a nonconstant rational function. Let* $k \ge 1$ *be an integer, and let* b *be a non-zero finite value. If* $f \ne 0$*, then* $f^{(k)}(z) - b$ *has at least two distinct zeros in the plane.*

Lemma 2.4. *Let* $f(z)$ *be a nonconstant rational function. Let* $k \ge 1$ *be an integer, and let* l *be a positive integer. If* $f \ne 0$, $f^{(k)} \ne 0$*, then* $f^{(k)}(z) - z^l$ *has at least two distinct zeros in the plane.*

Proof. Since $f \ne 0$ and $f^{(k)} \ne 0$, then f is a nonpolynomial rational function and has the form

$$f(z) = \frac{A}{(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_t)^{m_t}}, \tag{2.1}$$

where $A \ne 0$ is a constant, and $m_1, m_2, \dots, m_t$ are positive integers. Set $m = m_1 + m_2 + \cdots + m_t$. Then,

$$f'(z) = \frac{-A\left(mz^{t-1} + b_{t-2}z^{t-2} + \cdots + b_0\right)}{(z - z_1)^{m_1+1}(z - z_2)^{m_2+1} \cdots (z - z_t)^{m_t+1}}, \tag{2.2}$$

where $b_{t-2}, \dots, b_0$ are constants. For $k \ge 2$, by mathematical induction, we have

$$f^{(k)}(z) = \frac{Bz^{kt-k} + c_{kt-k-1}z^{kt-k-1} + \cdots + c_0}{(z - z_1)^{m_1+k}(z - z_2)^{m_2+k} \cdots (z - z_t)^{m_t+k}}, \tag{2.3}$$

where $B = (-1)^k m(m+1)(m+2)\cdots(m+k-1)A \ne 0$, $c_{kt-k-1}, \dots, c_0$ are constants. Since $f^{(k)} \ne 0$, we deduce that $t = 1$, and thus

$$f(z) = \frac{A}{(z - z_1)^{m_1}}, \tag{2.4}$$

$$f^{(k)}(z) = \frac{B}{(z - z_1)^{m_1+k}}. \tag{2.5}$$

Case 1 (if $f^{(k)} - z^l$ has exactly one zero z_0). From (2.5), we set

$$f^{(k)}(z) - z^l = \frac{B}{(z - z_1)^{m_1+k}} - z^l = \frac{B'(z - z_0)^{m_1+k+l}}{(z - z_1)^{m_1+k}}. \tag{2.6}$$

Obviously, B' is a nonzero constant and $l \ge 1$.

From (2.6), we obtain

$$f^{(k+l+1)}(z) = \frac{(z-z_0)^{m_1+k-1}P_1(z)}{(z-z_1)^{m_1+k+l+1}}, \tag{2.7}$$

where $P_1(z)\not\equiv 0$. By (2.4), we deduce

$$f^{(k+l+1)}(z) = \frac{A'}{(z-z_1)^{m_1+k+1+1}}, \tag{2.8}$$

where A' is nonzero constant.

Comparing (2.7) and (2.8), we obtain that $\deg A' = 0 \geq m_1 + k - 1$ is impossible.

Case 2 (if $f^{(k)}(z) - z^l \neq 0$). By (2.5), clearly Case 2 is impossible.

Lemma 2.4 is proved. □

Lemma 2.5 (see [7]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let $\psi(\neq 0)$ be a holomorphic function in D. If, for any $f \in \mathcal{F}$ satisfying $f \neq 0$ and if $f^{(k)}, g^{(k)}$ share ψ IM for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In this paper, by the same method of [7], we consider the differential polynomial in Lemma 2.5 and prove a more general result.

