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Research Article

On the Well-Posedness of the Boussinesq Equation in the Triebel-Lizorkin-Lorentz Spaces

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We establish the local well-posedness and obtain a blow-up criterion of smooth solutions for the Boussinesq equations in the framework of Triebel-Lizorkin-Lorentz spaces. The main ingredients of our proofs are Littlewood-Paley decomposition and the paradifferential calculus.

1. Introduction

In this paper, we consider the following inviscid Boussinesq equations:

$$\begin{gathered}
\partial_t u + u \cdot \nabla u + \nabla p = \theta e_d, \quad \text{in } \mathbb{R}^d \times (0,T), \\
\partial_t \theta + u \cdot \nabla \theta = 0, \quad \text{in } \mathbb{R}^d \times (0,T), \\
\operatorname{div} u = 0, \quad \text{in } \mathbb{R}^d \times (0,T), \\
u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \quad \text{in } \mathbb{R}^d,
\end{gathered} \tag{1.1}$$

with $\operatorname{div} u_0 = 0$ and $e_d = (0,\ldots,0,1)$. This system describes the motion of lighter or denser incompressible fluid under the influence of gravitational forces, for instance, in the atmospheric sciences, where the vector $u = (u_1, u_2, \ldots, u_d)$ is the velocity, the scalar function θ denotes the temperature, and p stands for the pressure in the fluid (see [1]). The term θe_d takes into account the influence of the gravity and the stratification on the motion of the fluid. In case $d = 2$, it is also used as a simplified model for the 3D axisymmetric Euler equations with swirl away from the symmetric axis $r = 0$.

The regularity or singularity questions of (1.1) are an outstanding open problem in the mathematical fluid mechanics. Over the past few years, (1.1) has been studied extensively both theoretically (see [1–6] and references therein) and numerically (see [7–9]). In particular, the local well-posedness and some blow-up criteria of (1.1) are established by Chae and Nam [3] in the Sobolev spaces, Liu et al. [5, 6] and Xiang [10] in the Besov spaces, and Cui et al. [11] in the Hölder spaces. The purpose of this paper is to establish the local well-posedness for the Boussineq equations (1.1) and to obtain a blow-up criterion of the smooth solutions in the framework of Triebel-Lizorkin-Lorentz spaces, which contain the classical Triebel-Lizorkin spaces and Lorentz spaces. Indeed, it is more natural to consider the well-posedness in the Triebel-Lizorkin-type spaces than in the Besov spaces in the sense that $F^s_{p,2} = W^{s,p}$ for $1 < p < \infty$.

Before proceeding further, we mention several local well-posedness results for other fluid equations in the Triebel-Lizorkin spaces. Recently, Chae [12] introduced a family trajectory mapping $\{X_j(\alpha, t)\}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t} X_j(\alpha, t) = S_{j-2} v(X_j(\alpha, t), t), \qquad X_j(\alpha, 0) = \alpha, \tag{1.2}$$

where S_{j-2} is a frequency projection to the ball $\{\xi \in \mathbb{R}^2 \mid |\xi| \lesssim 2^j\}$ and used the following equivalent relation:

$$\left\| 2^{js} \| \Delta_j \theta(X_j(\alpha, t), t) \|_{L^q(\mathbb{Z})} \right\|_{L^p(\mathrm{d}\alpha)} \sim \left\| 2^{js} \| \Delta_j \theta(x, t) \|_{L^q(\mathbb{Z})} \right\|_{L^p(\mathrm{d}x)} = \| \theta(t) \|_{\dot{F}^s_{p,q}}, \tag{1.3}$$

to estimate the frequency-localized solutions of the Euler equations in the Triebel-Lizorkin spaces. However, it seems difficult to give a strict proof for (1.3)-type equivalent relation due to the lack of the uniform change of the coordinates independent of j. To avoid this trouble, Chen et al. [13] introduced a particle trajectory mapping $X(\alpha, t)$ independent of j defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} X(\alpha, t) = v(X(\alpha, t), t), \qquad X(\alpha, 0) = \alpha, \tag{1.4}$$

and then they established a new commutator estimate to obtain the local well-posedness of the ideal MHD equations in the Triebel-Lizorkin spaces.

In this paper, we will adapt the method of Chen et al. [13] to establish the local well-posedness and to obtain a blow-up criterion of the smooth solutions for the Boussineq equations (1.1) in the framework of Triebel-Lizorkin-Lorentz spaces. Precisely, we first define the particle trajectory mapping $X(\alpha, t)$ by (1.4). Then to show the well-posedness in the new framework, we have to establish the commutator estimate (Proposition 2.5) and the product estimate (Proposition 2.6). For the blow-up criterion, we also need the logarithmic inequality (Proposition 2.7). Fortunately, these preliminary estimates and inequalities have been obtained in our recent work [14].

Our main results can be formulated in the following way.

Theorem 1.1. *(i) Local Existence. Let* $1 < r \le p < \infty$, $1 < q < \infty$, *and* $s > d/p + 1$. *If* $(u_0, \theta_0) \in (F^{s,r}_{p,q}(\mathbb{R}^d))^{d+1}$, *then there exists* $T_1 = T_1(\|u_0\|_{F^{s,r}_{p,q}}, \|\theta_0\|_{F^{s,r}_{p,q}}) > 0$ *such that the Boussinesq equation* (1.1) *has a unique solution* $(u, \theta) \in (C([0, T_1]; F^{s,r}_{p,q}(\mathbb{R}^d)))^{d+1}$.

(ii) Blow-Up Criterion. The local-in-time solution $(u, \theta) \in (C([0, T_1]; F^{s,r}_{p,q}(\mathbb{R}^d)))^{d+1}$ *constructed in (i) blows up at finite time* $T^* > T_1$, *that is,*

$$\limsup_{t \to T^*-} \left(\|u(t)\|_{F^{s,r}_{p,q}} + \|\theta(t)\|_{F^{s,r}_{p,q}} \right) = \infty, \tag{1.5}$$

if and only if

$$\int_0^{T^*} \left(\|u(t)\|_{\dot{F}^1_{\infty,\infty}} + \|\theta(t)\|_{\dot{F}^1_{\infty,\infty}} \right) \mathrm{d}t = \infty. \tag{1.6}$$

Remark 1.2. In case $\theta_0 \equiv 0$, we have $\theta \equiv 0$ and thus get the local well-posedness and blow-up criterion for the incompressible Euler equations in the Triebel-Lizorkin-Lorentz spaces, which generalize the corresponding results in [12] because of $F^{s,p}_{p,q} = F^s_{p,q}$. For general θ_0, we also extend the results of [3] from Sobolev spaces to Triebel-Lizorkin-Lorentz spaces.

Remark 1.3. For the MHD equations, we can also generalize the local well-posedness and blow-up criterion of [13] from the Triebel-Lizorkin spaces to the Triebel-Lizorkin-Lorentz spaces by using similar arguments.

The rest of this paper is organized as follows. We state some preliminaries on functional settings in Section 2 and then prove the local well-posedness and blow-up criterion in Section 3.

