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Research Article

Existence Results for Quasilinear Elliptic Equations with Indefinite Weight

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We provide the existence of a solution for quasilinear elliptic equation $-\mathrm{div}(a_\infty(x)|\nabla u|^{p-2}\nabla u + \tilde{a}(x,|\nabla u|)\nabla u) = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x)$ in Ω under the Neumann boundary condition. Here, we consider the condition that $\tilde{a}(x,t) = o(t^{p-2})$ as $t \to +\infty$ and $f(x,u) = o(|u|^{p-1})$ as $|u| \to \infty$. As a special case, our result implies that the following p-Laplace equation has at least one solution: $-\Delta_p u = \lambda m(x)|u|^{p-2}u + \mu|u|^{r-2}u + h(x)$ in $\Omega, \partial u/\partial \nu = 0$ on $\partial\Omega$ for every $1 < r < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \neq 0$ and $m, h \in L^\infty(\Omega)$ with $\int_\Omega m\, dx \neq 0$. Moreover, in the nonresonant case, that is, λ is not an eigenvalue of the p-Laplacian with weight m, we present the existence of a solution of the above p-Laplace equation for every $1 < r < p < \infty$, $\mu \in \mathbb{R}$ and $m, h \in L^\infty(\Omega)$.

1. Introduction

In this paper, we consider the existence of a solution for the following quasilinear elliptic equation:

$$-\mathrm{div}\, A(x,\nabla u) = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \qquad (P;\lambda,m,h)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$, ν denotes the outward unit normal vector on $\partial\Omega$, $\lambda \in \mathbb{R}$, $1 < p < \infty$ and $m, h \in L^\infty(\Omega)$. We assume that f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = 0 \quad \text{uniformly in } x \in \Omega, \tag{1.1}$$

and that $f(x,t)$ is bounded on a bounded set (admitting $f \equiv 0$ in the nonresonant case). Here, $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption (A)). The equation $(P;\lambda,m,h)$ contains the corresponding p-Laplacian problem as a special case. Although the operator A is nonhomogeneous in the second variable in general, we assume that $A(x,y)$ is asymptotically $(p-1)$-homogeneous at infinity in the following sense (AH).

Throughout this paper, we assume that the map A satisfies the following assumptions (AH) and (A):

(AH) there exist a positive function $a_\infty \in C^1(\overline{\Omega},\mathbb{R})$ and a continuous function $\widetilde{a}(x,t)$ on $\overline{\Omega} \times \mathbb{R}$ such that

$$A(x,y) = a_\infty(x)|y|^{p-2}y + \widetilde{a}(x,|y|)y \quad \text{for every } x \in \Omega, y \in \mathbb{R}^N,$$
$$\lim_{t\to+\infty} \frac{\widetilde{a}(x,t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}. \tag{1.2}$$

(A) $A(x,y) = a(x,|y|)y$, where $a(x,t) > 0$ for all $(x,t) \in \overline{\Omega} \times (0,+\infty)$ and

(i) $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$;

(ii) there exists $C_1 > 0$ such that

$$|D_y A(x,y)| \le C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\}; \tag{1.3}$$

(iii) there exists $C_0 > 0$ such that

$$D_y A(x,y)\xi \cdot \xi \ge C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N; \tag{1.4}$$

(iv) there exists $C_2 > 0$ such that

$$|D_x A(x,y)| \le C_2 \left(1 + |y|^{p-1}\right) \quad \text{for every } x \in \overline{\Omega}, y \in \mathbb{R}^N \setminus \{0\}. \tag{1.5}$$

A similar hypothesis to (A) is considered in the study of quasilinear elliptic problems (cf. [1, Example 2.2], [2–6]). It is easily seen that many examples as in the above references satisfy the condition (AH). Also, the following example satisfies our hypotheses:

$$\operatorname{div}\left(\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\left(1 + |\nabla u|^q\right)^{(p-q)/q} \nabla u\right) \quad \text{for } 1 < p \le q < \infty. \tag{1.6}$$

In particular, for $A(x,y) = |y|^{p-2}y$, that is, $\operatorname{div} A(x,\nabla u)$ stands for the usual p-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p-1$ in (A). Conversely, in the case where $C_0 = C_1 = p-1$ holds in (A), by the inequalities in Remark 1.4 (ii) and (iii), we see $a(x,t) = |t|^{p-2}$ whence $A(x,y) = |y|^{p-2}y$.

Concerning the weight m, throughout this paper, we assume that

$$|\{m > 0\}| := |\{x \in \Omega;\ m(x) > 0\}| > 0 \tag{1.7}$$

holds, where $|X|$ denotes the Lebesgue measure of a measurable set X.

Because $A(x, y)$ is asymptotically $(p-1)$-homogeneous at infinity, the solvability of our equation is related to the following homogeneous equation (see Theorem 1.1):

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{$EV; m$}$$

where a_∞ is the positive function as in (AH). We say that $\lambda \in \mathbb{R}$ is an eigenvalue of $(EV; m)$ if the equation $(EV; m)$ has a nontrivial solution.

There are few existence results of a solution to our equation (and also the p-Laplace equation). For example, if $\lambda < 0$ and $m \equiv 1$ hold, then the standard argument guarantees the existence of a solution. For the p-Laplacian as a special case of our problem, it is shown in [7] that the equation

$$-\Delta_p u = \lambda m|u|^{p-2}u + h \quad \text{in } \Omega \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{1.8}$$

has a unique positive solution provided $0 < \lambda < \lambda^*(m)$, $\int_\Omega m\,dx < 0$ and $0 \not\equiv h \in L^\infty(\Omega)_+$, where $\lambda^*(m)$ is the principal eigenvalue defined in Section 2.1 with $a_\infty \equiv 1$. In [8], although the resonant case where $\lambda = \lambda_1(m)$ or $\lambda = \lambda_2(m)$ is considered under the assumptions to $f(x, u) = f(u)$, its result does not cover the case of $f(u) = |u|^{r-2}u$ with $1 < r < p$, where $\lambda_i(m)$ $(i = 1, 2)$ is ith eigenvalue of the p-Laplacian with weight m. For the Laplace problem under the Neumann boundary condition, we can refer to [9, 10]. Under the Dirichlet boundary condition, the existence results for the Laplace problem are well known when $m \equiv 1$ and λ is not an eigenvalue of the Laplacian (cf. [11]). Moreover, under the Dirichlet (or blow-up) boundary condition, many authors study various equations involving the p-Laplace (Laplace) operator with (indefinite) weight. For example, we refer to [12] for boundary blow-up problems with Laplacian, [13] for periodic reaction-diffusion problems and [14, 15] for singular quasilinear elliptic problems.

Recently, the present author shows the existence of a solution for our problem in the case where λ is between the principal eigenvalue and the second eigenvalue in [6] (for $f \equiv 0$). In addition, a similar situation is treated in [5]. However, existence results are not seen in the case when λ is greater than the second eigenvalue for our problem. Therefore, the first purpose of this paper is to present an existence result of a solution in the nonresonant case where λ is not an eigenvalue of $(EV; m)$. Then, it studied the existence of at least one solution in the resonant case under assumptions that cover the case $f(u) = \mu|u|^{r-2}u$ with $1 < r < p$ and $\mu \neq 0$.

For the proof of our result, it is necessary to study the weighted eigenvalue problem $(EV; m)$. Thus, in Section 2, we introduce two sequences $\{\lambda_n(m)\}_n$ and $\{\mu_n(m)\}_n$ of an eigenvalue of $(EV; m)$ defined by Ljusternik-Schnirelman theory or Drábek-Robinson's method (cf. [16]), respectively. Then, we show several properties of above eigenvalues. In Section 3, we give the proof in the nonresonant case by using $\{\mu_n(m)\}_n$. In Sections 4 and 5, we handle the resonant case.

1.1. Statements of Our Existence Results

First, we state the existence result of a solution in the nonresonant case.

Theorem 1.1. *Assume that $\lambda \in \mathbb{R}$ is not an eigenvalue of $(EV;m)$. Then, $(P;\lambda,m,h)$ has at least one solution.*

To state our existence result in the resonant case, we introduce some conditions. Set

$$F(x,u) := \int_0^u f(x,s)\,ds, \qquad \widetilde{G}(x,y) := \int_0^{|y|} \widetilde{a}(x,t)t\,dt, \tag{1.9}$$

where $\widetilde{a}$ is the function as in (AH).

