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Research Article

Positive Solutions for Sturm-Liouville Boundary Value Problems in a Banach Space

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We consider the existence of single and multiple positive solutions for a second-order Sturm-Liouville boundary value problem in a Banach space. The sufficient condition for the existence of positive solution is obtained by the fixed point theorem of strict set contraction operators in the frame of the ODE technique. Our results significantly extend and improve many known results including singular and nonsingular cases.

1. Introduction

Boundary value problems for ordinary differential equations play a very important role in both theoretical study and practical application in many fields. They are used to describe a large number of physical, biological, and chemical phenomena. In this paper, we study the existence of positive solutions for the following second-order nonlinear Sturm-Liouville boundary value problem (BVP) in a Banach Space E

$$\begin{gathered}\frac{1}{p(t)}\left(p(t)u'(t)\right)' + f(u(t)) = 0, \quad 0 < t < 1,\\ \alpha u(0) - \beta \lim_{t\to 0+} p(t)u'(t) = 0,\\ \gamma u(1) + \delta \lim_{t\to 1-} p(t)u'(t) = 0,\end{gathered} \tag{1.1}$$

where $\alpha, \beta, \delta, \gamma \geq 0$ are constants such that $B(t,s) = \int_t^s d\tau/p(\tau)$, $\omega = \beta\gamma + \alpha\gamma B(0,1) + \alpha\delta > 0$, and $p \in C^1((0,1),(0,+\infty))$. Moreover, p may be singular at $t = 0$ and/or 1.

The BVP (1.1) is often referred to as a model for the deformation of an elastic beam under a variety of boundary conditions [1–12]. We notice that previous work is limited to use the completely continuous operators and the function f is required to satisfy some growth condition or assumptions of monotonicity.

The aim of this paper is to consider the existence of positive solutions for the more general Sturm-Liouville boundary value problem (1.1) by using the fixed point theorem of strict set contraction operators. Here we allow p to have singularity at $t = 0, 1$. The results obtained in this paper improve and generalize many well-known results.

The rest of the paper is organized as follows. In Section 2, we first present some properties of Green's functions to be used to define a positive operator. Then we approximate the singular second-order boundary value problem by constructing an integral operator. In Section 3, the sufficient condition for the existence of single and multiple positive solutions for the BVP (1.1) is established. In Section 4, we give a example to demonstrate the application of our results.

2. Preliminaries and Lemmas

In this paper, we denote by $(E, \|\cdot\|_1)$ a real Banach space. A nonempty closed convex subset P in E is said to be a cone if $\lambda P \in P$ for $\lambda \geq 0$ and $P \bigcap \{-P\} = \{\theta\}$, where θ denotes the zero element of E. The cone P defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. Recall that the cone P is said to be normal if there exists a positive constant λ such that $0 \leq x \leq y$ implies $\|x\|_1 \leq \lambda \|y\|_1$.

In this paper, we assume $P \subseteq E$ is normal, and without loss of generality, we may assume that the normal constant of P is 1. Let $J = [0, 1]$, and

$$
\begin{gathered}
C(J, E) = \{u : J \longrightarrow E \mid u(t) \text{ continuous}\}, \\
C^i(J, E) = \left\{u : J \longrightarrow E \mid u(t) \text{ is } i\text{-order continuously differentiable}\right\}, \quad i = 1, 2, \ldots.
\end{gathered}
\tag{2.1}
$$

For $u = u(t) \in C(J, E)$, let $\|u\| = \max_{t \in J} \|u(t)\|_1$, then $C(J, E)$ is a Banach space with the norm $\|\cdot\|$.

Definition 2.1. A function $u(t)$ is said to be a positive solution of the boundary value problem (1.1) if $u \in C([0, 1], E) \bigcap C^1((0, 1), E)$ satisfies $u(t) > 0,\ t \in (0, 1]$, $pu' \in C^1((0, 1), E)$ and the BVP (1.1).

We notice that if $u(t)$ is a positive solution of the BVP (1.1) and $p \in C^1(0, 1)$, then $u \in C^2(0, 1)$.