Lemma 2.6. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let k be a positive integer, and let $\psi(\neq 0)$ be a holomorphic function in D. If, for any $f \in \mathcal{F}$ satisfying $f \neq 0$ and if $G(f), G(g)$ share ψ IM for every pair of functions $f, g \in \mathcal{F}$, where $G(f)$ is a differential polynomial of f as the definition 1 satisfying $q \geq \gamma_H$, and $\Gamma/\gamma|_H < k+1$, then $\mathcal{F}$ is normal in D, where q, $\Gamma/\gamma|_H$ are as in Definitions 1.1 and 1.2.*

Proof. We may assume that $D = \Delta = \{|z| < 1\}$. Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exists a number $r \in (0,1)$; a sequence of complex numbers z_j, $z_j \to 0$ $(j \to \infty)$; a sequence of functions $f_j \in \mathcal{F}$; a sequence of positive numbers $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi)$ converges uniformly with respect to the spherical metric to a nonconstant meromorphic functions $\overline{g}(\xi)$ in C. Moreover, $\overline{g}(\xi)$ is of order at most 2. Hurwitz's theorem implies that $\overline{g}(\xi) \neq 0$.

We have

$$\begin{aligned} G(f_j)(z_j+\rho_j\zeta) &= P\left(f_j^{(k)}(z_j+\rho_j\zeta)\right) + H\left(f_j, f_j', \dots, f_j^{(k)}\right)(z_j+\rho_j\zeta), \\ H\left(f_j, f_j', \dots, f_j^{(k)}\right)(z_j+\rho_j\zeta) &= \sum_{i=1}^{n} b_i(z_j+\rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i}-\Gamma_{M_i}} M_i\left(g_j, g_j', \dots, g_j^{(k)}\right)(\zeta). \end{aligned} \tag{2.9}$$

Considering $b_i(z)$ is analytic on $D(i = 1, 2, \ldots, n)$, we have

$$\left|b_i(z_j + \rho_j\zeta)\right| \le M\left(\frac{1+r}{2}, b_i(z)\right) < \infty, \qquad (i = 1, 2, \ldots, n) \tag{2.10}$$

for sufficiently large j.

Hence, we deduce from $\Gamma/\gamma|_H < k + 1$ that

$$\sum_{i=1}^{n} b_i(z_j + \rho_j\zeta)\rho_j^{(k+1)\gamma_{M_i} - \Gamma_{M_i}} M_i\left(g_j, g_j', \ldots, g_j^{(k)}\right)(\zeta) \tag{2.11}$$

converges uniformly to 0 on every compact subset of $\mathbb{C}$ which contains no poles of $\overline{g}(\xi)$.

Thus, we have

$$\begin{gathered} G(f_j)(z_j + \rho_j\zeta) \longrightarrow P\left(\overline{g}^{(k)}\right)(\zeta), \\ G(f_j)(z_j + \rho_j\zeta) - \psi(z_j + \rho_j\zeta) \longrightarrow P\left(\overline{g}^{(k)}\right)(\zeta) - \psi(z_0) \end{gathered} \tag{2.12}$$

on every compact subset of $\mathbb{C}$ which contains no poles of $\overline{g}(\zeta)$.

Next, we will prove that $G(f_j)(\zeta) - \psi(z_0)$ has just a unique zero. By way of contradiction, let ζ_0 and ζ_0^* be two distinct solutions of $G(f_j)(\zeta) - \psi(z_0)$, and choose $\delta(> 0)$ small enough such that $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$ where $D(\zeta_0, \delta) = \{\zeta : |\zeta - \zeta_0| < \delta\}$ and $D(\zeta_0^*, \delta) = \{\zeta : |\zeta - \zeta_0^*| < \delta\}$. By Hurwitz's theorem, there exist points $\zeta_j \in D(\zeta_0, \delta)$, $\zeta_j^* \in D(\zeta_0^*, \delta)$ such that, for sufficiently large j,

$$\begin{gathered} G(f_j)(z_j + \rho_j\zeta_j) - \psi(z_0) = 0, \\ G(f_j)\left(z_j + \rho_j\zeta_j^*\right) - \psi(z_0) = 0. \end{gathered} \tag{2.13}$$

By the hypothesis that for each pair of functions f and g in $\mathcal{F}$, $G(f)$ and $G(g)$ share $\psi(z_0)$ in D, we know that, for any positive integer m,

$$\begin{gathered} G(f_m)(z_j + \rho_j\zeta_j) - \psi(z_0) = 0, \\ G(f_m)\left(z_j + \rho_j\zeta_j^*\right) - \psi(z_0) = 0. \end{gathered} \tag{2.14}$$