2. Preliminaries

In this section, we give some definitions and basic estimates (see [14, 15] for details). First of all, we let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. For a fixed radially symmetric bump function $\chi \in C_0^\infty(B(0, 4/3))$ with value 1 over the ball $B(0, 3/4)$, we set $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$ and then have the following dyadic partition of unity:

$$\begin{gathered} \chi(\xi) + \sum_{j \ge 0} \varphi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbb{R}^d, \\ \sum_{j \in \mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \end{gathered} \tag{2.1}$$

The frequency localization operators Δ_j and S_j can be defined as follows:

$$\begin{gathered} \Delta_j u := \varphi\left(2^{-j}D\right)u = \mathcal{F}^{-1}\left(\varphi\left(2^{-j}\cdot\right)\mathcal{F}(u)\right) = 2^{jd}\left(\mathcal{F}^{-1}\varphi\left(2^j\cdot\right)\right) * u, \quad j \in \mathbb{Z}, \\ S_j u := \sum_{j' \le j-1} \Delta_{j'} u = \chi\left(2^{-j}D\right)u = 2^{jd}\left(\mathcal{F}^{-1}\chi\left(2^j\cdot\right)\right) * u, \quad j \in \mathbb{Z}, \end{gathered} \tag{2.2}$$

where $\mathcal{F}$ is the Fourier transform and D is the Fourier multiplier with symbol $|\xi|$. The frequency localization operators Δ_j and S_j have nice almost orthogonal properties in L^2:

$$\Delta_j \Delta_{j'} u \equiv 0, \quad \text{if } |j - j'| \geq 2, \qquad \Delta_j (S_{j'-1} u \Delta_{j'} v) \equiv 0, \quad \text{if } |j - j'| \geq 5 \tag{2.3}$$

for any $u, v \in \mathcal{S}'(\mathbb{R}^d)$.

Next we recall several function spaces used in this paper. The first one is *the Triebel-Lizorkin spaces*. For $s \in \mathbb{R}$, $(p, q) \in [1, +\infty) \times [1, +\infty]$, we define the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^d)$ as the set of tempered distributions $u \in \mathcal{S}' / \mathcal{P}(\mathbb{R}^d)$ such that

$$\|u\|_{\dot{F}^s_{p,q}(\mathbb{R}^d)} := \left\| \left\| 2^{js} |\Delta_j u| \right\|_{l^q(\mathbb{Z})} \right\|_{L^p(\mathbb{R}^d)} < \infty, \tag{2.4}$$

where $\mathcal{P}$ is the polynomial space. Then for $s > 0$, $(p, q) \in [1, +\infty) \times [1, +\infty]$, we can define the inhomogeneous Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d)$ as the set of tempered distributions u such that

$$\|u\|_{F^s_{p,q}(\mathbb{R}^d)} := \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{\dot{F}^s_{p,q}(\mathbb{R}^d)} < \infty. \tag{2.5}$$

The inhomogeneous Triebel-Lizorkin spaces $F^s_{p,q}$ are Banach spaces with norms $\|\cdot\|_{F^s_{p,q}}$.

The second one is *the Lorentz spaces*. Let $u_*(\tau)$ be the distribution function of a function $u(x)$ and let $u^*(t)$ be its nonincreasing rearrangement, that is,

$$u_*(\tau) = \operatorname{meas}\{x \mid |u(x)| > \tau\}, \qquad u^*(\tau) = \inf\{\tau > 0 \mid u_*(\tau) \leq t\}. \tag{2.6}$$

Then for $1 < p,\ r < \infty$, the Lorentz spaces $L^{p,r}(\mathbb{R}^d)$ are defined as the set of those measurable functions u satisfying

$$\|u\|_{L^{p,r}} = \left(\frac{r}{p} \int_0^\infty \left(t^{1/p} u^{**}(t) \right)^r \frac{\mathrm{d}t}{t} \right)^{1/r} < \infty, \tag{2.7}$$

where $u^{**}(t) := (1/t) \int_0^t u^*(s) \mathrm{d}s$ is the average rearrangement. The Lorentz spaces $L^{p,r}$ are Banach spaces with norms $\|\cdot\|_{L^{p,r}}$. Since $u_*(\tau)$ is nonincreasing in τ, it is easy to verify that the following equivalent characterization holds:

$$\|u\|_{L^{p,r}} \sim \left\| 2^j u_*^{1/p}\left(2^j\right) \right\|_{l^r}. \tag{2.8}$$

Moreover, it is well known that there exist $1 \leq p_1$ and $p_2 < \infty$ such that $L^{p,r}$ are the interpolations spaces between L^{p_1} and L^{p_2}. Thus we can deduce the following basic fact, which is standard in the usual L^p spaces.

Lemma 2.1. *Assume $u \in L^{p,r}$ for $1 < p,\ r < \infty$. If $X : \alpha \to X(\alpha)$ is a volume-preserving diffeomorphism, then*

$$\|u(\alpha)\|_{L^{p,r}} = \|u(X(\alpha))\|_{L^{p,r}}. \tag{2.9}$$

The last one is called *the Triebel-Lizorkin-Lorentz spaces* and is introduced recently by Yang et al. [16] to unify the Triebel-Lizorkin spaces and the Lorentz spaces.

Definition 2.2. For $s \in \mathbb{R}$ and $1 < p, q,\ r < \infty$, a distribution $u \in \mathcal{S}'$ is said to be in the homogeneous Triebel-Lizorkin-Lorentz spaces $\dot{F}^{s,r}_{p,q}(\mathbb{R}^d)$ if

$$\left\| 2^k \left(\text{meas}\left\{ x \in \mathbb{R}^d \mid \left\| 2^{jS} \left| \Delta_j u \right| \right\|_{l^q_j(\mathbb{Z})} > 2^k \right\} \right)^{1/p} \right\|_{l^r_k(\mathbb{Z})} < \infty. \tag{2.10}$$

Indeed we can deduce that $u \in \dot{F}^{s,r}_{p,q}(\mathbb{R}^d)$ *if and only if* $\| \| 2^{js} |\Delta_j u| \|_{l^q(\mathbb{Z})} \|_{L^{p,r}} < \infty$ by the equivalent characterization (2.8) of Lorentz space $L^{p,r}$. Then $\dot{F}^{s}_{p,q}(\mathbb{R}^d) \backsimeq \dot{F}^{s,p}_{p,q}(\mathbb{R}^d)$ and $L^{p,r}(\mathbb{R}^d) \backsimeq \dot{F}^{0,r}_{p,2}(\mathbb{R}^d)$ (see [16, Theorem 5]). For $s > 0$, the inhomogeneous Triebel-Lizorkin-Lorentz spaces $F^{s,r}_{p,q}(\mathbb{R}^d)$ are defined by

$$F^{s,r}_{p,q}\left(\mathbb{R}^d\right) = L^{p,r} \cap \dot{F}^{s,r}_{p,q}\left(\mathbb{R}^d\right) \tag{2.11}$$

with norm

$$\|u\|_{F^{s,r}_{p,q}(\mathbb{R}^d)} = \|u\|_{L^{p,r}} + \|u\|_{\dot{F}^{s,r}_{p,q}(\mathbb{R}^d)}. \tag{2.12}$$

Clearly, $F^{s,r}_{p,q}(\mathbb{R}^d)$ is a Banach space. It has been proved that there exist $1 < p_1$ and $p_2 < \infty$ such that $F^{s,r}_{p,q}(\mathbb{R}^d)$ are the interpolation spaces between $F^{s}_{p_1,q}(\mathbb{R}^d)$ and $F^{s}_{p_2,q}(\mathbb{R}^d)$ (see [16, Theorem 6]). Thus by the boundedness of Riesz transform on the Triebel-Lizorkin spaces (see Frazier et al. [17]) we have the following.