$(H+)$ there exist $0 \le q \le p-1$ and $H_0 > 0$ such that

$$\begin{gathered}\lim_{|y|\to\infty} \frac{p\widetilde{G}(x,y) - \widetilde{a}(x,|y|)|y|^2}{|y|^{1+q}} = +\infty \quad \text{uniformly in a.e. } x \in \Omega,\\ f(x,t)t - pF(x,t) \ge -H_0\left(1+|t|^{1+q}\right) \quad \text{for a.e. } x\in\Omega, \text{ every } t \in \mathbb{R};\end{gathered} \tag{1.10}$$

$(H-)$ there exist $0 \le q \le p-1$ and $H_0 > 0$ such that

$$\begin{gathered}\lim_{|y|\to\infty} \frac{p\widetilde{G}(x,y) - \widetilde{a}(x,|y|)|y|^2}{|y|^{1+q}} = -\infty \quad \text{uniformly in a.e. } x \in \Omega,\\ f(x,t)t - pF(x,t) \le H_0\left(|t|^{1+q}+1\right) \quad \text{for a.e. } x\in\Omega, \text{every } t \in \mathbb{R};\end{gathered} \tag{1.11}$$

$(HF+)$ there exist $0 \le q \le p-1$ and $H_0 > 0$ such that

$$\begin{gathered}p\widetilde{G}(x,y) - \widetilde{a}(x,|y|)|y|^2 \ge -H_0\left(1+|y|^{1+q}\right) \quad \text{for every } x\in\Omega,\ y \in \mathbb{R}^N,\\ \lim_{|t|\to\infty} \frac{f(x,t)t - pF(x,t)}{|t|^{1+q}} = +\infty \quad \text{uniformly in a.e. } x \in \Omega;\end{gathered} \tag{1.12}$$

$(HF-)$ there exist $0 \le q \le p-1$ and $H_0 > 0$ such that

$$\begin{gathered}p\widetilde{G}(x,y) - \widetilde{a}(x,|y|)|y|^2 \le H_0\left(1+|y|^{1+q}\right) \quad \text{for every } x\in\Omega,\ y \in \mathbb{R}^N,\\ \lim_{|t|\to\infty} \frac{f(x,t)t - pF(x,t)}{|t|^{1+q}} = -\infty \quad \text{uniformly in a.e. } x \in \Omega.\end{gathered} \tag{1.13}$$

Theorem 1.2. *Assume one of the following conditions:*

(i) $\lambda = 0$ *and* $(HF+)$ *or* $(HF-)$ *hold;*

(ii) $\lambda \neq 0$, $\int_\Omega m\, dx \neq 0$ *and one of* $(H+)$, $(H-)$, $(HF+)$ *and* $(HF-)$ *hold;*

(iii) $\lambda \neq 0$, $\int_\Omega m\, dx = 0$ *and* $(H+)$ *or* $(HF+)$ *hold;*

Then, $(P; \lambda, m, h)$ *has at least one solution.*

In the special case where $\tilde{a}(x,t) \equiv 0$ and $f(x,u) = \mu|u|^{r-2}u$ for $1 < r < p$, we easily see that $(HF+)$ or $(HF-)$ holds with $0 \leq q < r-1$ provided $\mu < 0$ or $\mu > 0$, respectively. Therefore, the following result is proved according to Theorem 1.2.

Corollary 1.3. *Let* $1 < r < p < \infty$, $\mu \neq 0$ *and* $\int_\Omega m\, dx \neq 0$. *Then, the following equation has at least one solution:*

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda m(x)|u|^{p-2}u + \mu|u|^{r-2}u + h(x) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.14}$$

1.2. Properties of the Map A

In what follows, the norm on $W^{1,p}(\Omega)$ is given by $\|u\|^p := \|\nabla u\|_p^p + \|u\|_p^p$, where $\|u\|_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \leq q \leq \infty)$. Setting $G(x,y) := \int_0^{|y|} a(x,t)t\, dt$, then we can easily see that

$$\nabla_y G(x,y) = A(x,y), \qquad G(x,0) = 0 \tag{1.15}$$

for every $x \in \overline{\Omega}$.

Remark 1.4. It is easily seen that the following assertions hold under condition (A):

(i) for all $x \in \overline{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in y;

(ii) $|A(x,y)| \leq (C_1/(p-1))|y|^{p-1}$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;

(iii) $A(x,y)y \geq (C_0/(p-1))|y|^p$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;

(iv) $G(x,y)$ is convex in y for all x and satisfies the following inequalities:

$$A(x,y)y \geq G(x,y) \geq \frac{C_0}{p(p-1)}|y|^p, \qquad G(x,y) \leq \frac{C_1}{p(p-1)}|y|^p, \tag{1.16}$$

for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$, where C_0 and C_1 are the positive constants in (A).

The following result is proved in [3]. It plays an important role for our poof.

Proposition 1.5 (see [3, Proposition 1]). *Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the map defined by*

$$\langle A(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v \, dx, \tag{1.17}$$

for $u, v \in W^{1,p}(\Omega)$. Then, A has the $(S)_+$ property, that is, any sequence $\{u_n\}$ weakly convergent to u with $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0$ strongly converges to u.

2. The Weighted Eigenvalue Problems

2.1. Preliminaries

The following lemmas can be easily shown by way of contradiction because $\int_\Omega a_\infty |\nabla u|^p \, dx$ is equivalent to $\|\nabla u\|_p^p$ (note that a_∞ is positive). Here, we omit the proofs (refer to [7]).

Lemma 2.1. *Assume $\int_\Omega m \, dx < 0$. Then, there exists a constant $c > 0$ such that $\int_\Omega a_\infty |\nabla u|^p \, dx \ge c\|u\|_p^p$ for every $u \in W^{1,p}(\Omega)$ with $\int_\Omega m|u|^p \, dx > 0$.*

Lemma 2.2. *Assume that $\int_\Omega m \, dx \neq 0$ and $\xi > 0$. Then, there exists a constant $b(m, \xi) > 0$ such that*

$$\int_\Omega a_\infty |\nabla u|^p \, dx - \xi \int_\Omega m|u|^p \, dx \ge b(m, \xi) \int_\Omega |u|^p \, dx \tag{2.1}$$

for every $u \in B(m) := \{u \in W^{1,p}(\Omega); \int_\Omega m|u|^p \, dx \le 0\}$.

Lemma 2.3. *Assume that $m \ge 0$ in Ω. Then, for every $\xi > 0$ there existed $d(m, \xi) > 0$ such that*

$$\int_\Omega a_\infty |\nabla u|^p \, dx - \xi \int_\Omega m|u|^p \, dx \ge d(m, \xi) \int_\Omega |u|^p \, dx \tag{2.2}$$

for every $u \in W^{1,p}(\Omega)$.

First, we recall the following principle eigenvalue $\lambda^*(m)$:

$$\lambda^*(m) := \inf\left\{ \int_\Omega a_\infty |\nabla u|^p \, dx; \; u \in W^{1,p}(\Omega), \int_\Omega m|u|^p \, dx = 1 \right\}. \tag{2.3}$$

Because of $\infty > \sup_{x\in\Omega} a_\infty(x) \ge \inf_{x\in\Omega} a_\infty(x) > 0$, we have the following result as the same argument as in the case of the p-Laplacian.

Proposition 2.4 (see [7, Proposition 2.2]). *The following assertions hold:*

- (i) *If $\int_\Omega m \, dx \ge 0$ holds, then $\lambda^*(m) = 0$;*
- (ii) *If $\int_\Omega m \, dx < 0$ holds, then $\lambda^*(m) > 0$ is a simple eigenvalue and it admits a positive eigenfunction. In addition, the open interval $(0, \lambda^*(m))$ contains no eigenvalues of $(EV; m)$.*

Lemma 2.5. *Assume* $\int_\Omega m\,dx < 0$. *Then, one has* $\lambda^*(m+\varepsilon) < \lambda^*(m) < \lambda^*(m-\varepsilon')$ *for every* $\varepsilon > 0$ *and* $\varepsilon' > 0$ *with* $|\{m-\varepsilon' > 0\}| > 0$.