Fix m, take $j \to \infty$, and note $z_j + \rho_j\zeta_j \to 0$, $z_j + \rho_j\zeta_j^* \to 0$, then

$$G(f_m)(0) - \psi(z_0) = 0. \tag{2.15}$$

Since the zeros of $G(f_m)(0) - \psi(z_0) = 0$ have no accumulation point, so $z_j + \rho_j\zeta_j = 0$, $z_j + \rho_j\zeta_j^* = 0$.

Hence,

$$\zeta_j = -\frac{z_j}{\rho_j}, \qquad \zeta_j^* = -\frac{z_j}{\rho_j}. \tag{2.16}$$

This contradicts with $\zeta_j \in D(\zeta_0, \delta)$, $\zeta_j^* \in D(\zeta_0^*, \delta)$, and $D(\zeta_0, \delta) \cap D(\zeta_0^*, \delta) = \emptyset$. So $G(f_j) - \psi(z_0)$ has just a unique zero. By Hurwitz's theorem, we know $P(\overline{g}^{(k)})(\zeta) - \psi(z_0)$ has just a unique zero.

By Lemmas 2.2 and 2.3, we know $\overline{g}^{(k)}(\zeta) - \psi(z_0)$ has at least two distinct zeros. From the definition of $P(w)$, we deduce that $P(\overline{g}^{(k)}(\zeta)) - \psi(z_0)$ has more than two distinct zeros, a contradiction.

So $\mathcal{F}$ is normal in D. Lemma 2.6 is proved. □

By Lemma 2.6, we immediately deduce the following lemma.

Lemma 2.7. *Let $\mathcal{F}$ be a family of meromorphic functions defined in D, let $\psi(\neq 0)$, $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions in D, and let k be a positive integer. Suppose that, for every function $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f \neq 0$ and, for every pair functions $(f, g) \in \mathcal{F}$, $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_1 f' + a_0 f$, $g^{(k)} + a_{k-1}g^{(k-1)} + \cdots + a_1 g' + a_0 g$ share ψ, then $\mathcal{F}$ is normal in D.*

Lemma 2.8 (see [1]). *Let $f(z)$ be a meromorphic function. Let k be a positive integer. If $f(z) \neq 0$, then $f^{(k)}(z) \neq 1$, then f is a constant.*

Lemma 2.9 (see [13, 14]). *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$, and let $P(\not\equiv 0)$ be a polynomial. Let k be a positive integer. If all zeros (except at most finite zeros) of $f(z)$ have the multiplicity at least 3, then $f^{(k)}(z) - P(z)$ has infinite zeros.*

3. Proof of Theorem 1.4

Proof. Since normality is a local property, without loss of generality, we may assume $D = \Delta = \{z : |z| < 1\}$, and

$$\psi(z) = z^l \varphi(z) \quad (z \in \Delta), \tag{3.1}$$

where l is a positive integer, $\varphi(0) = 1$, $\varphi(z) \neq 0$ on $\Delta' = \{z : 0 < |z| < 1\}$. By Lemma 2.6, we only need to prove that $\mathcal{F}$ is normal at $z = 0$.

If $f \in \mathcal{F}$, $P(f)(0) \neq \psi(0)$, then there exists $\delta > 0$ such that $P(f)(z) \neq \psi(z)$ on Δ_δ. By condition of Theorem, for every $g \in \mathcal{F}$, we know $P(g)(z) \neq \psi(z)$ on Δ_δ. By theorem D, $\mathcal{F}$ is normal on Δ_δ, so $\mathcal{F}$ is normal on $z = 0$.