Lemma 2.3. *Riesz transform is bounded from the Triebel-Lizorkin-Lorentz space* $F^{s,r}_{p,q}(\mathbb{R}^d)$ *into itself.*

Also by $L^{p,r} \hookrightarrow L^p$ *for* $r \leq p$, *we have the following Sobolev's embedding theorem:*

$$F^{s,r}_{p,q} \hookrightarrow F^{s,p}_{p,q} = F^{s}_{p,q} \hookrightarrow L^{\infty}, \quad \text{for } s > \frac{d}{p}. \tag{2.13}$$

The following lemma is referred as Bernstein's inequality, which describes the way derivatives act on spectrally localized functions.

Lemma 2.4 (Bernstein's inequality). *Let* $1 < p, r < \infty$, *and* $u \in L^{p,r}(\mathbb{R}^d)$. *Then*

$$\text{supp}\, \mathcal{F}(u) \subset \left\{ \xi \in \mathbb{R}^d : |\xi| \sim 2^j \right\} \Longrightarrow \|\partial^{\alpha} u\|_{L^{p,r}} \sim 2^{j|\alpha|} \|u\|_{L^{p,r}}. \tag{2.14}$$

Here and thereafter, we use $A \lesssim B$ and $A \sim B$ to denote $A \leq CB$ and $cB \leq A \leq CB$ for some constants $C > c > 0$, respectively. Thus Bernstein's inequality together with the Littlewood-Paley decomposition gives that

$$\|\partial^{\alpha} u\|_{\dot{F}^{s,r}_{p,q}} \sim \|u\|_{\dot{F}^{s+|\alpha|,r}_{p,q}}, \qquad 1 < p,\ r < \infty \tag{2.15}$$

for any $u \in \dot{F}^{s+|\alpha|,r}_{p,q}(\mathbb{R}^d)$.

To conclude this section, we recall three important propositions in the Triebel-Lizorkin-Lorentz spaces, which will be frequently used in the proof of Theorem 1.1. For their proofs, we refer to our recent work [14] (for the case in the Triebel-Lizorkin spaces, see also Chae [12] and Chen et al. [13]). The first one is related to the product estimates.

Proposition 2.5. *Let $1 < p, q, \ r < \infty$. There exists a constant $C > 0$ such that for $s > 0$*

$$\|u\theta\|_{\dot{F}^{s,r}_{p,q}} \le C\Big(\|u\|_{L^\infty}\|\theta\|_{\dot{F}^{s,r}_{p,q}} + \|\theta\|_{L^\infty}\|u\|_{\dot{F}^{s,r}_{p,q}}\Big), \tag{2.16}$$

and for $s > -1$

$$\|u\theta\|_{F^{s,r}_{p,q}} \le C\Big(\|u\|_{L^\infty}\|\theta\|_{F^{s,r}_{p,q}} + \|\theta\|_{L^\infty}\|u\|_{F^{s,r}_{p,q}}\Big), \tag{2.17}$$

as long as the right-hand sides of (2.16) *and* (2.17) *are finite.*

The next one concerns the commutator estimates.

Proposition 2.6. *Let $s > 0$, $1 < p, q, \ r < \infty$, and let the vector field v be divergence free. There exists a constant $C > 0$ such that*

$$\Big\|\Big\|2^{js}\big([v, \Delta_j]\cdot\nabla\theta\big)\Big\|_{l^q(\mathbb{Z})}\Big\|_{L^{p,r}} \le C\Big(\|\nabla v\|_{L^\infty}\|\theta\|_{\dot{F}^{s,r}_{p,q}} + \|\nabla\theta\|_{L^\infty}\|v\|_{\dot{F}^{s,r}_{p,q}}\Big), \tag{2.18}$$

$$\Big\|\Big\|2^{js}\big([v, \Delta_j]\cdot\nabla\theta\big)\Big\|_{l^q(\mathbb{Z})}\Big\|_{L^{p,r}} \le C\Big(\|\nabla v\|_{L^\infty}\|\theta\|_{\dot{F}^{s,r}_{p,q}} + \|\theta\|_{L^\infty}\|\nabla v\|_{\dot{F}^{s,r}_{p,q}}\Big), \tag{2.19}$$

as long as the right-hand sides of (2.18) *and* (2.19) *are finite.*

The last one is the logarithmic Triebel-Lizorkin-Lorentz inequality with features similar to the classical logarithmic Sobolev inequality [18] and to the logarithmic Triebel-Lizorkin inequality [12].

Proposition 2.7. *Let $1 < r \le p < \infty, \ 1 < q < \infty$, and $s > d/p$. There exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty} \le C\Big(1 + \|u\|_{\dot{F}^0_{\infty,\infty}}\Big(\log^+\|u\|_{F^{s,r}_{p,q}} + 1\Big)\Big). \tag{2.20}$$

3. Proof of Theorem 1.1

In this section, we prove the local well-posedness of solutions to (1.1) by constructing the successive approximations and using the *a priori* estimate and establish the blow-up criterion of smooth solutions by the logarithmic Triebel-Lizorkin-Lorentz inequality. The proof is divided into five steps.

Step 1 (*a priori* estimate). For a given fluid velocity $u(x,t)$, we let $X(\alpha,t)$ be the particle-trajectory mapping defined by the solutions of the following ordinary differential equations:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}X(\alpha,t)=u(X(\alpha,t),t),\\ X(\alpha,0)=\alpha.\end{gathered}\tag{3.1}$$

Then from the Boussinesq equations (1.1) and the ODEs (3.1), we have

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}u(X(\alpha,t),t)=-\nabla p(X(\alpha,t),t)+\theta(X(\alpha,t),t)e_d,\\ \frac{\mathrm{d}}{\mathrm{d}t}\theta(X(\alpha,t),t)=0,\\ u(X(\alpha,0),0)=u_0(\alpha),\qquad \theta(X(\alpha,0),0)=\theta_0(\alpha),\end{gathered}\tag{3.2}$$

from which we get

$$\begin{gathered}u(X(\alpha,t),t)=u_0(\alpha)-\int_0^t\nabla p(X(\alpha\tau),\tau)\mathrm{d}\tau+\int_0^t\theta(X(\alpha,\tau),\tau)\ e_d\mathrm{d}\tau,\\ \theta(X(\alpha,t),t)=\theta_0(\alpha).\end{gathered}\tag{3.3}$$

It follows from $\operatorname{div}u=0$ that the mapping $\alpha\to X(\alpha,t)$ is a volume-preserving diffeomorphism for any given $t>0$. Thus by Lemma 2.1 we have

$$\begin{aligned}&\|u(t)\|_{L^{p,r}}+\|\theta(t)\|_{L^{p,r}}\\ &\quad=\|u(X(\alpha,t),t)\|_{L^{p,r}}+\|\theta(X(\alpha,t),t)\|_{L^{p,r}}\\ &\quad\le\|u_0\|_{L^{p,r}}+\|\theta_0\|_{L^{p,r}}+\int_0^t\left\|\nabla p(X(\alpha,\tau),\tau)\right\|_{L^{p,r}}\mathrm{d}\tau+\int_0^t\|\theta(X(\alpha,\tau),\tau)\|_{L^{p,r}}\mathrm{d}\tau\\ &\quad=\|u_0\|_{L^{p,r}}+\|\theta_0\|_{L^{p,r}}+\int_0^t\left\|\nabla p(\tau)\right\|_{L^{p,r}}\mathrm{d}\tau+\int_0^t\|\theta(\tau)\|_{L^{p,r}}\mathrm{d}\tau.\end{aligned}\tag{3.4}$$