Proof. We choose a minimizer u for $\lambda^*(m)$ because Proposition 2.4 guarantees the existence of it. Then, for every $\varepsilon > 0$, we have

$$\lambda^*(m+\varepsilon) \le \frac{\int_\Omega a_\infty|\nabla u|^p\,dx}{\int_\Omega (m+\varepsilon)|u|^p\,dx} < \frac{\int_\Omega a_\infty|\nabla u|^p\,dx}{\int_\Omega m|u|^p\,dx} = \int_\Omega a_\infty|\nabla u|^p\,dx = \lambda^*(m) \tag{2.4}$$

by the definition of $\lambda^*(m+\varepsilon)$. By applying the same argument to a minimizer for $\lambda^*(m-\varepsilon)$, we obtain $\lambda^*(m) < \lambda^*(m-\varepsilon')$ for $\varepsilon' > 0$ with $|\{m-\varepsilon' > 0\}| > 0$. □

2.2. Other Eigenvalues

Here, we introduce two unbounded sequences $\{\lambda_n(m)\}_n$ and $\{\mu_n(m)\}_n$ as follows:

$$\begin{aligned}
J(u) &:= \int_\Omega a_\infty|\nabla u|^p\,dx \quad \text{for } u \in W^{1,p}(\Omega), \qquad \widetilde{J} := J|_{S(m)}, \\
S(m) &:= \left\{u \in W^{1,p}(\Omega);\ \int_\Omega m|u|^p\,dx = 1\right\}, \\
\mathcal{S}_n(m) &:= \{X \subset S(m);\ \text{compact, symmetric and } \gamma(X) \ge n\}, \\
\mathcal{F}_n(m) &:= \left\{g \in C\left(S^{n-1}, S(m)\right); g \text{ is odd}\right\}, \\
\lambda_n(m) &:= \inf_{X\in\mathcal{S}_n(m)} \max_{u\in X} \widetilde{J}(u), \\
\mu_n(m) &:= \inf_{g\in\mathcal{F}_n(m)} \max_{z\in S^{n-1}} \widetilde{J}(g(z)),
\end{aligned} \tag{2.5}$$

where $\gamma(X)$ denotes the Krasnoselskii genus of X (see [17, Definition 5.1] for the definition) and S^{n-1} denotes the usual unit sphere in $\mathbb{R}^n$. We see that $\lambda_n(m)$ is defined by Ljusternik-Schnirelman theory and it is known that the definition of $\mu_n(m)$ is introduced by Drábek and Robinson ([16]) under the p-Laplace Dirichlet problem with $m \equiv 1$.

Remark 2.6. The following assertions can be shown easily:

(i) $\lambda_1(m) = \mu_1(m) = \lambda^*(m)$;

(ii) $\mathcal{S}_n(m) \ne \emptyset$ and $\mathcal{F}_n(m) \ne \emptyset$ for every $n \in \mathbb{N}$;

(iii) $g(S^{n-1}) \subset \mathcal{S}_n(m)$ for every $g \in \mathcal{F}_n(m)$;

(iv) $\mu_n(m) \ge \lambda_n(m)$ for every $n \in \mathbb{N}$;

(v) $\lambda_{n+1}(m) \ge \lambda_n(m)$ and $\mu_{n+1}(m) \ge \mu_n(m)$ for every $n \in \mathbb{N}$,

see [18] for the proof of (ii).

Define a C^1 functional Φ_m on $W^{1,p}(\Omega)$ by $\Phi_m(u) := \int_\Omega m|u|^p\,dx$ for $u \in W^{1,p}(\Omega)$. Because $1 \in \mathbb{R}$ is a regular value of Φ_m, it is well known that the norm of the derivative at $u \in S(m)$ of the restriction of J to $S(m)$ is defined as follows:

$$\begin{aligned}\left\|\tilde{J}'(u)\right\|_* &:= \min\left\{\|J'(u) - t\Phi'_m(u)\|_{W^{1,p}(\Omega)^*};\, t \in \mathbb{R}\right\}\\ &= \sup\{\langle J'(u), v\rangle;\, v \in T_u(S(m)),\, \|v\| = 1\},\end{aligned} \tag{2.6}$$

where $T_u(S(m))$ denotes the tangent space of $S(m)$ at u, that is, $T_u(S(m)) = \{v \in W^{1,p}(\Omega);\ \int_\Omega m|u|^{p-2}uv\,dx = 0\}$. Here, we recall the definition of the Palais-Smale condition for $\tilde{J}$.

Definition 2.7. $\tilde{J}$ is said to satisfy the bounded Palais-Smale condition if any bounded sequence $u_n \in S(m)$ such that $\|\tilde{J}'(u_n)\|_* \to 0$ has a convergent subsequence. Moreover, we say that $\tilde{J}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ if any sequence $u_n \in S(m)$ such that $\tilde{J}(u_n) \to c$ and $\|\tilde{J}'(u_n)\|_* \to 0$ as $n \to \infty$ has a convergent subsequence. In addition, we say that $\tilde{J}$ satisfies the Palais-Smale condition if $\tilde{J}$ satisfies the Palais-Smale condition for every $c \in \mathbb{R}$.

The following result can be proved by the same argument as in [19, Proposition 3.3] (which treats the case of the p-Laplacian, i.e., $a_\infty \equiv 1$) because of $\infty > \sup_{x\in\Omega} a_\infty(x) \geq \inf_{x\in\Omega} a_\infty(x) > 0$. Here, we omit the proof.

Lemma 2.8. *The following assertions hold:*

(i) *$\tilde{J}$ satisfies the bounded Palais-Smale condition;*

(ii) *$\tilde{J}$ satisfies the Palais-Smale condition provided $\int_\Omega m\,dx \neq 0$.*

Proposition 2.9. *$\lambda_n(m)$ and $\mu_n(m)$ are eigenvalues of $(EV;m)$ such that*

$$\lim_{n\to\infty} \lambda_n(m) = \lim_{n\to\infty} \mu_n(m) = +\infty. \tag{2.7}$$

Proof. In the case of $\int_\Omega m\,dx \neq 0$, since $\tilde{J}$ satisfies the Palais-Smale condition, we can apply the first deformation lemma on C^1 manifold (refer to [20]). Thus, by the standard argument, we can prove that $\lambda_n(m)$ and $\mu_n(m)$ are critical values of $\tilde{J}$. This means that $\lambda_n(m)$ and $\mu_n(m)$ are eigenvalues of $(EV;m)$ by the Lagrange multiplier rule. In addition, we can easily show $\lim_{n\to\infty} \lambda_n(m) = +\infty$ by the standard argument via the first deformation lemma on C^1 manifold (refer to [21, Proposition 3.14.7], [22] or [17] in the case of a Banach space). Hence, $\lim_{n\to\infty} \mu_n(m) = +\infty$ holds because of $\mu_n(m) \geq \lambda_n(m)$ for every $n \in \mathbb{N}$.

In the case of $\int_\Omega m\,dx = 0$, by the same argument as in [18], our conclusion can be proved. For readers' convenience, we give a sketch of the proof. For $\varepsilon > 0$, we define $J_\varepsilon(u) := J(u) + \varepsilon\|u\|_p^p$ and $\tilde{J}_\varepsilon := J_\varepsilon|_{S(m)}$. Moreover, we set minimax values $\lambda_n^\varepsilon(m)$ and $\mu_n^\varepsilon(m)$ of $\tilde{J}_\varepsilon$ by

$$\lambda_n^\varepsilon(m) := \inf_{X\in\mathcal{S}_n(m)} \max_{u\in X} \tilde{J}_\varepsilon(u), \qquad \mu_n^\varepsilon(m) := \inf_{g\in\mathcal{F}_n(m)} \max_{z\in S^{n-1}} \tilde{J}_\varepsilon(g(z)). \tag{2.8}$$

Because any Palais-Smale sequence of $\tilde{J}_\varepsilon$ is bounded, it is easily shown that $\tilde{J}_\varepsilon$ satisfies the Palais-Smale condition (refer to [19, Proposition 3.3]) Hence, it can be proved that $\lambda_n^\varepsilon(m)$

and $\mu_n^\varepsilon(m)$ are critical values of $\widetilde{J}_\varepsilon$. Furthermore, it follows from the argument as in [18, Lemma 3.5] that $\lambda_n^\varepsilon(m) \to \lambda_n(m)$ and $\mu_n^\varepsilon(m) \to \mu_n(m)$ as $\varepsilon \to 0+$. Therefore, by noting that J_ε is p-homogeneous, we can obtain a solution u_ε with $\|u_\varepsilon\| = 1$ for $-\operatorname{div}(a_\infty|\nabla u|^{p-2}\nabla u) = c_\varepsilon m|u|^{p-2}u$ in Ω, $\partial u/\partial \nu = 0$ on $\partial\Omega$, where $c_\varepsilon = \lambda_n^\varepsilon(m)$ or $\mu_n^\varepsilon(m)$. Because of $\|u_\varepsilon\| = 1$, it follows from the standard argument that u_ε has a subsequence strongly convergent to a solution u for

$$-\operatorname{div}\left(a_\infty|\nabla u|^{p-2}\nabla u\right) = cm|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$

where $c = \lim_{\varepsilon\to 0+} c_\varepsilon$. Thus, $\lambda_n(m)$ and $\mu_n(m)$ are eigenvalues of $(EV;m)$. To prove $\lim_{n\to\infty}\lambda_n(m) = +\infty$, by considering a function $m_\delta(x) := \max\{m(x), \delta\}$ for $\delta > 0$, we have $\lambda_n(m_\delta) \le \lambda_n(m)$ (refer to Proposition 2.10). Because we can apply our fist assertion to m_δ (note $\int_\Omega m_\delta\, dx > 0$), we obtain $\lim_{n\to\infty}\mu_n(m) \ge \lim_{n\to\infty}\lambda_n(m) \ge \lim_{n\to\infty}\lambda_n(m_\delta) = +\infty$. □