Now, we consider $P(f)(0) = \psi(0)$. Suppose $P(f)(z) \not\equiv \psi(z)$ on the neighborhood $|z| < \delta$ (where δ is a small positive number) (otherwise, $P(f)(z) \equiv \psi(z)$ on the neighborhood $|z| < \delta$, by condition of theorem, for every $g \in \mathcal{F}$, we also obtain $P(g)(z) \equiv \psi(z)$. So $P(g)(z) \neq \psi(z)+1$. By Theorem D, $\mathcal{F}$ is normal at $z = 0$. So Theorem 1.4 is proved), there exists $\delta > 0$ such that $P(f)(z) \neq \psi(z)$ on $(z \in \Delta'_\delta)$. So, for every $g \in \mathcal{F}$, we obtain

$$P(g)(z) \neq \psi(z) \quad (z \in \Delta'_\delta). \tag{3.2}$$

By Theorem D, $\mathcal{F}$ is normal on Δ'.

Next, we will prove $\mathcal{F}$ is normal at $z = 0$. Suppose, on the contrary, that $\mathcal{F}$ is not normal at $z = 0 \in \Delta$, then there exists a sequence functions (we also denote $\mathcal{F}$) that has no any normal subsequence on $z = 0$.

Consider the family $\Im = \{g(z) = (f(z)/\psi(z)) : f \in \mathcal{F}, z \in \Delta\}$. Since $f \neq 0$ for $f \in \mathcal{F}$, we have that $g(0) = \infty$ for each $g \in \Im$.

We first prove that $\Im$ is normal in Δ. Suppose, on the contrary, that $\Im$ is not normal at $z_0 \in \Delta$. By Lemma 2.1, there exist a sequence of functions $g_n \in \Im$, a sequence of complex numbers $z_n \to z_0$, and a sequence of positive numbers $\rho_n \to 0$, such that

$$G_n(\xi) = \frac{g_n(z_n + \rho_n\xi)}{\rho_n^k} \longrightarrow G(\xi) \tag{3.3}$$

converges spherically uniformly on compact subsets of $\mathbb{C}$ where $G(\xi)$ is a nonconstant meromorphic function on $\mathbb{C}$, and $G(\xi) \neq 0$.

We distinguish two cases.

Case 1 ($z_n/\rho_n \to \infty$). By a simple calculation, for $0 \leq i \leq k$, we have

$$\begin{aligned} g_n^{(i)}(z) &= \frac{f_n^{(i)}(z)}{\psi(z)} - \sum_{j=1}^{i} C_i^j g_n^{(i-j)}(z) \frac{\psi^{(j)}(z)}{\psi(z)} \\ &= \frac{f_n^{(i)}(z)}{\psi(z)} - \sum_{j=1}^{i}\left[C_i^j g_n^{(i-j)}(z) \sum_{t=0}^{j} A_{jt} \frac{1}{z^{j-t}} \frac{\varphi^{(t)}(z)}{\varphi(z)}\right], \end{aligned} \tag{3.4}$$

where $A_{jt} = l(l-1)\cdots(l-j+t+1)C_j^t$ if $l < j$, for $t = 0, 1, \ldots, j-1$ and $A_{jj} = 1$.

Thus, from (3.4), we have

$$\begin{aligned} \rho_n^{k-i} G_n^{(i)}(\xi) &= g_n^{(i)}(z_n + \rho_n\xi) \\ &= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\psi(z_n + \rho_n\xi)} - \sum_{j=1}^{i}\left[C_i^j g_n^{(i-j)}(z_n + \rho_n\xi) \sum_{t=0}^{j} A_{jt} \frac{1}{(z_n + \rho_n\xi)^{j-t}} \frac{\varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)}\right] \\ &= \frac{f_n^{(i)}(z_n + \rho_n\xi)}{\psi(z_n + \rho_n\xi)} - \sum_{j=1}^{i}\left[C_i^j \frac{g_n^{(i-j)}}{\rho_n^j}(z_n + \rho_n\xi) \sum_{t=0}^{j} A_{jt} \frac{1}{(z_n + \rho_n\xi)^{j-t}} \frac{\rho_n^t \varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)}\right]. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{(z_n/\rho_n) + \xi} = 0, \\ &\lim_{n\to\infty} \frac{\rho_n^t \varphi^{(t)}(z_n + \rho_n\xi)}{\varphi(z_n + \rho_n\xi)} = 0, \end{aligned} \tag{3.6}$$

for $t \geq 1$. Noting that $g_n^{(i-j)}(z_n + \rho_n\xi)/\rho_n^j$ is locally bounded on $\mathbb{C}$ minus the set of poles of $G(\xi)$ since $g_n(z_n + \rho_n\xi)/\rho_n^k \to G(\xi)$. Therefore, on every subset of $\mathbb{C}$ which contains no poles of $G(\xi)$, we have