To deal with the pressure term, we take the divergence on both sides of $(1.1)_1$ and then obtain the representation of the pressure:

$$p=(-\Delta)^{-1}\partial_iu_j\partial_ju_i-(-\Delta)^{-1}\partial_d\theta=(-\Delta)^{-1}\partial_i\partial_j\left(u_iu_j\right)-(-\Delta)^{-1}\partial_d\theta,\tag{3.5}$$

where we used the Einstein convention on the summation over repeated indices and the fact that $\operatorname{div}u=0$. Thus by the boundedness of Riesz transform on $L^{p,r}$, we have

$$\begin{aligned}\left\|\nabla p(\tau)\right\|_{L^{p,r}}&\le\left\|\nabla(-\Delta)^{-1}\partial_i\partial_j\left(u_iu_j\right)(\tau)\right\|_{L^{p,r}}+\left\|\nabla(-\Delta)^{-1}\partial_d\theta(\tau)\right\|_{L^{p,r}}\\ &\lesssim\|u\cdot\nabla u(\tau)\|_{L^{p,r}}+\|\theta(\tau)\|_{L^{p,r}}\\ &\le\|u(\tau)\|_{L^{p,r}}\|\nabla u(\tau)\|_{\infty}+\|\theta(\tau)\|_{L^{p,r}}.\end{aligned}\tag{3.6}$$

Substituting the last inequality into (3.4), we obtain

$$\|u(t)\|_{L^{p,r}} + \|\theta(t)\|_{L^{p,r}} \lesssim \|u_0\|_{L^{p,r}} + \|\theta_0\|_{L^{p,r}} + \int_0^t \|u(\tau)\|_{L^{p,r}} \|\nabla u(\tau)\|_\infty \mathrm{d}\tau + \int_0^t \|\theta(\tau)\|_{L^{p,r}} \mathrm{d}\tau. \quad (3.7)$$

On the other hand, we apply the frequency projection Δ_j to both sides of (1.1) to obtain

$$\begin{gathered} \partial_t \Delta_j u + u \cdot \nabla \Delta_j u = [u, \Delta_j] \cdot \nabla u - \nabla \Delta_j p + \Delta_j \theta e_d, \\ \partial_t \Delta_j \theta + u \cdot \nabla \Delta_j \theta = [u, \Delta_j] \cdot \nabla \theta. \end{gathered} \quad (3.8)$$

Then by (3.1), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \Delta_j u(X(\alpha,t),t) = [u, \Delta_j] \cdot \nabla u(X(\alpha,t),t) - \nabla \Delta_j p(X(\alpha,t),t) + \Delta_j \theta(X(\alpha,t),t) e_d, \\ &\frac{\mathrm{d}}{\mathrm{d}t} \Delta_j \theta(X(\alpha,t),t) = [u, \Delta_j] \cdot \nabla \theta(X(\alpha,t),t). \end{aligned} \quad (3.9)$$

Integrating the previous two equations from 0 to t gives

$$\begin{aligned} &\Delta_j u(X(\alpha,t),t) + \Delta_j \theta(X(\alpha,t),t) \\ &\quad = \Delta_j u_0(\alpha) + \Delta_j \theta_0(\alpha) + \int_0^t \left([u, \Delta_j] \cdot \nabla u(X(\alpha,\tau),\tau) + [u, \Delta_j] \cdot \nabla \theta(X(\alpha,\tau),\tau) \right) \mathrm{d}\tau \\ &\qquad - \int_0^t \nabla \Delta_j p(X(\alpha,\tau),\tau) \mathrm{d}\tau + \int_0^t \theta(X(\alpha,\tau),\tau) e_d \mathrm{d}\tau. \end{aligned} \quad (3.10)$$

We first multiply both sides by 2^{js}, take the $l^q(\mathbb{Z})$ norm, and then use Minkowski's inequality to obtain

$$\begin{aligned} &\left\| 2^{js} \left| \Delta_j u(X(\alpha,t),t) \right| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} \left| \Delta_j \theta(X(\alpha,t),t) \right| \right\|_{l^q(\mathbb{Z})} \\ &\quad \leq \left\| 2^{js} \left| \Delta_j u_0(\alpha) \right| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} \left| \Delta_j \theta_0(\alpha) \right| \right\|_{l^q(\mathbb{Z})} \\ &\qquad + \int_0^t \left(\left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla u(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} + \left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla \theta(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} \right) \mathrm{d}\tau \\ &\qquad + \int_0^t \left\| 2^{js} \left| \nabla \Delta_j p(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} \mathrm{d}\tau + \int_0^t \left\| 2^{js} |\theta(X(\alpha,\tau),\tau)| \right\|_{l^q(\mathbb{Z})} \mathrm{d}\tau, \end{aligned} \quad (3.11)$$

and then we take the $L^{p,r}$ norm with respect to α and use Minkowski's inequality to get

$$\begin{aligned}
&\left\| \left\| 2^{js} \left| \Delta_j u(X(\alpha,t),t) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} + \left\| \left\| 2^{js} \left| \Delta_j \theta(X(\alpha,t),t) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} \\
&\quad \leq \|u_0\|_{\dot{F}^{s,r}_{p,q}} + \|\theta_0\|_{\dot{F}^{s,r}_{p,q}} + \int_0^t \left\| \left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla u(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
&\qquad + \int_0^t \left\| \left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla \theta(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
&\qquad + \int_0^t \left\| \left\| 2^{js} \left| \nabla \Delta_j p(X(\alpha,\tau),\tau) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau + \int_0^t \left\| \left\| 2^{js} |\theta(X(\alpha,\tau),\tau)| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau.
\end{aligned} \tag{3.12}$$

Recalling the fact that $\alpha \to X(\alpha,t)$ is a volume-preserving diffeomorphism for any given $t > 0$ and using Lemma 2.1, we have

$$\begin{aligned}
\|u(t)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(t)\|_{\dot{F}^{s,r}_{p,q}} \lesssim{}& \|u_0\|_{\dot{F}^{s,r}_{p,q}} + \|\theta_0\|_{\dot{F}^{s,r}_{p,q}} \\
&+ \int_0^t \left\| \left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla u(\tau) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} + \left\| \left\| 2^{js} \left| [u, \Delta_j] \cdot \nabla \theta(\tau) \right| \right\|_{l^q(\mathbb{Z})} \right\|_{L^{p,r}} d\tau \\
&+ \int_0^t \|\nabla p(\tau)\|_{\dot{F}^{s,r}_{p,q}} d\tau + \int_0^t \|\theta(\tau)\|_{\dot{F}^{s,r}_{p,q}} d\tau.
\end{aligned} \tag{3.13}$$

Thanks to the commutator estimates (2.18), the integrand in the first integral is dominated by

$$\|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{\dot{F}^{s,r}_{p,q}} + \|\nabla u(\tau)\|_{L^\infty} \|\theta(\tau)\|_{\dot{F}^{s,r}_{p,q}} + \|\nabla \theta(\tau)\|_{L^\infty} \|u(\tau)\|_{\dot{F}^{s,r}_{p,q}}, \tag{3.14}$$

and thus

$$\begin{aligned}
\|u(t)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(t)\|_{\dot{F}^{s,r}_{p,q}} \lesssim{}& \|u_0\|_{\dot{F}^{s,r}_{p,q}} + \|\theta_0\|_{\dot{F}^{s,r}_{p,q}} + \int_0^t \|\nabla p(\tau)\|_{\dot{F}^{s,r}_{p,q}} d\tau \\
&+ \int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla \theta(\tau)\|_{L^\infty}) \left(\|u(\tau)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(\tau)\|_{\dot{F}^{s,r}_{p,q}} \right) d\tau.
\end{aligned} \tag{3.15}$$