Now we denote by $G(t, s)$ the Green's functions for the following boundary value problem:

$$
\begin{gathered}
\frac{1}{p(t)} \left(p(t) u'(t)\right)' = 0, \quad 0 < t < 1, \\
u(0) - \lim_{t \to 0+} \beta p(t) u'(t) = 0, \\
\gamma u(1) + \lim_{t \to 1-} \delta p(t) u'(t) = 0.
\end{gathered}
\tag{2.2}
$$

It is well known that $G(t,s)$ can be written by

$$G(t,s)=\frac{1}{\omega}\begin{cases}(\beta+\alpha B(0,s))(\delta+\gamma B(t,1)), & 0\le s\le t\le 1,\\ (\beta+\alpha B(0,t))(\delta+\gamma B(s,1)), & 0\le t\le s\le 1,\end{cases} \tag{2.3}$$

where $B(t,s)=\int_t^s d\tau/p(\tau)$, $\omega=\alpha\delta+\alpha\gamma B(0,1)+\beta\gamma>0$.

It is easy to verify the following properties of $G(t,s)$:

(I) $G(t,s)\le G(s,s)\le (1/\omega)(\beta+\alpha B(0,1))(\delta+\gamma B(0,1))<+\infty$;

(II) $G(t,s)\ge \rho G(s,s)$, for any $t\in[a,b]\subset(0,1)$, $s\in[0,1]$, where

$$\rho=\min\left\{\frac{\delta+\gamma B(b,1)}{\delta+\gamma B(0,1)},\frac{\beta+\alpha B(0,a)}{\beta+\alpha B(0,1)}\right\}. \tag{2.4}$$

Throughout this paper, we adopt the following assumptions:

(H_1) $p\in C^1((0,1),(0,+\infty))$ and satisfies

$$0<\int_0^1\frac{ds}{p(s)}<+\infty,\qquad 0<e=\int_0^1 G(s,s)p(s)ds<+\infty. \tag{2.5}$$

(H_2) $f:P\to P$ is a uniformly continuous function and there exists $M>0$ such that for any bounded set $B\subset C(J,E)$, we have

$$\alpha(f(B(t)))\le M\alpha(B(t)),\quad t\in J,\ 2Me\rho<1, \tag{2.6}$$

where $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness in E.

The following Lemmas play an important role in this paper (see [13]).

Lemma 2.2. *Let $B\subset C(J,E)$ be bounded and equicontinuous on J, then $\alpha_c(B)=\sup_{t\in J}\alpha(B(t))$.*

Lemma 2.3. *Let $B\subset C(J,E)$ be bounded and equicontinuous on J, then $\alpha(B(t))$ is continuous on J and*

$$\alpha\left(\left\{\int_J u(t)dt : u\in B\right\}\right)\le\int_J\alpha(B(t))dt. \tag{2.7}$$

Lemma 2.4. *Let $B\subset C(J,E)$ be a bounded set on J. Then $\alpha(B(t))\le 2\alpha_c(B)$.*

Now, for the given $[a,b]\subset(0,1)$ and the ρ as in (II), we introduce

$$K=\{u\in C(J,P): u(t)\ge\rho u(s),\ t\in[a,b],\ s\in[0,1]\}. \tag{2.8}$$

It is easy to check that K is a cone in $C(J,E)$ and for $u(t)\in K$, $t\in[a,b]$, we have $\|u(t)\|_1\ge\rho\|u\|$.

Next, we define an operator $T : K \longrightarrow C(J, P)$ given by

$$Tu(t) = \int_0^1 G(t,s)p(s)f(u(s))ds, \quad \forall u \in K, \ t \in [0,1]. \tag{2.9}$$

Clearly, u is a solution of the BVP (1.1) if and only if u is a fixed point of the operator T.

Through direct calculation, by (II) and for $v \in K, \ t \in [a,b], \ s \in J$, we have

$$Tu(t) = \int_0^1 G(t,s)p(s)f(u(s))ds \geq \rho \int_0^1 G(s,s)p(s)f(u(s))ds = \rho Tu(s). \tag{2.10}$$

So, this implies that $TK \subset K$.

Lemma 2.5. *Assume that, hold. Then $T : K \longrightarrow K$ is a strict set contraction operator.*

Proof. Firstly, The continuity of T is easily obtained. In fact, if $v_n, v \in K$ and $v_n \longrightarrow v$ in the norm in $C(J, E)$, then for any $t \in J$, we get

$$\|Tv_n(t) - Tv(t)\|_1 \leq \int_0^1 G(s,s)p(s)\left\|f(v_n(s)) - f(v(s))\right\|_1 ds, \tag{2.11}$$

so, by the uniformly continuity of f, we have

$$\|Tv_n - Tv\| = \sup_{t\in J}\|Tv_n(t) - Tv(t)\|_1 \longrightarrow 0. \tag{2.12}$$

This implies that $Tv_n \longrightarrow Tv$ in $C(J, E)$, that is, T is continuous.