Proposition 2.10. *Let $1 < r < \infty$ if $N \le p$ and $p^*/(p^* - p) \le r < \infty$ if $N > p$. Then, the following assertions hold:*

(i) *if $m' \ge m$ in Ω, then $\mu_k(m') \le \mu_k(m)$;*

(ii) *if $\lim_{n\to\infty} m_n = m$ in $L^r(\Omega)$, then $\limsup_{n\to\infty}\mu_k(m_n) \le \mu_k(m)$;*

(iii) *if $\int_\Omega m\, dx \ne 0$ and $\lim_{n\to\infty} m_n = m$ in $L^r(\Omega)$, then $\lim_{n\to\infty}\mu_k(m_n) = \mu_k(m)$.*

Moreover, the same conclusion holds for $\lambda_k(m)$.

Proof. We only treat $\mu_k(m)$ because we can give the proof for $\lambda_k(m)$ similarly.

(i) Let $m' \ge m$ in Ω. Fix an arbitrary $\varepsilon > 0$. Then, by the definition of $\mu_k(m)$, there exists a $g \in \mathcal{F}_k(m)$ such that $\max_{z\in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon$. Set $\widetilde{g}(z) := g(z)/(\int_\Omega m'|g(z)|^p\, dx)^{1/p}$ for $z \in S^{k-1}$ (note $\int_\Omega m'|g(z)|^p dx \ge \int_\Omega m|g(z)|^p dx = 1$), then $\widetilde{g} \in \mathcal{F}_k(m')$ holds. Therefore, by the definition of $\mu_k(m')$, we have

$$\mu_k(m') \le \max_{z\in S^{k-1}} J(\widetilde{g}(z)) = \max_{z\in S^{k-1}} \frac{J(g(z))}{\int_\Omega m'|g(z)|^p dx} \le \max_{z\in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon. \tag{2.10}$$

because of $\int_\Omega m'|g(z)|^p dx \ge \int_\Omega m|g(z)|^p dx = 1$ for every $z \in S^{k-1}$. Since $\varepsilon > 0$ is arbitrary, we obtain $\mu_k(m') \le \mu_k(m)$.

(ii) Let $\lim_{n\to\infty} m_n = m$ in $L^r(\Omega)$ and fix an arbitrary $\varepsilon > 0$. By the definition of $\mu_k(m)$, there exists a $g \in \mathcal{F}_k(m)$ such that $\max_{z\in S^{k-1}} J(g(z)) < \mu_k(m) + \varepsilon/2$. Since $g(S^{k-1})$ is compact and $pr' := pr/(r-1) \le p^*$, we set $M := \max_{u\in g(S^{k-1})}\|u\|_{pr'}$. Then, due to Hölder's inequality and $m_n \to m$ in $L^r(\Omega)$, there exists an $n_0 \in \mathbb{N}$ such that

$$\int_\Omega m_n|u|^p dx = 1 + \int_\Omega (m_n - m)|u|^p dx \ge 1 - \|m_n - m\|_r M^p > 0 \tag{2.11}$$

for every $u \in g(S^{k-1})$ and $n \ge n_0$. Therefore, by a similar argument to (i), we obtain

$$\mu_k(m_n) \le \max_{z\in S^{k-1}} \frac{J(g(z))}{\int_\Omega m_n|g(z)|^p dx} \le \frac{\mu_k(m) + \varepsilon/2}{1 - \|m_n - m\|_r M^p} < \mu_k(m) + \varepsilon \tag{2.12}$$

for sufficiently large n. Hence, $\limsup_{n\to\infty}\mu_k(m_n)\le\mu_k(m)+\varepsilon$ follows. Since $\varepsilon>0$ is arbitrary, our conclusion is proved.

(iii) Let $\lim_{n\to\infty}m_n = m$ in $L^r(\Omega)$ and $\int_\Omega m\,dx\neq 0$. We fix an arbitrary $\varepsilon>0$. Due to our assertion (ii), there exists an $n_1\in\mathbb{N}$ such that $\mu_k(m_n)\le\mu_k(m)+\varepsilon/2$. For every $n\ge n_1$, by the definition of $\mu_k(m_n)$, we can take $g_n\in\mathcal{F}_k(m_n)$ satisfying $\max_{z\in S^{k-1}}J(g_n(z))<\mu_k(m_n)+\varepsilon/2$.

Here, we will prove

$$\sup_{n\ge n_1}\max\left\{\|u\|_p;\,u\in g_n\left(S^{k-1}\right)\right\}<\infty. \tag{2.13}$$

If $u\in g_n(S^{k-1})$ satisfies $\int_\Omega m|u|^p\,dx\le 0$, then we obtain

$$\begin{aligned}
b(m,1)\|u\|_p^p &\le J(u)-\int_\Omega m|u|^p\,dx = J(u)-\int_\Omega m_n|u|^p\,dx+\int_\Omega (m_n-m)|u|^p\,dx\\
&\le \mu_k(m_n)+\frac{\varepsilon}{2}-1+\|m_n-m\|_r\|u\|_{pr'}^p\\
&\le \mu_k(m)+\varepsilon+C\|m_n-m\|_r\|u\|_p^p+\frac{CJ(u)\|m_n-m\|_r}{\inf_\Omega a_\infty}\\
&\le\left(1+\frac{C\|m_n-m\|_r}{\inf_\Omega a_\infty}\right)\left(\mu_k(m)+\varepsilon\right)+C\|m_n-m\|_r\|u\|_p^p
\end{aligned} \tag{2.14}$$

by Lemma 2.2 and Hölder's inequality (note $\|\nabla u\|_p^p\le J(u)/\inf_\Omega a_\infty$ and $\mu_k(m_n)\le\mu_k(m)+\varepsilon/2$), where $C>0$ is a constant (independent of n and u) obtained by the continuity of $W^{1,p}(\Omega)$ into $L^{pr'}(\Omega)$. Therefore, if we take an $n_2\ge n_1$ satisfying $C\|m_n-m\|_r\le b(m,1)/2$ for every $n\ge n_2$, then we obtain

$$\|u\|_p^p\le\frac{2}{b(m,1)}\left(1+\frac{b(m,1)}{2\inf_\Omega a_\infty}\right)\left(\mu_k(m)+\varepsilon\right) \tag{2.15}$$

for every $u\in g_n(S^{k-1})$ provided $\int_\Omega m|u|^p\,dx\le 0$ and $n\ge n_2$. Similarly, in the case where m changes sign, for every $u\in g_n(S^{k-1})$ satisfying $\int_\Omega m|u|^p\,dx>0$, we have

$$\begin{aligned}
b(-m,1)\|u\|_p^p &\le J(u)-\int_\Omega(-m)|u|^p\,dx\\
&\le\left(1+\frac{C\|m_n-m\|_r}{\inf_\Omega a_\infty}\right)\left(\mu_k(m)+\varepsilon\right)+1+C\|m_n-m\|_r\|u\|_p^p.
\end{aligned} \tag{2.16}$$

Hence, by taking a sufficiently large $n_3\ge n_2$, we get the inequality

$$\|u\|_p^p\le\frac{2}{b(-m,1)}\left(1+\frac{b(-m,1)}{2\inf_\Omega a_\infty}\right)\left(\mu_k(m)+\varepsilon+1\right), \tag{2.17}$$

for every $u \in g_n(S^{k-1})$ with $\int_\Omega m|u|^p\,dx > 0$ and $n \geq n_3$. In the case of $m \geq 0$ in Ω, by using Lemma 2.3 instead of Lemma 2.2, we have a similar inequality

$$\|u\|_p^p \leq \frac{2}{d(m,1)}\left(1 + \frac{d(m,1)}{2\inf_\Omega a_\infty}\right)(\mu_k(m) + \varepsilon + 1), \tag{2.18}$$

for every $u \in g_n(S^{k-1})$ provided $n \geq n_4$ (some sufficiently large $n_4 \geq n_3$). Consequently, our claim follows from (2.15), (2.17), and (2.18).