For the term related to the pressure, we use (3.5) to obtain

$$\partial_k \partial_l p = \partial_k \partial_l (-\Delta)^{-1} \partial_i u_j \partial_j u_i - \partial_k \partial_l (-\Delta)^{-1} \partial_d \theta := R_k R_l (\partial_i u_j \partial_j u_i) - R_k R_l (\partial_d \theta), \tag{3.16}$$

where R_k and R_l are Riesz transform, and then

$$\|\nabla p\|_{\dot{F}^{s,r}_{p,q}} \leq \|\partial_k \partial_l p\|_{\dot{F}^{s-1,r}_{p,q}} \leq \|\partial_i u_j \partial_j u_i\|_{\dot{F}^{s-1,r}_{p,q}} + \|\partial_d \theta\|_{\dot{F}^{s-1,r}_{p,q}} \leq \|u\|_{\dot{F}^{s,r}_{p,q}} \|\nabla u\| + \|\theta\|_{\dot{F}^{s,r}_{p,q}}, \tag{3.17}$$

where we used the boundedness of Riesz transform on $\dot{F}^{s,r}_{p,q}$ and the product estimate (2.16). Substituting the last inequality into (3.15), we have

$$\begin{aligned}&\|u(t)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(t)\|_{\dot{F}^{s,r}_{p,q}} \\ &\quad\lesssim \|u_0\|_{\dot{F}^{s,r}_{p,q}} + \|\theta_0\|_{\dot{F}^{s,r}_{p,q}} + \int_0^t \Big(1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty}\Big)\Big(\|u(\tau)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(\tau)\|_{\dot{F}^{s,r}_{p,q}}\Big)\mathrm{d}\tau.\end{aligned} \tag{3.18}$$

This together with Gronwall's inequality gives the *a priori* estimates in the homogeneous Triebel-Lizorkin-Lorentz spaces:

$$\|u(t)\|_{\dot{F}^{s,r}_{p,q}} + \|\theta(t)\|_{\dot{F}^{s,r}_{p,q}} \le C\Big(\|u_0\|_{\dot{F}^{s,r}_{p,q}} + \|\theta_0\|_{\dot{F}^{s,r}_{p,q}}\Big) \exp\left(C\int_0^t (1 + \|\nabla v(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty})\mathrm{d}\tau\right). \tag{3.19}$$

Similarly, by (3.7) and (3.18), we get

$$\begin{aligned}&\|u(t)\|_{F^{s,r}_{p,q}} + \|\theta(t)\|_{F^{s,r}_{p,q}} \\ &\quad\lesssim \|u_0\|_{F^{s,r}_{p,q}} + \|\theta_0\|_{F^{s,r}_{p,q}} + \int_0^t \Big(1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty}\Big)\Big(\|u(\tau)\|_{F^{s,r}_{p,q}} + \|\theta(\tau)\|_{F^{s,r}_{p,q}}\Big)\mathrm{d}\tau,\end{aligned} \tag{3.20}$$

and thus Gronwall's inequality yields the *a priori* estimates in the inhomogeneous spaces:

$$\|u(t)\|_{F^{s,r}_{p,q}} + \|\theta(t)\|_{F^{s,r}_{p,q}} \le C\Big(\|u_0\|_{F^{s,r}_{p,q}} + \|\theta_0\|_{F^{s,r}_{p,q}}\Big) \exp\left(C\int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty})\mathrm{d}\tau\right). \tag{3.21}$$

Step 2 (approximate solutions and uniform estimates). In order to construct the approximate solutions of (1.1), we first set $(u^{(0)}, \theta^{(0)}) = (0,0)$ and then define $\{(u^{(n)}, \theta^{(n)})\}_{n\in\mathbb{N}}$ as the solutions of the linear equations:

$$\begin{gathered}\partial_t u^{(n+1)} + u^{(n)} \cdot \nabla u^{(n+1)} + \nabla p^{(n+1)} = \theta^{(n+1)} e_d, \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla\theta^{(n+1)} = 0, \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ \operatorname{div} u^{(n)} = 0, \quad \text{in } \mathbb{R}^d \times (0,\infty),\end{gathered} \tag{3.22}$$

with initial data $u^{(n+1)}(x,0) = u_0^{(n+1)}(x) := S_{n+2}u_0(x)$, $\theta^{(n+1)}(x,0) = \theta_0^{(n+1)}(x) := S_{n+2}\theta_0(x)$.

For each $n \in \mathbb{N}$, similar to Step 1, we let the particle-trajectory mapping $X^{(n)}(\alpha,t)$ be the solution of the following ordinary differential equation:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}X^{(n)}(\alpha,t) = u^{(n)}\Big(X^{(n)}(\alpha,t),t\Big), \\ X^{(n)}(\alpha,0) = \alpha.\end{gathered} \tag{3.23}$$

Then following the same procedure of estimate leading to (3.20), we obtain

$$\begin{aligned}\left\|u^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}} \lesssim{} & \|S_{n+2}u_0\|_{F^{s,r}_{p,q}}+\|S_{n+2}\theta_0\|_{F^{s,r}_{p,q}}\\ &+\int_0^t\left(1+\left\|\nabla u^{(n)}(\tau)\right\|_{L^\infty}\right)\\ &\times\left(\left\|u^{(n+1)}(\tau)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\mathrm{d}\tau\mathrm{d}\tau.\\ &+\int_0^t\left(\left\|\nabla u^{(n+1)}(\tau)\right\|_{L^\infty}+\left\|\nabla\theta^{(n+1)}(\tau)\right\|_{L^\infty}\right)\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}.\end{aligned} \tag{3.24}$$

Note that

$$\|S_{n+2}u_0\|_{F^{s,r}_{p,q}}\le\|u_0\|_{F^{s,r}_{p,q}},\qquad \|S_{n+2}\theta_0\|_{F^{s,r}_{p,q}}\le\|\theta_0\|_{F^{s,r}_{p,q}}. \tag{3.25}$$

This together with Sobolev embedding theorem $F^{s-1,r}_{p,q}\hookrightarrow L^\infty$ for $s-1>d/p$ yields that

$$\begin{aligned}\left\|u^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}} \lesssim{} & \|u_0\|_{F^{s,r}_{p,q}}+\|\theta_0\|_{F^{s,r}_{p,q}}\\ &+\int_0^t\left(1+\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\left(\left\|u^{(n+1)}(\tau)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\mathrm{d}\tau.\end{aligned} \tag{3.26}$$

It follows from Gronwall's inequality that

$$\left\|u^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}\le C\left(\|u_0\|_{F^{s,r}_{p,q}}+\|\theta_0\|_{F^{s,r}_{p,q}}\right)\exp\left(C\int_0^t\left(1+\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\mathrm{d}\tau\right) \tag{3.27}$$

for some $C>0$ independent of n. Choosing $T_0:=T_0(\|u_0\|_{F^{s,r}_{p,q}},\|\theta_0\|_{F^{s,r}_{p,q}})>0$ such that

$$C\left(1+2C\|u_0\|_{F^{s,r}_{p,q}}+2C\|\theta_0\|_{F^{s,r}_{p,q}}\right)T_0\le\log 2, \tag{3.28}$$

and using the standard induction arguments, we obtain for any $n\in\mathbb{N}_0$

$$\sup_{0\le t\le T_0}\left(\left\|u^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n+1)}(t)\right\|_{F^{s,r}_{p,q}}\right)\le 2C\left(\|u_0\|_{F^{s,r}_{p,q}}+\|\theta_0\|_{F^{s,r}_{p,q}}\right). \tag{3.29}$$