Now, let $B \subset K$ be a bounded set. It follows from that there exists a positive number L such that $\|f(v)\| \leq L$ for any $v \in B$. Then, we can get

$$\|Tv(t)\|_1 \leq Le < \infty, \quad \forall t \in J, \ v \in B, \tag{2.13}$$

where e is as defined in. So, $T(B) \subset K$ is a bounded set in K.

For any $\varepsilon > 0$, by (H_1), there exists a $\delta' > 0$ such that

$$\int_0^{\delta'} G(s,s)p(s) \leq \frac{\varepsilon}{6L}, \qquad \int_{1-\delta'}^1 G(s,s)p(s) \leq \frac{\varepsilon}{6L}. \tag{2.14}$$

Let $P = \max_{t\in[\delta', 1-\delta']} p(t)$. It follows from the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$ that there exists $\delta > 0$ such that

$$\left|G(t,s) - G(t',s)\right| \leq \frac{\varepsilon}{3PL}, \quad \left|t - t'\right| < \delta, \quad t, t' \in [0,1], \ s \in [0,1]. \tag{2.15}$$

Consequently, when $|t-t'|<\delta,\ t,t'\in[0,1],\ v\in B$, we have

$$\begin{aligned}
\|Tv(t)-Tv(t')\|_1 &= \left\|\int_0^1 (G(t,s)-G(t',s))p(s)f(v(s))ds\right\|_1 \\
&\le \int_0^{\delta'} |G(t,s)-G(t',s)|p(s)\|f(v(s))\|_1 ds \\
&\quad + \int_{\delta'}^{1-\delta'} |G(t,s)-G(t',s)|p(s)\|f(v(s))\|_1 ds \\
&\quad + \int_{1-\delta'}^{1} |G(t,s)-G(t',s)|p(s)\|f(v(s))\|_1 ds \\
&\le 2L\int_0^{\delta'} G(s,s)p(s)ds + 2L\int_{1-\delta'}^{1} G(s,s)p(s)ds \\
&\quad + PL\int_0^1 |G(t,s)-G(t',s)|ds \\
&\le \varepsilon.
\end{aligned} \tag{2.16}$$

This implies that $T(B)$ is an equicontinuous set on J. Therefore, by Lemma 2.2, we have

$$\alpha(T(B)) = \sup_{t\in J}\alpha(T(B)(t)). \tag{2.17}$$

Without loss of generality, by condition, we may assume that $p(t)$ is singular at $t=0,1$. So, there exist $\{a_{n_i}\},\{b_{n_i}\}\subset(0,1),\{n_i\}\subset N$ with $\{n_i\}$ being a strictly increasing sequence and $\lim_{i\to+\infty}n_i=+\infty$ such that

$$\begin{gathered}
0<\cdots<a_{n_i}<\cdots<a_{n_1}<b_{n_1}<\cdots<b_{n_i}<\cdots<1, \\
p(t)\ge n_i,\quad t\in(0,a_{n_i}]\cup[b_{n_i},1),\qquad p(a_{n_i})=p(b_{n_i})=n_i,
\end{gathered} \tag{2.18}$$

$$\lim_{i\to+\infty}a_{n_i}=0,\qquad \lim_{i\to+\infty}b_{n_i}=1. \tag{2.19}$$

Next, we let

$$p_{n_i}(t)=\begin{cases} n_i, & t\in(0,a_{n_i}]\cup[b_{n_i},1), \\ p(t), & t\in[a_{n_i},b_{n_i}]. \end{cases} \tag{2.20}$$

Then, from the above discussion we know that p_{n_i} is continuous on J for every $i\in N$ and

$$p_{n_i}(t)\ge p(t),\quad p_{n_i}(t)\longrightarrow p(t),\quad \forall t\in(0,1),\ \text{as } i\longrightarrow+\infty. \tag{2.21}$$

For any $\varepsilon > 0$, by (2.19) and, there exists an i_0 such that for any $i > i_0$, we have

$$2L\int_0^{a_{n_i}} G(s,s)p(s)ds < \frac{\varepsilon}{2}, \qquad 2L\int_{b_{n_i}}^1 G(s,s)p(s)ds < \frac{\varepsilon}{2}. \tag{2.22}$$