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi),$$
$$\frac{f_n^{(i)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow 0, \tag{3.7}$$

for $i=0,1,\ldots,k-1$, and thus

$$\frac{f_n^{(k)}(z_n+\rho_n\xi)+\Sigma_{i=0}^{k-1}a_i(z_n+\rho_n\xi)f_n^{(i)}(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi),$$
$$\frac{f_n^{(k)}(z_n+\rho_n\xi)+\Sigma_{i=0}^{k-1}a_i(z_n+\rho_n\xi)f_n^{(i)}(z_n+\rho_n\xi)-\psi(z_n+\rho_n\xi)}{\psi(z_n+\rho_n\xi)} \longrightarrow G^{(k)}(\xi)-1, \tag{3.8}$$

since $a_0,\ldots,a_{k-1}$ are analytic in D.

By $G(\xi)\neq 0$, we know $G^{(k)}(\xi)\not\equiv 1$. In fact, if $G^{(k)}(\xi_0)=1$, by Hurwitz's theorem, then exists $\xi_n \to \xi_0$, for n sufficiently large,

$$P(f)(z_n+\rho_n\xi_n)=\psi(z_n+\rho_n\xi_n). \tag{3.9}$$

By the condition of theorem, for every positive number m, we obtain $P(f_m)(z_n+\rho_n\xi_n)=\psi(z_n+\rho_n\xi_n)$. We know $z_n+\rho_n\xi_n \to z_0\in\Delta_\delta$, and, for sufficiently large n, $z_n+\rho_n\xi_n\in\Delta_\delta$. However, $z_n+\rho_n\xi_n\neq 0$ (otherwise, $z_n+\rho_n\xi_n=0$, so $\xi_n=-(z_n/\rho_n)\to\infty$, a contradiction), so for sufficiently large n, $z_n+\rho_n\xi_n\in\Delta'_\delta$. This contradicts with (3.2).

So $G(\xi)\neq 0$ and $G^{(k)}(\xi)\neq 1$, by Lemma 2.8, we obtain G is a constant, a contradiction.

Case 2. $z_n/\rho_n \to \alpha$ is a finite complex number. Then,

$$\frac{g_n(\rho_n\xi)}{\rho_n^k}=\frac{g_n(z_n+\rho_n(\xi-(z_n/\rho_n)))}{\rho_n^k}=G_n\left(\xi-\frac{z_n}{\rho_n}\right)\longrightarrow G(\xi-\alpha)=\mathbb{G}(\xi). \tag{3.10}$$

Obviously, $\mathbb{G}(\xi)\neq 0$, and $\xi=0$ is a pole of $\mathbb{G}$ with order at least l.

Set

$$H_n(\xi)=\frac{f_n(\rho_n\xi)}{\rho_n^{k+l}}. \tag{3.11}$$

Then,

$$H_n(\xi)=\frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{f_n(\rho_n\xi)}{\rho_n^k\psi(\rho_n\xi)}=\frac{\psi(\rho_n\xi)}{\rho_n^l}\frac{g_n(\rho_n\xi)}{\rho_n^k}. \tag{3.12}$$

Noting that $\psi(\rho_n\xi)/\rho_n^l \to \xi^l$, thus

$$H_n(\xi) \longrightarrow \xi^l \mathbb{G}(\xi) = H(\xi), \tag{3.13}$$

uniformly on compact subsets of $\mathbb{C}$. Since $\mathbb{G}$ has a pole of order at least at $\xi = 0$, we have $H(0) \neq 0$, so that $H(\xi) \neq 0$.