Step 3 (existence). To prove the local existence, we show that there exists $T_1 \in (0, T_0]$ independent of n such that $(u^{(n)}, \theta^{(n)})$ is a Cauchy sequence in $\left(C([0,T_1]; F^{s-1,r}_{p,q}(\mathbb{R}^d))\right)^{d+1}$. For this purpose, we denote

$$\delta u^{(n+1)} := u^{(n+1)} - u^{(n)}, \qquad \delta\theta^{(n+1)} := \theta^{(n+1)} - \theta^{(n)}, \qquad \delta p^{(n+1)} := p^{(n+1)} - p^{(n)}. \tag{3.30}$$

It follows that $(\delta u^{(n+1)}, \delta\theta^{(n+1)}, \delta p^{(n+1)})$ satisfies the following:

$$\begin{gathered}
\partial_t \delta u^{(n+1)} + u^{(n)} \cdot \nabla \delta u^{(n+1)} + \nabla \delta p^{(n+1)} = -\delta u^{(n)} \cdot \nabla u^{(n)} + \delta\theta^{(n+1)} e_d, \quad \text{in } \mathbb{R}^d \times (0,\infty), \\
\partial_t \delta\theta^{(n+1)} + u^{(n)} \cdot \nabla \delta\theta^{(n+1)} = -\delta u^{(n)} \cdot \nabla \theta^{(n)}, \quad \text{in } \mathbb{R}^d \times (0,\infty), \\
\operatorname{div} u^{(n)} = 0, \quad \text{in } \mathbb{R}^d \times (0,\infty), \\
\delta u^{(n+1)}(x,0) = \Delta_{n+1} u_0(x), \quad \delta\theta^{(n+1)}(x,0) = \Delta_{n+1}\theta_0(x), \quad \text{in } \mathbb{R}^d.
\end{gathered} \tag{3.31}$$

Since $X^{(n)}(\alpha, t)$ is a solution of the ordinary differential equations (3.23), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \delta u^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) &= -\delta u^{(n)} \cdot \nabla u^{(n)}\left(X^{(n)}(\alpha,t),t\right) - \nabla \delta p^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) \\
&\quad + \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) e_d, \\
\frac{\mathrm{d}}{\mathrm{d}t} \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) &= -\delta u^{(n)} \cdot \nabla \theta^{(n)}\left(X^{(n)}(\alpha,t),t\right).
\end{aligned} \tag{3.32}$$

Integrating the previous equations on $[0, t]$, we have

$$\begin{aligned}
&\delta u^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) + \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) \\
&\quad = \Delta_{n+1} u_0(\alpha) + \Delta_{n+1}\theta_0(\alpha) - \int_0^t \delta u^{(n)} \cdot \nabla\left(u^{(n)} + \theta^{(n)}\right)\left(X^{(n)}(\alpha,\tau),\tau\right) \mathrm{d}\tau \\
&\qquad - \int_0^t \nabla \delta p^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right) \mathrm{d}\tau + \int_0^t \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right) e_d \,\mathrm{d}\tau.
\end{aligned} \tag{3.33}$$

Similar to Step 1, we take the $L^{p,r}$ norm on both sides, use Minkowski's inequality and Hölder's inequality in Lorentz spaces, and then obtain

$$\begin{aligned}
&\left\| \delta u^{(n+1)}(t) \right\|_{L^{p,r}} + \left\| \delta\theta^{(n+1)}(t) \right\|_{L^{p,r}} \\
&\quad \lesssim \|\Delta_{n+1} u_0\|_{L^{p,r}} + \|\Delta_{n+1}\theta_0\|_{L^{p,r}} + \int_0^t \left\| \delta u^{(n)} \cdot \nabla\left(u^{(n)} + \theta^{(n)}\right)(\tau) \right\|_{L^{p,r}} \mathrm{d}\tau \\
&\qquad + \int_0^t \left\| \nabla \delta p^{(n+1)} \right\|_{L^{p,r}} \mathrm{d}\tau + \int_0^t \left\| \delta\theta^{(n+1)}(\tau) \right\|_{L^{p,r}} \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\Delta_{n+1}u_0\|_{L^{p,r}} + \|\Delta_{n+1}\theta_0\|_{L^{p,r}} + \int_0^t \left\|\delta u^{(n)}\right\|_{L^{p,r}} \left(\left\|\nabla u^{(n)}(\tau)\right\|_{\infty} + \left\|\nabla\theta^{(n)}(\tau)\right\|_{L^\infty}\right)\mathrm{d}\tau \\
&\quad + \int_0^t \left(1 + \left\|\nabla u^{(n)}(\tau)\right\|_{L^\infty}\right)\left(\left\|\delta u^{(n+1)}(\tau)\right\|_{L^{p,r}} + \left\|\delta\theta^{(n+1)}(\tau)\right\|_{L^{p,r}}\right)\mathrm{d}\tau \\
&\lesssim \|\Delta_{n+1}u_0\|_{L^{p,r}} + \|\Delta_{n+1}\theta_0\|_{L^{p,r}} + \int_0^t \left\|\delta u^{(n)}\right\|_{L^{p,r}} \left(\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}} + \left\|\theta^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\mathrm{d}\tau \\
&\quad + \int_0^t \left(1 + \left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\left(\left\|\delta u^{(n+1)}(\tau)\right\|_{L^{p,r}} + \left\|\delta\theta^{(n+1)}(\tau)\right\|_{L^{p,r}}\right)\mathrm{d}\tau,
\end{aligned}
\tag{3.34}
$$

where we used $F^{s-1,r}_{p,q} \hookrightarrow L^\infty$ for $s-1 > d/p$ in the last inequality.

Next, we apply Δ_j to the first equation of (3.31) to obtain

$$
\begin{aligned}
\partial_t \Delta_j \delta u^{(n+1)} + u^{(n)} \cdot \nabla \Delta_j \delta u^{(n+1)} &= \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta u^{(n+1)} - \Delta_j\left(\delta u^{(n)} \cdot \nabla u^{(n)}\right) \\
&\quad - \nabla \Delta_j \delta p^{(n+1)} + \Delta_j \delta\theta^{(n+1)} e_d, \\
\partial_t \Delta_j \delta\theta^{(n+1)} + u^{(n)} \cdot \nabla \Delta_j \delta\theta^{(n+1)} &= \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta\theta^{(n+1)} - \Delta_j\left(\delta u^{(n)} \cdot \nabla \theta^{(n)}\right).
\end{aligned}
\tag{3.35}
$$

It follows from the definition of $X^{(n)}(\alpha, t)$ that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Delta_j \delta u^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) &= \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta u^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) \\
&\quad - \Delta_j\left(\delta u^{(n)} \cdot \nabla u^{(n)}\right)\left(X^{(n)}(\alpha,t),t\right) \\
&\quad - \nabla \Delta_j \delta p^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) + \Delta_j \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) e_d, \\
\frac{\mathrm{d}}{\mathrm{d}t}\Delta_j \delta \theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) &= \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta \theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) \\
&\quad - \Delta_j\left(\delta u^{(n)} \cdot \nabla \theta^{(n)}\right)\left(X^{(n)}(\alpha,t),t\right).
\end{aligned}
\tag{3.36}
$$