Therefore, for any bounded set $B \subset C(J,E)$, by Lemmas 2.3 and 2.4,, the above discussion and noting that $p_{n_i}(t) \le p(t),\ t \in (0,1)$, as $t \in J,\ i > i_0$, we have that

$$\begin{aligned}
\alpha(T(B)(t)) &= \alpha\left(\left\{\int_0^1 G(t,s)p(s)f(v(s))ds \mid v \in B\right\}\right)\\
&\le \alpha\left(\left\{\int_0^1 G(t,s)\left[p(s)-p_{n_i}(s)\right]f(v(s))ds \mid v \in B\right\}\right)\\
&\quad + \alpha\left(\left\{\int_0^1 G(t,s)p_{n_i}(s)f(v(s))ds \mid v \in B\right\}\right)\\
&\le 2L\int_0^{a_{n_i}} G(s,s)p(s)ds + 2L\int_{b_{n_i}}^1 G(s,s)p(s)ds\\
&\quad + \int_0^1 \alpha\big(G(t,s)p_{n_i}(s)f(v(s)) \mid v \in B\big)ds\\
&\le \varepsilon + \rho\int_0^1 G(s,s)p(s)\alpha\big(f(v(s)) \mid v \in B\big)ds\\
&\le \varepsilon + 2Me\rho\alpha(B).
\end{aligned} \tag{2.23}$$

As ε is arbitrarily, we get

$$\alpha(T(B)(t)) \le 2Me\rho\alpha_c(B), \quad t \in J. \tag{2.24}$$

So, it follows from (2.17) and (2.24) that for any bounded set $B \subset C(J,E)$, we have

$$\alpha_c(T(B)) \le 2Me\rho\alpha_c(B). \tag{2.25}$$

And note that $2Me\rho < 1$, we have $T : K \to K$ is a strict set contraction operator. The proof is completed. □

Remark 2.6. When $E = R$, (2.6) naturally holds. In this case, we may take M as 0, consequently, $T : K \to K$ is a completely continuous operator. Furthermore, if $p(t) \equiv 1,\ t \in J$, Dalmasso [1] used the following condition:

$$0 < e = \int_0^1 s^{\alpha}(1-s)^{\beta}p(s)ds < +\infty, \quad \alpha,\beta \in [0,1). \tag{2.26}$$

Clearly, our condition is weaker than (2.26).

Our main tool used in this paper is the following fixed point index theorem of cone.

Theorem 2.7 (see [13]). *Suppose that E is a Banach space, $K \subset E$ is a cone, and let the Ω_1, Ω_2 be two bounded open sets of E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be strict set contraction. Suppose that one of the following two conditions holds:*

(i) $\|Tx\| \leq \|x\|$, *for all* $x \in K \cap \partial\Omega_1$, $\|Tx\| \geq \|x\|$, *for all* $x \in K \cap \partial\Omega_2$;

(ii) $\|Tx\| \geq \|x\|$, *for all* $x \in K \cap \partial\Omega_1$, $\|Tx\| \leq \|x\|$, *for all* $x \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 2.8 (see [13]). *Suppose E is a real Banach space, $K \subset E$ is a cone. Let $\Omega_r = \{u \in K : \|u\| \leq r\}$, and let the operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \neq x$, for all $x \in \partial\Omega_r$. Then*

(i) *If* $\|Tx\| \leq \|x\|$, *for all* $x \in \partial\Omega_r$, *then* $i(T, \Omega_r, K) = 1$;

(ii) *If* $\|Tx\| \geq \|x\|$, *for all* $x \in \partial\Omega_r$, *then* $i(T, \Omega_r, K) = 0$.

3. The Main Results

Denote

$$f_0 = \lim_{\|u\|_1 \to 0^+} \frac{\|f(u)\|_1}{\|u\|_1}, \qquad f_\infty = \lim_{\|u\|_1 \to \infty} \frac{\|f(u)\|_1}{\|u\|_1}. \tag{3.1}$$

We now present our main results by Theorems 3.1 and 3.2.