Let us return to the proof of (iii). Because

$$\sup\left\{\|u\|_{pr'};\, u \in g_n\left(S^{k-1}\right),\, n \geq n_1\right\} =: R < +\infty \tag{2.19}$$

holds by (2.13), $J(u) \leq \mu_k(m) + \varepsilon/2$ and the continuity of $W^{1,p}(\Omega)$ into $L^{pr'}(\Omega)$, we see the inequality

$$\int_\Omega m|u|^p\,dx = 1 - \int_\Omega (m_n - m)|u|^p\,dx > 1 - \|m_n - m\|_r R^p > 0, \tag{2.20}$$

for every $u \in g_n(S^{k-1})$ and $n \geq n_5$ (some sufficiently large $n_5 \geq n_4$). By considering $\widetilde{g}_n(\cdot) := g_n(\cdot)/(\int_\Omega m|g_n(\cdot)|^p dx)^{1/p} \in \mathcal{F}_k(m)$, we obtain

$$\mu_k(m) \leq \max_{z\in S^{k-1}} J(\widetilde{g}_n(z)) \leq \frac{\max_{z\in S^{k-1}} J(g_n(z))}{1 - \|m_n - m\|_r R^p} \leq \frac{\mu_k(m_n) + \varepsilon/2}{1 - \|m_n - m\|_r R^p}. \tag{2.21}$$

Because of $\|m_n - m\|_r \to 0$, we get $\mu_k(m_n) \geq \mu_k(m) - \varepsilon$ for sufficiently large n, and hence our conclusion holds. □

Finally, we recall the second eigenvalue of $(EV;m)$ obtained by the mountain pass theorem.

$$\begin{aligned} \Sigma(m) &:= \{\eta \in C([0,1], S(m));\, \eta(0) \in P,\, \eta(1) \in (-P)\}, \\ c(m) &:= \inf_{\eta\in\Sigma(m)} \max_{t\in[0,1]} \widetilde{J}(\eta(t)), \end{aligned} \tag{2.22}$$

where $P := \{u \in W^{1,p}(\Omega); u(x) \geq 0 \text{ for a.e. } x \in \Omega\}$.

Since $\infty > \sup_{x\in\Omega} a_\infty(x) \geq \inf_{x\in\Omega} a_\infty(x) > 0$ holds, the following result can be shown by the same argument as in [19] (although they handle the asymmetry case, it is sufficient to consider the case of $m \equiv n$ in this paper). See [19, Theorem 3.2] for the proof.

Theorem 2.11. *$c(m)$ is an eigenvalue of $(EV;m)$ which satisfies $\lambda^*(m) < c(m)$. Moreover, there is no eigenvalues of $(EV;m)$ between $\lambda^*(m)$ and $c(m)$.*

Now, we have the following result.

Proposition 2.12.

$$\lambda_2(m) = \mu_2(m) = c(m) \tag{2.23}$$

holds, where $c(m)$ is a minimax value defined by (2.22).

Proof. First, we prove the inequality $c(m) \geq \mu_2(m)$. Because $c(m)$ is an eigenvalue (note that the following equation is homogeneous), we can choose a solution $u \in W^{1,p}(\Omega)$ with $\int_\Omega m|u|^p\, dx = 1$ for

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = c(m)m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.24}$$

Note that u is a sign-changing function because any eigenfunction associated with any eigenvalue greater than the principal eigenvalue changes sign (refer to [18, Proposition 4.3]). Thus, we have

$$0 < \int_\Omega a_\infty |\nabla u_\pm|^p\, dx = c(m)\int_\Omega m u_\pm^p\, dx \tag{2.25}$$

by taking $\pm u_\pm$ as test function (recall that $u_\pm := \max\{\pm u, 0\}$). Hence, we may assume that $\int_\Omega m u_\pm^p\, dx = 1$ by the normalization. Set $X := \{su_+ - tu_-;\ |s|^p + |t|^p = 1\} \subset S(m)$. Then, because X is homeomorphic to S^1, there exists $g \in \mathcal{F}_2(m)$ such that $g(S^1) = X$. Since the value of J is equal to $c(m)$ on X, we obtain

$$\mu_2(m) \leq \max_{z\in S^1} \widetilde{J}(g(z)) = c(m) \tag{2.26}$$

by the definition of $\mu_2(m)$ and X.

Next, we will prove the inequality $c(m) \leq \lambda_2(m)$ by dividing into two cases: $\int_\Omega m\, dx \neq 0$ and $\int_\Omega m\, dx = 0$.

Case of $\int_\Omega m\, dx \neq 0$: by way of contradiction, we assume that $\lambda_2(m) < c(m)$. Then, $\lambda^*(m) = \lambda_1(m) = \lambda_2(m)$ follows from Theorem 2.11. Note that $\widetilde{J}$ satisfies the Palais-Smale condition in this case (see Lemma 2.8), and hence we can apply the first deformation lemma to $\widetilde{J}$. Therefore, by the standard argument (cf. [22], [17, Lemma 5.6]), we see that $\gamma(K) \geq 2$, where $K := \{u \in S(m); \widetilde{J}'(u) = 0, \widetilde{J}(u) = \lambda^*(m)\}$. This means that K is an infinite set, that is, the following equation has infinite many solutions:

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda^*(m)m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{2.27}$$

due to the Lagrange multiplier's rule. This contradicts to the fact described as in Proposition 2.4 that $\lambda^*(m)$ is simple. As a result, we have shown that $c(m) = \lambda_2(m) = \mu_2(m)$ holds in the case of $\int_\Omega m\, dx \neq 0$ (note $\lambda_n(m) \leq \mu_n(m)$).

Case of $\int_\Omega m\,dx = 0$: According to Proposition 2.10 (i) for $\lambda_2(m)$, we have $\lambda_2(m) \geq \lambda_2(m+\varepsilon) = c(m+\varepsilon)$ for every $\varepsilon > 0$ since we can apply the first result to $m+\varepsilon$. Because we prove $\lim_{\varepsilon \to 0+} c(m+\varepsilon) = c(m)$ by the same argument as in [6, Lemma 2.9] (for the case $a_\infty \equiv 1$), our conclusion is proved by taking $\varepsilon \downarrow 0$ in the inequality $\lambda_2(m) \geq c(m+\varepsilon)$. □

3. Proof of Theorem 1.1

We define a functional $I_{\lambda,m}$ on $W^{1,p}(\Omega)$ as follows:

$$\begin{aligned} I_{\lambda,m}(u) &= \int_\Omega G(x,\nabla u)\,dx - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega hu\,dx \\ &= \frac{1}{p}\int_\Omega a_\infty |\nabla u|^p\,dx + \int_\Omega \widetilde{G}(x,\nabla u)dx - \frac{\lambda}{p}\int_\Omega m|u|^p\,dx \\ &\quad - \int_\Omega F(x,u)\,dx - \int_\Omega hu\,dx \end{aligned} \tag{3.1}$$

for $u \in W^{1,p}(\Omega)$ ((1.15) or (1.9) for the definition of G, $\widetilde{G}$, and F). It is easily seen that $I_{\lambda,m}$ is well defined and class of C^1 on $W^{1,p}(\Omega)$ by (1.1), (1.16) and the continuity of $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$.

Remark 3.1. Let $u \in W^{1,p}(\Omega)$ be a critical point of $I_{\lambda,m}$, namely, u satisfies the equality

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx = \lambda\int_\Omega m|u|^{p-2}u\varphi\,dx + \int_\Omega f(x,u)\varphi\,dx + \int_\Omega h\varphi\,dx \tag{3.2}$$

for every $\varphi \in W^{1,p}(\Omega)$. Then, $u \in L^\infty(\Omega)$ by the Moser iteration process (refer to Theorem C in [4]). Therefore, $u \in C^{1,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ follows from the regularity result in [23]. Furthermore, due to [24, Theorem 3], u satisfies $(P;\lambda,m,h)$ in the distribution sense and the boundary condition

$$0 = \frac{\partial u}{\partial \nu_A} = A(\cdot,\nabla u)\nu = a(\cdot,|\nabla u|)\frac{\partial u}{\partial \nu} \quad \text{in } W^{-1/q,q}(\partial\Omega) \tag{3.3}$$

for every $1 < q < \infty$ (see [24] for the definition of $W^{-1/q,q}(\partial\Omega)$). Since $u \in C^{1,\alpha}(\overline{\Omega})$ and $a(x,t) > 0$ for every $t \neq 0$, u satisfies the Neumann boundary condition, that is, $(\partial u/\partial\nu)(x) = 0$ for every $x \in \partial\Omega$.