Integrating the previous equations on $[0,t]$ yields

$$
\begin{aligned}
&\Delta_j \delta u^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) + \Delta_j \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,t),t\right) \\
&\quad = \Delta_j \Delta_{n+1} u_0(\alpha) + \Delta_j \Delta_{n+1}\theta_0(\alpha) + \int_0^t \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta u^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right)\mathrm{d}\tau \\
&\qquad + \int_0^t \left[u^{(n)}, \Delta_j\right] \cdot \nabla \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right)\mathrm{d}\tau - \int_0^t \Delta_j\left(\delta u^{(n)} \cdot \nabla u^{(n)}\right)\left(X^{(n)}(\alpha,\tau),\tau\right)\mathrm{d}\tau \\
&\qquad - \int_0^t \Delta_j\left(\delta u^{(n)} \cdot \nabla \theta^{(n)}\right)\left(X^{(n)}(\alpha,\tau),\tau\right)\mathrm{d}\tau - \int_0^t \nabla \Delta_j \delta p^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right)\mathrm{d}\tau \\
&\qquad + \int_0^t \Delta_j \delta\theta^{(n+1)}\left(X^{(n)}(\alpha,\tau),\tau\right) e_d \mathrm{d}\tau.
\end{aligned}
\tag{3.37}
$$

Then following the similar procedure of estimate leading to (3.20), we get

$$\begin{aligned}
&\left\|\delta u^{(n+1)}(t)\right\|_{\dot{F}^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{\dot{F}^{s-1,r}_{p,q}}\\
&\quad\lesssim \|\Delta_{n+1}u_0\|_{\dot{F}^{s-1,r}_{p,q}}+\|\Delta_{n+1}\theta_0\|_{\dot{F}^{s-1,r}_{p,q}}+\int_0^t\left\|\left\|2^{j(s-1)}\left|\left[u^{(n)},\Delta_j\right]\cdot\nabla\delta u^{(n+1)}(\tau)\right|\right\|_{l^q(\mathbb{Z})}\right\|_{L^{p,r}}\mathrm{d}\tau\\
&\qquad+\int_0^t\left\|\left\|2^{j(s-1)}\left|\left[u^{(n)},\Delta_j\right]\cdot\nabla\delta\theta^{(n+1)}(\tau)\right|\right\|_{l^q(\mathbb{Z})}\right\|_{L^{p,r}}\mathrm{d}\tau+\int_0^t\left\|\delta u^{(n)}\cdot\nabla u^{(n)}\right\|_{\dot{F}^{s-1,r}_{p,q}}\mathrm{d}\tau\\
&\qquad+\int_0^t\left\|\delta u^{(n)}\cdot\nabla\theta^{(n)}\right\|_{\dot{F}^{s-1,r}_{p,q}}\mathrm{d}\tau+\int_0^t\left\|\nabla\delta p^{(n+1)}\right\|_{\dot{F}^{s-1,r}_{p,q}}\mathrm{d}\tau+\int_0^t\left\|\delta\theta^{(n+1)}\right\|_{\dot{F}^{s-1,r}_{p,q}}\mathrm{d}\tau\\
&\quad\lesssim \|\Delta_{n+1}u_0\|_{\dot{F}^{s-1,r}_{p,q}}+\|\Delta_{n+1}\theta_0\|_{\dot{F}^{s-1,r}_{p,q}}\\
&\qquad+\int_0^t\left(1+\left\|\nabla u^{(n)}(\tau)\right\|_\infty\right)\left(\left\|\delta u^{(n+1)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}\right)\mathrm{d}\tau\\
&\qquad+\int_0^t\left(\left\|\delta u^{(n+1)}(\tau)\right\|_\infty+\left\|\delta\theta^{(n+1)}(\tau)\right\|_\infty\right)\left\|\nabla u^{(n)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}\mathrm{d}\tau\\
&\qquad+\int_0^t\left\|\delta u^{(n)}(\tau)\right\|_\infty\left(\left\|\nabla u^{(n)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}+\left\|\nabla\theta^{(n)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}\right)\mathrm{d}\tau\\
&\qquad+\int_0^t\left\|\delta u^{(n)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}\left(\left\|\nabla u^{(n)}(\tau)\right\|_\infty+\left\|\nabla\theta^{(n)}(\tau)\right\|_\infty\right)\mathrm{d}\tau,
\end{aligned}\tag{3.38}$$

where we used the commutator estimates (2.19) and the product estimates (2.16) in last inequality. This together with (3.34), Sobolev embedding theorem, and the boundedness of Riesz transform gives that

$$\begin{aligned}
&\left\|\delta u^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}\\
&\quad\lesssim \|\Delta_{n+1}u_0\|_{F^{s-1,r}_{p,q}}+\|\Delta_{n+1}\theta_0\|_{F^{s-1,r}_{p,q}}\\
&\qquad+\int_0^t\left(1+\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\left(\left\|\delta u^{(n+1)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(\tau)\right\|_{\dot{F}^{s-1,r}_{p,q}}\right)\mathrm{d}\tau\\
&\qquad+\int_0^t\left\|\delta u^{(n)}(\tau)\right\|_{F^{s-1,r}_{p,q}}\left(\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\mathrm{d}\tau\\
&\quad\lesssim 2^{-(n+1)}\left(\|u_0\|_{\dot{F}^{s,r}_{p,q}}+\|\theta_0\|_{\dot{F}^{s,r}_{p,q}}\right)\\
&\qquad+T_1\left(1+\sup_{t\in[0,T_1]}\left\|u^{(n)}(t)\right\|_{F^{s,r}_{p,q}}\right)\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n+1)}(t)\right\|_{\dot{F}^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{\dot{F}^{s-1,r}_{p,q}}\right)\\
&\qquad+T_1\sup_{t\in[0,T_1]}\left(\left\|u^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}+\left\|\theta^{(n)}(\tau)\right\|_{F^{s,r}_{p,q}}\right)\sup_{t\in[0,T_1]}\left\|\delta u^{(n)}(t)\right\|_{F^{s-1,r}_{p,q}}.
\end{aligned}\tag{3.39}$$

It follows from the uniform bounds (3.29) that

$$\begin{aligned}
&\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}\right)\\
&\quad\leq C2^{-(n+1)}\left(\|u_0\|_{\dot F^{s,r}_{p,q}}+\|\theta_0\|_{\dot F^{s,r}_{p,q}}\right)+CT_1\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}\right)\\
&\qquad+CT_1\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n)}(t)\right\|_{F^{s-1,r}_{p,q}}\right).
\end{aligned}\tag{3.40}$$

If we choose T_1 small enough such that $CT_1 \leq 1/2$, then

$$\begin{aligned}
&\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}\right)\\
&\quad\leq C2^{-(n+1)}\left(\|u_0\|_{\dot F^{s,r}_{p,q}}+\|\theta_0\|_{\dot F^{s,r}_{p,q}}\right)+2CT_1\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n)}(t)\right\|_{F^{s-1,r}_{p,q}}\right).
\end{aligned}\tag{3.41}$$

By the standard induction arguments, we obtain

$$\sup_{t\in[0,T_1]}\left(\left\|\delta u^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}+\left\|\delta\theta^{(n+1)}(t)\right\|_{F^{s-1,r}_{p,q}}\right)\leq 2C2^{-n}\left(\|u_0\|_{\dot F^{s,r}_{p,q}}+\|\theta_0\|_{\dot F^{s,r}_{p,q}}\right),\tag{3.42}$$

which implies that $(u^{(n)}(x,t),\theta^{(n)}(x,t))$ is a Cauchy sequence in $(C([0,T_1];F^{s-1,r}_{p,q}))^{d+1}$ with $T_1 \leq \min\{T_0, 1/2C\}$. Thus $u^{(n)}(x,t) \to u(x,t)$ and $\theta^{(n)}(x,t) \to \theta(x,t)$ in $(C([0,T_1];F^{s,r}_{p,q}))^{d+1}$ as $n \to \infty$. From (3.22) and the uniform estimates (3.29), we have actually $(u(x,t),\theta(x,t)) \in (C([0,T_1];F^{s,r}_{p,q}))^{d+1}$ solving the Boussinesq equation (1.1).