Theorem 3.1. *Suppose that conditions, hold. Assume that f also satisfies*

(A_1) $f(u) \geq ru^*$, $\rho r \leq \|u\|_1 \leq r$;

(A_2) $f(u) \leq Ru_*$, $0 \leq \|u\|_1 \leq R$,

where u^* and u_* satisfy

$$\rho \left\| \int_a^b G(s,s)p(s)u^*(s)ds \right\|_1 \geq 1, \qquad \|u_*\| \int_0^1 G(s,s)p(s)ds \leq 1. \tag{3.2}$$

Then, the boundary value problem (1.1) *has a positive solution u such that $\|u\|$ is between r and R.*

Proof of Theorem 3.1. Without loss of generality, we suppose that $r < R$. For any $u \in K$, we have

$$\|u(t)\|_1 \geq \rho\|u\|, \quad t \in [a,b]. \tag{3.3}$$

We now define two open subset Ω_1 and Ω_2 of $C(J,E)$

$$\Omega_1 = \{u \in C(J,E) : \|u\| < r\}, \qquad \Omega_2 = \{u \in C(J,E) : \|u\| < R\}. \tag{3.4}$$

For $u \in K \bigcap \partial\Omega_1$, by (3.3), we have

$$r = \|u\| \geq \|u(t)\|_1 \geq \rho\|u\| = \rho r, \quad t \in [a,b]. \tag{3.5}$$

For $u \in K \cap \partial\Omega_1$, if holds, we have

$$\begin{aligned}\|Tu(t)\|_1 &= \left\|\int_0^1 G(t,s)p(s)f(u(s))ds\right\|_1 \geq \left\|\int_a^b G(t,s)p(s)u^*(s)rds\right\|_1 \\ &\geq \rho r\left\|\int_a^b G(s,s)p(s)u^*(s)ds\right\|_1 \geq r = \|u\|, \quad t \in J.\end{aligned} \tag{3.6}$$

Therefore, we have

$$\|Tu\| = \max_{t\in[0,1]}\|Tu(t)\|_1 \geq \|u\|, \quad \forall u \in K \cap \partial\Omega_1. \tag{3.7}$$

On the other hand, as $u \in K \cap \partial\Omega_2$, we have $u(t) \leq \|u\| = R$ and by, we know

$$\begin{aligned}\|Tu(t)\|_1 &= \left\|\int_0^1 G(t,s)p(s)f(u(s))ds\right\|_1 \leq R\left\|\int_0^1 G(t,s)p(s)u_*(s)ds\right\|_1 \\ &\leq R\|u_*\|\int_0^1 G(s,s)p(s)ds \leq R = \|u\|.\end{aligned} \tag{3.8}$$

Thus

$$\|T(u)\| = \max_{t\in[0,1]}\|Tu(t)\|_1 \leq \|u\|, \quad \forall u \in K \cap \partial\Omega_2. \tag{3.9}$$

Therefore, by (3.7), (3.9), Theorem 2.7 and $r < R$, we have that T has a fixed point $u \in K \cap (\Omega_2 \setminus \overline{\Omega}_1)$. Obviously, u is a positive solution of the problem (1.1) and $r < \|u\| < R$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Suppose that conditions,, and in Theorem 3.1 hold. Assume that f also satisfies*

- (A_3) $f_0 = 0$;
- (A_4) $f_\infty = 0$.

Then, the boundary value problem (1.1) *has at least two solutions.*

Proof of Theorem 3.2. Firstly, by condition, we can have $\lim_{\|u\|_1 \to 0^+}(\|f(u)\|_1/\|u\|_1) = 0$. Then, there exists an adequately small positive number $m > 0$ such that $m\int_a^b G(s,s)p(s)ds \leq 1$ and there exists a constant $\rho_* \in (0,r)$ such that

$$\|f(u)\|_1 \leq m\|u\|_1, \quad 0 < \|u\|_1 \leq \rho_*, \; u \neq 0. \tag{3.10}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (3.10), we have

$$\|f(u)\|_1 \leq m\|u\|_1 \leq m\rho_*. \tag{3.11}$$

For $u \in \partial\Omega_{\rho_*}$, we have

$$\begin{aligned} \|Tu(t)\|_1 &= \left\|\int_0^1 G(t,s)p(s)f(u(s))ds\right\|_1 \leq \int_0^1 G(t,s)p(s)\|f(u(s))\|_1 \\ &\leq \int_a^b G(t,s)p(s)m\rho_* ds \leq \rho_* m\int_a^b G(s,s)p(s)ds \leq \rho_* = \|u\|. \end{aligned} \tag{3.12}$$