3.1. The Palais-Smale Condition in the Nonresonant Case

First, we recall the definition of the Palais-Smale condition.

Definition 3.2. A C^1 functional Ψ on a Banach space X is said to satisfy the Palais-Smale condition at $c \in \mathbb{R}$ if a Palais-Smale sequence $\{u_n\} \subset X$ at level c, namely,

$$\Psi(u_n) \longrightarrow c, \qquad \left\|\Psi'(u_n)\right\|_{X^*} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.4}$$

has a convergent subsequence. We say that Ψ satisfies the Palais-Smale condition if Ψ satisfies the Palais-Smale condition at any $c \in \mathbb{R}$. Moreover, we say that Ψ satisfies the bounded Palais-Smale condition if any bounded sequence $\{u_n\}$ such that $\{\Psi(u_n)\}$ is bounded and $\|\Psi'(u_n)\|_{X^*} \to 0$ as $n \to \infty$ has a convergent subsequence.

Concerning the Palais-Smale condition, we state the following result developed from [6, Proposition 7].

Proposition 3.3. *If λ is not an eigenvalue of $(EV; m)$, then $I_{\lambda,m}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence of $I_{\lambda,m}$, namely,

$$I_{\lambda,m}(u_n) \longrightarrow c, \qquad \left\| I'_{\lambda,m}(u_n) \right\|_{W^{1,p}(\Omega)^*} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{3.5}$$

for some $c \in \mathbb{R}$. It is sufficient to prove only the boundedness of $\|u_n\|$ because the operator $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ described in Proposition 1.5 has the $(S)_+$ property.

To prove the boundedness of $\|u_n\|$, it suffices to show that $\|u_n\|_p$ is bounded because of the inequality $|f(x,u)| \le C(|u|^{p-1} + 1)$ (obtained by (1.1)) and the following inequality:

$$\begin{aligned} &\left\langle I'_{\lambda,m}(u_n), u_n \right\rangle + \lambda \int_\Omega m|u_n|^p \, dx + \int_\Omega f(x,u_n)u_n \, dx + \int_\Omega h u_n \, dx, \\ &\quad = \int_\Omega A(x,\nabla u_n)\nabla u_n \, dx \ge \frac{C_0}{p-1} \|\nabla u_n\|_p^p, \end{aligned} \tag{3.6}$$

where we use Remark 1.4 (iii) in the last inequality. By way of contradiction, we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence if necessary. Set $v_n := u_n/\|u_n\|_p$. Then, since the inequality (3.6) guarantees that $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, we may suppose, by choosing a subsequence, that $v_n \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^p(\Omega)$ for some v_0.

Here, we will prove that

$$\lim_{n \to \infty} \frac{\|f(\cdot, u_n)\|_{p'}}{\|u_n\|_p^{p-1}} = 0, \tag{3.7}$$

where $p' = p/(p-1)$. Fix an arbitrary $\varepsilon > 0$. It follows from (1.1) that there exists a $C_\varepsilon > 0$ such that

$$\left| f(x,u) \right| \le \varepsilon |u|^{p-1} + C_\varepsilon \quad \text{for every } u \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \tag{3.8}$$

Then, we obtain

$$\int_\Omega \left| f(x,u_n) \right|^{p'} dx \le 2^{p'} \int_\Omega \left(\varepsilon^{p'} |u_n|^p + C_\varepsilon^{p'} \right) dx \le 2^{p'} \varepsilon^{p'} \|u_n\|_p^p + 2^{p'} C_\varepsilon^{p'} |\Omega|. \tag{3.9}$$

Since we are assuming that $\|u_n\|_p \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$

$$\frac{\|f(\cdot,u_n)\|_{p'}}{\|u_n\|_p^{p-1}} \leq 4\varepsilon \tag{3.10}$$

holds. This shows that $\lim_{n\to\infty} \|f(\cdot,u_n)\|_{p'}/\|u_n\|_p^{p-1} = 0$ because $\varepsilon > 0$ is arbitrary.

Here, we recall the following result proved in [6]:

$$\lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla(v_n - v_0)\,dx = \lim_{n\to\infty} \int_\Omega \frac{\tilde{a}(x,|\nabla u_n|)\nabla u_n}{\|u_n\|_p^{p-1}} \nabla\varphi\,dx = 0, \tag{3.11}$$

for every $\varphi \in W^{1,p}(\Omega)$. Thus, by considering

$$o(1) = \frac{\left\langle I'_{\lambda,m}(u_n), v_n - v_0\right\rangle}{\|u_n\|_p^{p-1}} = \int_\Omega a_\infty |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v_0)dx + o(1), \tag{3.12}$$

we see that v_n strongly converges to v_0 in $W^{1,p}(\Omega)$ (note that p-Laplacian has the $(S)_+$ property). Therefore, by taking a limit in $o(1) = \langle I'_{\lambda,m}(u_n),\varphi\rangle/\|u_n\|_p^{p-1}$ for any $\varphi \in W^{1,p}(\Omega)$ and by noting (3.7) and (3.11), we know that v_0 is a nontrivial solution (note $\|v_0\|_p = 1$) of

$$-\operatorname{div}\left(a_\infty |\nabla u|^{p-2}\nabla u\right) = \lambda m|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

This means that λ is an eigenvalue of $(EV;m)$. This is a contradiction. Hence, $\|u_n\|_p$ is bounded. □

3.2. Key Lemmas

To show the linking lemma, we define

$$Y(\mu,m) := \left\{u \in W^{1,p}(\Omega); \int_\Omega a_\infty |\nabla u|^p\,dx \geq \mu \int_\Omega m|u|^p\,dx\right\} \tag{3.14}$$

for $\mu \in \mathbb{R}$.

Lemma 3.4. *Let $g_0 \in C(S^{k-1}, W^{1,p}(\Omega) \setminus \{0\})$ be odd and $0 < \mu \leq \mu_{k+1}(m)$. Then, $g(S^k_+) \cap Y(\mu,m) \neq \emptyset$ for every $g \in C(S^k_+, W^{1,p}(\Omega))$ with $g|_{S^{k-1}} = g_0$, where $Y(\mu,m)$ is the set introduced in* (3.14) *and S^k_+ is the upper hemisphere in $\mathbb{R}^{k+1}$ with boundary S^{k-1}.*

Proof. Fix any $g \in C(S^k_+, W^{1,p}(\Omega))$ such that $g|_{S^{k-1}} = g_0$. If $u \in g(S^k_+)$ satisfies $\int_\Omega m|u|^p\,dx \le 0$, then $u \in Y(\mu, m)$ holds. So, we may assume that $\int_\Omega m|u|^p dx > 0$ for every $u \in g(S^k_+)$. Define $\widetilde{g} \in \mathcal{F}_{k+1}(m)$ as follows:

$$\widetilde{g}(z) := \begin{cases} \dfrac{g(z)}{\left(\int_\Omega m\left|g(z)\right|^p\,dx\right)^{1/p}} & \text{if } z \in S^k_+, \\ -\dfrac{g(-z)}{\left(\int_\Omega m\left|g(-z)\right|^p\,dx\right)^{1/p}} & \text{if } z \in S^k_-. \end{cases} \tag{3.15}$$

By the definition of $\mu_{k+1}(m)$, there exists $z_0 \in S^k$ such that $\widetilde{J}(\widetilde{g}(z_0)) \ge \mu_{k+1}(m)$. Since $\widetilde{g}$ is odd and J is even, we may suppose $z_0 \in S^k_+$. So, this yields the inequality $J(g(z_0)) \ge \mu_{k+1}(m)\int_\Omega m|g(z_0)|^p dx \ge \mu\int_\Omega m|g(z_0)|^p\,dx$, whence $g(z_0) \in Y(\mu, m)$ holds. □

Lemma 3.5. *Let $\mu_k(m) < \lambda$. Then, there exists $g_0 \in \mathcal{F}_k(m)$ such that*

$$\max_{z\in S^{k-1}} J\left(g_0(z)\right) < \lambda, \qquad \max_{z\in S^{k-1}} I_{\lambda,m}\left(Tg_0(z)\right) \longrightarrow -\infty \quad \text{as } |T| \longrightarrow \infty, \tag{3.16}$$

where $\mu_k(m)$ is defined by (2.5).