From (3.11), we get

$$H_n^{(i)} = \frac{f_n^{(i)}(\rho_n\xi)}{\rho_n^{k+l-i}} \longrightarrow H^{(i)}(\xi), \tag{3.14}$$

spherically uniformly on compact subsets of $\mathbb{C}$ minus the set of poles of $\mathbb{G}(\xi)$. As the above, on every compact subset of $\mathbb{C}$ minus the set of poles of $G(\xi)$, we have

$$\frac{f_n^{(k)}(\rho_n\xi) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi) f_n^{(i)}(\rho_n\xi)}{\rho_n^l} \longrightarrow H^{(k)}(\xi), \tag{3.15}$$

$$\frac{f_n^{(k)}(\rho_n\xi) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi) f_n^{(i)}(\rho_n\xi) - \psi(\rho_n\xi)}{\rho_n^l} \longrightarrow H^{(k)}(\xi) - \xi^l, \tag{3.16}$$

locally uniformly on $\mathbb{C}$.

By the assumption of Theorem and (3.16), Hurwitz's theorem implies $H^{(k)}(\xi) \neq 0$.

Next, we proof that if $\xi \in \mathbb{C}/\{0\}$, then $H^{(k)}(\xi) \neq \xi^l$.

First, $H^{(k)}(\xi) \not\equiv \xi^l$, otherwise $H^{(k)}(\xi) \equiv \xi^l$, which contradicts with $H(\xi) \neq 0$. If there exists a $\xi_0 \neq 0$ such that $H^{(k)}(\xi_0) = \xi_0^l$, by Hurwitz's theorem and (3.16), there exists $\xi_n \to \xi_0$ such that $f_n^{(k)}(\rho_n\xi_n) + \Sigma_{i=0}^{k-1} a_i(\rho_n\xi_n) f_n^{(i)}(\rho_n\xi_n) = \psi(\rho_n\xi_n)$. By the assumption of Theorem 1.4, for every positive m such that $P(f_m)(\rho_n\xi_n) = \psi(\rho_n\xi_n)$. However, for n sufficiently large, $\rho_n\xi_n \in \Delta'_\delta$, all of these contradict with (3.2). So if $\xi \in \mathbb{C}/\{0\}$, then $H^{(k)}(\xi) \neq \xi^l$.

Noting $H(\xi) \neq 0$, By Lemma 2.9, we know H must be a rational function. If H is not a constant, By Lemma 2.4, we know $H^{(k)}(\xi) - \xi^l$ has at least two distinct zeros, a contradiction. So H must be a nonzero constant, also contradicts with $H^{(k)}(\xi) \neq 0$. Now, we have proved the $\mathfrak{I}$ is normal on Δ_δ.

It remains to show that $\mathcal{F}$ is normal at $z = 0$. Since $\mathfrak{I}$ is normal in Δ, then the family $\mathfrak{I}$ is equicontinuous on Δ with respect to the spherical distance. On the other hand, $g(0) = \infty$ for each $g \in \mathfrak{I}$, so there exists $\delta > 0$ such that $|g(z)| \geq 1$ for all $g \in \mathfrak{I}$ and each $z \in \Delta_\delta = \{z : |z| < \delta\}$. Suppose that $\mathcal{F}$ is not normal at $z = 0$. Since $\mathcal{F}$ is normal in $0 < |z| < 1$, the family $\mathcal{F}_1 = \{1/f : f \in \mathfrak{I}\}$ is normal in $\Delta = \{z : 0 < |z| < 1\}$, but it is not normal at $z = 0$. Then, there exists a sequence $\{1/f_n\} \subset \mathcal{F}_1$ which converges locally uniformly in Δ', but not in Δ. Noting that $f_n \neq 0$ in Δ, $1/f_n$ is holomorphic in Δ for each n. The maximum modulus principle implies that $1/f_n \to \infty$ in Δ'. Thus, $f_n \to 0$ converges locally uniformly in Δ', and hence so does $\{g_n\} \subset \mathfrak{I}$, where $g_n = f_n/\varphi$. But $|g_n(z)| \geq 1$ for each $z \in \Delta_\delta$, a contradiction. This finally completes the proof of Theorem 1.4. □

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