Therefore, we have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{3.13}$$

Then by Theorem 2.8, we have

$$i\left(T, \Omega_{\rho_*}, K\right) = 1. \tag{3.14}$$

Next, by condition (A_4), we have $\lim_{\|u\|_1\to\infty}(\|f(u)\|_1/\|u\|_1) = 0$. Then, there exists an adequately small positive number $\overline{m} > 0$ such that $\overline{m}\int_a^b G(s,s)p(s)ds \leq 1$, and there exists a constant $\rho_0 > 0$ such that

$$\|f(u)\|_1 \leq \overline{m}\|u\|_1, \quad \|u\|_1 > \rho_0. \tag{3.15}$$

We choose a constant $\rho^* > r$, obviously, $\rho_* < r < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, then for any $u \in \partial\Omega_{\rho^*}$, by (3.15), we have

$$\|f(u)\|_1 \leq \overline{m}\|u\|_1 \leq \overline{m}\rho^*. \tag{3.16}$$

For $u \in \partial\Omega_{\rho_*}$, we have

$$\begin{aligned} \|Tu(t)\|_1 &= \left\|\int_0^1 G(t,s)p(s)f(u(s))ds\right\|_1 \leq \int_0^1 G(t,s)p(s)\|f(u(s))\|_1 \\ &\leq \int_a^b G(t,s)p(s)\overline{m}\rho^* ds \leq \rho^*\overline{m}\int_a^b G(s,s)p(s)ds \leq \rho^* = \|u\|. \end{aligned} \tag{3.17}$$

Therefore, we have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{3.18}$$

Then by Theorem 2.8, we have

$$i(T, \Omega_{\rho^*}, K) = 1. \tag{3.19}$$

Finally, set $\Omega_r = \{u \in K : \|u\| < r\}$, For any $u \in \partial\Omega_r$, by, Lemma 2.3 and proceeding as for the proof of Theorem 3.1, we have

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_r. \tag{3.20}$$

Then by Theorem 2.8, we have

$$i(T, \Omega_r, K) = 0. \tag{3.21}$$

Therefore, by (3.14), (3.19), (3.21) and $\rho_* < R < \rho^*$, we have

$$i\left(T, \Omega_r \setminus \overline{\Omega}_{\rho_*}, k\right) = -1, i\left(T, \Omega_{\rho^*} \setminus \overline{\Omega}_r, k\right) = 1. \tag{3.22}$$

Then T have fixed points $u_1 \in \Omega_r \setminus \overline{\Omega}_{\rho_*}$ and $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_r$. Obviously, u_1 and u_2 are all positive solutions of the BVP (1.1). The proof of Theorem 3.2 is complete. □

4. Application

In order to illustrate the application of our results, we give an example in this section.

Example 4.1. Consider the following singular boundary value problem (SBVP):

$$\begin{gathered}\frac{3}{\sqrt[3]{t}}\left(\frac{1}{3}\sqrt[3]{t}u'(t)\right) + 160\left[u^{1/2} + u^{1/3}\right] = \theta, \quad 0 < t < 1, \\ u(0) - \lim_{t \to 0^+} \frac{1}{3}\sqrt[3]{t}u'(t) = \theta, \qquad u(1) + \lim_{t \to 1^-} \frac{1}{3}\sqrt[3]{t}u'(t) = \theta,\end{gathered} \tag{4.1}$$

where

$$\beta = \gamma = \delta = 1, \qquad \alpha = 3, \qquad p(t) = \frac{1}{3}\sqrt[3]{t}, f(u) = 160\left(u^{1/2} + u^{1/3}\right). \tag{4.2}$$

Then obviously,

$$\int_0^1 \frac{1}{p(t)} dt = \frac{3}{2}, \qquad f_\infty = 0, \qquad f_0 = 0. \tag{4.3}$$

By computing, we know that the Green's function is

$$G(t,s) = \frac{1}{7}\begin{cases}(1+3s)(2-t), & 0 \le s \le t \le 1,\\ (1+3t)(2-s), & 0 \le t \le s \le 1.\end{cases} \tag{4.4}$$

It is easy to note that $0 \le G(s,s) \le 1$ and conditions,,, hold.

Next, by computing, we know that $\rho = 0.44$. We choose $r = 3$, $u^* = 104$, as $1.32 = \rho r \le \|u\| = \max\{u(t), t \in J\} \le 3$ and $\rho\| \int_a^b G(s,s)p(s)u^*(s)ds\| = 1.3 > 1$, because of the monotone increasing of $f(u)$ on $[0,\infty)$, then

$$f(u) \ge f(1.32) = 359.3 \ge 312 = ru^*, \qquad 1.32 \le \|u\| \le 3. \tag{4.5}$$

Thus condition (A_1) holds. Hence by Theorem 3.2, SBVP (4.1) has at least two positive solutions u_1, u_2 and $0 < \|u_1\| < 3 < \|u_2\|$.

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