Proof. Choose $\varepsilon_0 > 0$ such that $\mu_k(m) + \varepsilon_0 < \lambda$. By the definition of $\mu_k(m)$, there exists $g_0 \in \mathcal{F}_k(m)$ such that

$$\max_{z\in S^{k-1}} J\left(g_0(z)\right) < \mu_k(m) + \varepsilon_0. \tag{3.17}$$

Due to the compactness of $g_0(S^{k-1})$, we put $M := \max_{z\in S^{k-1}} \|g_0(z)\|_p$. By the property of the function $\widetilde{a}$ as in (AH) and Young's inequality, for every $\varepsilon > 0$ there exist constants $C_\varepsilon > 0$ and $C'_\varepsilon > 0$ such that

$$\left|\widetilde{G}(x,y)\right| \le \frac{\varepsilon}{2}|y|^p + C_\varepsilon|y| \le \varepsilon|y|^p + C'_\varepsilon \le \frac{\varepsilon}{\inf_\Omega a_\infty} a_\infty(x)|y|^p + C'_\varepsilon \tag{3.18}$$

for every $x \in \Omega$ and $y \in \mathbb{R}^N$. Moreover, the hypothesis (1.1) ensures that for every $\varepsilon' > 0$ there exist constants $D_{\varepsilon'} > 0$ satisfying

$$|F(x,u)| \le \frac{\varepsilon'}{2}|u|^p + D_{\varepsilon'}|u| \le \varepsilon'|u|^p + D'_{\varepsilon'} \tag{3.19}$$

for every $u \in \mathbb{R}$ and a.e. $x \in \Omega$. Hence, we have

$$\begin{aligned} I_{\lambda,m}(Tu) &\le \frac{T^p}{p}\left(1 + \frac{p\varepsilon}{\underline{a}}\right)\int_\Omega a_\infty|\nabla u|^p dx - \frac{T^p\left(\lambda - p\varepsilon' M^p\right)}{p} + T\|h\|_\infty\|u\|_1 + C \\ &\le \frac{T^p}{p}\left\{\left(1 + \frac{p\varepsilon}{\underline{a}}\right)\left(\mu_k(m) + \varepsilon_0\right) - \lambda + pM^p\varepsilon'\right\} + TM\|h\|_\infty|\Omega|^{(p-1)/p} + C \end{aligned} \tag{3.20}$$

for every $T > 0$, $u \in g_0(S^{k-1})$, $\varepsilon > 0$ and $\varepsilon' > 0$ since $g_0(S^{k-1}) \subset S(m)$, (3.17), (3.18) and (3.19), where $C = (C'_\varepsilon + D'_{\varepsilon'})|\Omega|$ and $\underline{a} = \inf_{x\in\Omega} a_\infty(x) > 0$. By taking $\varepsilon > 0$ and $\varepsilon' > 0$ satisfying $(1+p\varepsilon/\underline{a})(\mu_k(m)+\varepsilon_0)-\lambda+pM^p\varepsilon' < 0$, we show that $\max_{z\in S^{k-1}} I_{\lambda,m}(Tg_0(z)) \to -\infty$ as $T \to +\infty$. Thus, our conclusion follows because $g_0(S^{k-1})$ is symmetric. □

3.3. The Case $\int_\Omega m\,dx \neq 0$

Lemma 3.6. *Let* $\int_\Omega m\,dx < 0$ *and* $0 < \lambda < \lambda^*(m)$. *Then,* $I_{\lambda,m}$ *is bounded from below, coercive and weakly lower semicontinuous (w.l.s.c.) on* $W^{1,p}(\Omega)$.

Proof. $\Phi(u) := \int_\Omega G(x,\nabla u)\,dx$ is w.l.s.c. on $W^{1,p}(\Omega)$ because Φ is convex and continuous on $W^{1,p}(\Omega)$ (cf. [25, Theorem 1.2]). Thus, $I_{\lambda,m}$ is also w.l.s.c. on $W^{1,p}(\Omega)$ since the inclusion from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ is compact.

Choose $\varepsilon > 0$ such that $p\varepsilon < \underline{a}(1-\lambda/\lambda^*(m))$, where $\underline{a} := \inf_\Omega a_\infty$. By an easy estimation, (3.18) and (3.19) as in Lemma 3.5, we have

$$\begin{aligned} I_{\lambda,m}(u) \geq \frac{\underline{a}-\varepsilon p}{p\underline{a}} \int_\Omega a_\infty |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega m|u|^p dx - \varepsilon' \|u\|_p^p \\ - \|h\|_\infty \|u\|_p |\Omega|^{(p-1)/p} - \left(C'_\varepsilon + D'_{\varepsilon'}\right)|\Omega| \end{aligned} \tag{3.21}$$

for every $u \in W^{1,p}(\Omega)$ and $\varepsilon' > 0$.

Let $u \in W^{1,p}(\Omega)$ satisfy $\int_\Omega m|u|^p dx \leq 0$. Then, the following inequality follows from Lemma 2.2:

$$D_0 \int_\Omega a_\infty |\nabla u|^p dx - \lambda \int_\Omega m|u|^p dx \geq \frac{D_0}{2}\int_\Omega a_\infty |\nabla u|^p dx + b(m,\xi)\|u\|_p^p, \tag{3.22}$$

where $b(m,\xi)$ is a positive constant independent of u with $\xi = 2\lambda/D_0$ and $D_0 = (\underline{a}-\varepsilon p)/\underline{a}$.

For every $u \in W^{1,p}(\Omega)$ such that $\int_\Omega m|u|^p dx > 0$, we obtain

$$\begin{aligned} D_0 \int_\Omega a_\infty |\nabla u|^p dx - \lambda \int_\Omega m|u|^p dx &\geq \left(D_0 - \frac{\lambda}{\lambda^*(m)}\right)\int_\Omega a_\infty |\nabla u|^p dx \\ &\geq \frac{1}{2}\left(D_0 - \frac{\lambda}{\lambda^*(m)}\right)\int_\Omega a_\infty |\nabla u|^p dx + \frac{c}{2}\left(D_0 - \frac{\lambda}{\lambda^*(m)}\right)\|u\|_p^p \end{aligned} \tag{3.23}$$

by the definition of $\lambda^*(m)$, Lemma 2.1 and $D_0 - \lambda/\lambda^*(m) > 0$, where $c > 0$ is a constant obtained by Lemma 2.1.

Consequently, if we choose a $\varepsilon' > 0$ satisfying $\varepsilon' < \min\{b(m,\xi)/p, c(D_0 - \lambda/\lambda^*(m))/(2p)\}$, then we obtain positive constants d_1 and d_2 (independent of u) such that

$$\begin{aligned} I_{\lambda,m}(u) &\geq d_1 \int_\Omega a_\infty |\nabla u|^p dx + d_2\|u\|_p^p - \|h\|_\infty \|u\|_p |\Omega|^{(p-1)/p} - \left(C'_\varepsilon + D'_{\varepsilon'}\right)|\Omega| \\ &\geq \min\{\underline{a}d_1, d_2\}\|u\|^p - \|h\|_\infty \|u\| |\Omega|^{(p-1)/p} - \left(C'_\varepsilon + D'_{\varepsilon'}\right)|\Omega| \end{aligned} \tag{3.24}$$

for every $u \in W^{1,p}(\Omega)$ by (3.21), (3.22), and (3.23). Because of $p > 1$, our conclusion is shown. □

Lemma 3.7. *Let $m \geq 0$ in Ω and $m \not\equiv 0$. If $\lambda < 0$ holds, then $I_{\lambda,m}$ is bounded from below, coercive and w.l.s.c. on $W^{1,p}(\Omega)$.*

Proof. First, as the same reason in Lemma 3.6, it follows that $I_{\lambda,m}$ is w.l.s.c. on $W^{1,p}(\Omega)$. By a similar argument to Lemma 3.6, for every $\varepsilon' > 0$ and $0 < \varepsilon < \underline{a}/p$ where $\underline{a} = \inf_\Omega a_\infty$, we obtain

$$\begin{aligned} I_{\lambda,m}(u) \geq{}& \frac{\underline{a} - \varepsilon p}{p\underline{a}} \int_\Omega a_\infty |\nabla u|^p dx + \frac{|\lambda|}{p} \int_\Omega m|u|^p dx - \varepsilon' \|u\|_p^p \\ &- \|h\|_\infty \|u\|_p |\Omega|^{(p-1)/p} - \left(C'_\varepsilon + D'_{\varepsilon'}\right)|\Omega| \end{aligned} \tag{3.25}$$

for every $u \in W^{1,p}(\Omega)$ (note $\lambda < 0$). Here, from Lemma 2.3,

$$D_0 \int_\Omega a_\infty |\nabla u|^p dx + |\lambda| \int_\Omega m|u|^p dx \geq \frac{D_0}{2} \int_\Omega a_\infty |\nabla u|^p dx + \frac{D_0}{2} b(\xi, m)\|u\|_p^p \tag{3.26}$$

for every $u \in W^{1,p}(\Omega)$ follows, where $D_0 := (\underline{a} - \varepsilon p)/\underline{a}$, $\xi := 2|\lambda|/D_0$ and $b(\xi, m)$ is a constant obtained in Lemma 2.3. Therefore, by choosing a ε' such that $0 < \varepsilon' < D_0 b(\xi, m)/2$, we can prove our conclusion. □

Lemma 3.8. *Let $\int_\Omega m\,dx \neq 0$ and $0 < \lambda < \mu$. Then, $I_{\lambda,m}$ is bounded from below on $Y(\mu, m)$, where $Y(\mu, m)$ is the set introduced in* (3.14).