Step 4 (uniqueness). To get the uniqueness we closely follow the arguments in last step. Assume that (u,θ) and $(\widetilde{u},\widetilde{\theta})$ are two solutions of the Boussinesq equations (1.1) with the same initial data (u_0,θ_0) in the class $L^\infty([0,T_1];F^{s,r}_{p,q}(\mathbb{R}^d))$. Denote

$$\delta u := \widetilde{u}-u,\qquad \delta\theta := \widetilde{\theta}-\theta,\qquad \delta p := \widetilde{p}-p.\tag{3.43}$$

Then it follows that $\delta\theta$ satisfies the following:

$$\begin{gathered}
\partial_t\delta u+\widetilde{u}\cdot\nabla\delta u+\nabla\delta p=-\delta u\cdot\nabla u+\delta\theta e_d,\quad \text{in } \mathbb{R}^d\times(0,T_1),\\
\partial_t\delta\theta+\widetilde{u}\cdot\nabla\delta\theta=-\delta u\cdot\nabla\theta,\quad \text{in } \mathbb{R}^d\times(0,T_1),\\
\operatorname{div}\widetilde{u}=0,\quad \text{in } \mathbb{R}^d\times(0,T_1),\\
\delta u(x,0)=0,\quad \delta\theta(x,0)=0,\quad \text{in } \mathbb{R}^d.
\end{gathered}\tag{3.44}$$

We follow the strategy used to derive (3.40) to obtain

$$\sup_{t\in[0,T]}\Big(\|\delta u(t)\|_{F^{s-1,r}_{p,q}}+\|\delta\theta(t)\|_{F^{s-1,r}_{p,q}}\Big)\le CT\sup_{t\in[0,T]}\Big(\|\delta u(t)\|_{F^{s-1,r}_{p,q}}+\|\delta\theta(t)\|_{F^{s-1,r}_{p,q}}\Big) \tag{3.45}$$

for any $T\le T_1$. Whenever T is small enough such that $CT<1$, we have $\delta u(x,t)\equiv\delta\theta(x,t)\equiv 0$ for any $t\le T$, that is, $\tilde{u}(x,t)\equiv u(x,t)$ and $\tilde{\theta}(x,t)\equiv\theta(x,t)$. By using the standard continuity argument, we see $\tilde{u}(x,t)\equiv u(x,t)$ and $\tilde{\theta}(x,t)\equiv\theta(x,t)$ for any $t\le T_1$.

Step 5 (blow-up criterion). By the *a priori* estimate (3.21), we only need to dominate $\|\nabla u\|_\infty$ and $\|\nabla\theta\|_\infty$. Indeed, it follows from the logarithmic Triebel-Lizorkin-Lorentz inequality (2.20) with $s-1>d/p$ that

$$\begin{aligned}\|\nabla u\|_{L^\infty}&\lesssim 1+\|\nabla u\|_{\dot{F}^0_{\infty,\infty}}\Big(\log^+\|\nabla u\|_{F^{s-1,r}_{p,q}}+1\Big)\lesssim 1+\|u\|_{\dot{F}^1_{\infty,\infty}}\Big(\log^+\|u\|_{F^{s,r}_{p,q}}+1\Big),\\ \|\nabla\theta\|_{L^\infty}&\lesssim 1+\|\nabla\theta\|_{\dot{F}^0_{\infty,\infty}}\Big(\log^+\|\nabla\theta\|_{F^{s-1,r}_{p,q}}+1\Big)\lesssim 1+\|\theta\|_{\dot{F}^1_{\infty,\infty}}\Big(\log^+\|\theta\|_{F^{s,r}_{p,q}}+1\Big).\end{aligned} \tag{3.46}$$

Thus, the *a priori* estimate (3.21) gives that

$$\begin{aligned}\|u(t)\|_{F^{s,r}_{p,q}}+\|\theta(t)\|_{F^{s,r}_{p,q}}\le C\Big(\|u_0\|_{F^{s,r}_{p,q}}+\|\theta_0\|_{F^{s,r}_{p,q}}\Big)\exp\Bigg(C\int_0^t&\Big(\|u(\tau)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(\tau)\|_{\dot{F}^1_{\infty,\infty}}\Big)\\ &\times\Big(\log^+\Big(\|u(\tau)\|_{F^{s,r}_{p,q}}+\|\theta(\tau)\|_{F^{s,r}_{p,q}}\Big)+1\Big)d\tau\Bigg).\end{aligned} \tag{3.47}$$

By using Gronwall's inequality, we have

$$\|u(t)\|_{F^{s,r}_{p,q}}+\|\theta(t)\|_{F^{s,r}_{p,q}}\le C\Big(\|u_0\|_{F^{s,r}_{p,q}}+\|\theta_0\|_{F^{s,r}_{p,q}}\Big)\exp\Bigg(C\exp\Bigg(C\int_0^t\Big(\|u(\tau)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(\tau)\|_{\dot{F}^1_{\infty,\infty}}\Big)d\tau\Bigg)\Bigg). \tag{3.48}$$

Thus if $\limsup_{t\to T^*-}(\|u(t)\|_{F^{s,r}_{p,q}}+\|\theta(t)\|_{F^{s,r}_{p,q}})=\infty$, then $\int_0^{T^*}(\|u(t)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(t)\|_{\dot{F}^1_{\infty,\infty}})dt=\infty$.

On the other hand, it follows from Sobolev embedding $F^{s,r}_{p,q}\hookrightarrow W^{1,\infty}\hookrightarrow\dot{F}^1_{\infty,\infty}$ for $s-1>d/p$ that

$$\begin{aligned}\int_0^{T^*}\Big(\|u(t)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(t)\|_{\dot{F}^1_{\infty,\infty}}\Big)dt&\le T^*\sup_{0\le\tau\le T^*}\Big(\|u(\tau)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(\tau)\|_{\dot{F}^1_{\infty,\infty}}\Big)\\ &\lesssim T^*\sup_{0\le\tau\le T^*}(\|\nabla u(\tau)\|_{L^\infty}+\|\nabla\theta(\tau)\|_{L^\infty})\\ &\lesssim T^*\sup_{0\le\tau\le T^*}\Big(\|u(\tau)\|_{F^{s,r}_{p,q}}+\|\theta(\tau)\|_{F^{s,r}_{p,q}}\Big).\end{aligned} \tag{3.49}$$

Thus $\int_0^{T^*}(\|u(t)\|_{\dot{F}^1_{\infty,\infty}}+\|\theta(t)\|_{\dot{F}^1_{\infty,\infty}})dt=\infty$ implies $\limsup_{t\to T^*-}(\|u(t)\|_{F^{s,r}_{p,q}}+\|\theta(t)\|_{F^{s,r}_{p,q}})=\infty$.

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