Proof. Due to the same inequalities concerning G and F as in Lemma 3.5, for every $\varepsilon > 0$ and $\varepsilon' > 0$, there exists $C = C(\varepsilon, \varepsilon') > 0$ such that

$$I_{\lambda,m}(u) \geq \frac{\underline{a} - p\varepsilon}{p\underline{a}} \int_\Omega a_\infty |\nabla u|^p dx - \frac{\lambda}{p} \int_\Omega m|u|^p dx - \varepsilon' \|u\|_p^p - \|h\|_\infty \|u\|_1 - C|\Omega| \tag{3.27}$$

for every $u \in W^{1,p}(\Omega)$, where $\underline{a} := \inf_{x\in\Omega} a_\infty(x)$. Choose positive constants ε and δ such that $D_0 := 1 - p\varepsilon/\underline{a} > \delta > \lambda/\mu$ (note $\lambda/\mu < 1$).

First, we consider the case of $m \geq 0$ in Ω. For every $u \in Y(\mu, m)$, we obtain

$$\begin{aligned} &D_0 \int_\Omega a_\infty |\nabla u|^p dx - \lambda \int_\Omega m|u|^p dx \\ &\quad\geq (D_0 - \delta) \int_\Omega a_\infty |\nabla u|^p dx + (\delta\mu - \lambda) \int_\Omega m|u|^p dx \geq d(m, \xi_1)(D_0 - \delta)\|u\|_p^p \end{aligned} \tag{3.28}$$

by Lemma 2.3 with $\xi_1 = (\delta\mu - \lambda)/(D_0 - \delta)$ (note $\delta\mu - \lambda > 0$ and $D_0 - \delta > 0$).

Next, we handle with the case where m changes sign. Let $u \in W^{1,p}(\Omega)$ satisfy $\int_\Omega m|u|^p\,dx \le 0$. Then, we have for such u

$$D_0 \int_\Omega a_\infty |\nabla u|^p dx - \lambda \int_\Omega m|u|^p dx \ge b(m, \xi_2) D_0 \|u\|_p^p \tag{3.29}$$

by Lemma 2.2, where $D_0 = 1 - p\varepsilon/\underline{a}$ and $\xi_2 := \lambda/D_0$.

On the other hand, for $u \in Y(\mu, m)$ with $\int_\Omega m|u|^p dx > 0$, the following inequality follows from Lemma 2.2:

$$\begin{aligned} D_0 \int_\Omega a_\infty |\nabla u|^p dx - \lambda \int_\Omega m|u|^p dx & \\ \ge (D_0 - \delta) \int_\Omega a_\infty |\nabla u|^p dx - (\delta\mu - \lambda) \int_\Omega (-m)|u|^p\,dx & \\ \ge b(-m, \xi_1)(D_0 - \delta)\|u\|_p^p. & \end{aligned} \tag{3.30}$$

Consequently, by (3.27), (3.29), (3.28), and (3.30), there exists $d > 0$ independent of u such that

$$I_{\lambda,m}(u) \ge (d - \varepsilon')\|u\|_p^p - \|h\|_\infty \|u\|_p |\Omega|^{(p-1)/p} - C|\Omega| \tag{3.31}$$

for every $u \in Y(\mu, m)$. Hence, our conclusion is shown by taking $\varepsilon' > 0$ satisfying $\varepsilon' < d$. □

Proof of Theorem 1.1 in the Case $\int_\Omega m\,dx \ne 0$. First, if either $m \ge 0$ on Ω and $\lambda < 0$ or $0 < \lambda < \lambda^*(m) = \mu_1(m)$ (i.e., $\int_\Omega m\,dx < 0$) holds, then Lemma 3.7 or Lemma 3.6 guarantees the existence of a global minimizer of $I_{\lambda,m}$, respectively (cf. [25, Theorem 1.1]). Hence, $(P;\lambda,m,h)$ has a solution.

Since λ is an eigenvalue of $(EV;m)$ if and only if $-\lambda$ is one of $(EV;-m)$, it suffices to consider the case of $\lambda > \lambda^*(m) \ge 0$. Furthermore, by Proposition 2.9, Remark 2.6 (i), and our hypothesis that λ is not an eigenvalue of $(EV;m)$, we may assume that there exists a $k \in \mathbb{N}$ such that $\mu_k(m) < \lambda < \mu_{k+1}(m)$. By Lemmas 3.5 and 3.8, we can choose $T > 0$ and $g_0 \in \mathcal{F}_k(m)$ satisfying

$$\max_{z \in S^{k-1}} I_{\lambda,\mathrm{m}}(Tg_0(z)) < \inf\{I_{\lambda,m}(u);\ u \in Y(\mu_{k+1}(m), m)\} =: \alpha. \tag{3.32}$$

Put

$$\begin{aligned} \Sigma &:= \left\{ g \in C\left(S_+^k, W^{1,p}(\Omega)\right);\ g|_{S^{k-1}} = Tg_0 \right\}, \\ c &:= \inf_{g \in \Sigma} \max_{z \in S_+^k} I_{\lambda,m}(g(z)). \end{aligned} \tag{3.33}$$

Then, it follows from Lemma 3.4 and (3.32) that $c \ge \alpha > \max_{z \in S^{k-1}} I_{\lambda,m}(Tg_0(z))$ holds. Since $I_{\lambda,m}$ satisfies the Palais-Smale condition by Proposition 3.3, the minimax theorem guarantees (cf. [25, Theorem 4.6]) that c is a critical value of $I_{\lambda,m}$. Hence, $(P;\lambda,m,h)$ has at least one solution. □

3.4. The Case $\int_\Omega m\,dx = 0$

First, we introduce an approximate functional $I^+_{\lambda,m,n}$ as follows:

$$I^+_{\lambda,m,n}(u) := I_{\lambda,m}(u) + \frac{1}{pn}\|u\|_p^p = I_{\lambda,m-1/(\lambda n)}(u) \quad \text{for } u \in W^{1,p}(\Omega). \tag{3.34}$$

Lemma 3.9. *Let $0 < \lambda < \mu$. Then, there exists an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $I^+_{\lambda,m,n}$ is bounded from below on $\Upsilon(\mu, m - 1/\lambda n)$, where $\Upsilon(\mu, m - 1/\lambda n)$ is the set introduced in* (3.14).

Proof. Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \lambda \operatorname{ess\,sup}_{x\in\Omega} m(x)/2$. Then, for every $n \geq n_0$, Lemma 3.8 guarantees that $I^+_{\lambda,m,n} = I_{\lambda,m-1/(\lambda n)}$ bounded from below on $\Upsilon(\mu, m - 1/(\lambda n))$ because of $\int_\Omega (m - 1/(\lambda n))\,dx < 0$ and $|\{m - 1/(\lambda n) > 0\}| > 0$. □

Proof of Theorem 1.1 in the Case $\int_\Omega m\,dx = 0$. By noting that $\lambda m = (-\lambda)(-m)$ and $\mu_1(m) = \lambda^*(m) = 0$, we may assume that $\mu_k(m) < \lambda < \mu_{k+1}(m)$ for some $k \in \mathbb{N}$. Let n_0 be a natural number obtained by Lemma 3.9. Due to Proposition 2.10 (i) and (ii), there exists an $n_1 \geq n_0$ such that

$$\mu_k(m) \leq \mu_k\left(m - \frac{1}{n\lambda}\right) \leq \mu_k\left(m - \frac{1}{n_1\lambda}\right) < \lambda < \mu_{k+1}(m) \leq \mu_{k+1}\left(m - \frac{1}{n\lambda}\right) \tag{3.35}$$

for every $n \geq n_1$. Thus, for every $n \geq n_1$, we can take $T_n > 0$ and $g_n \in \mathcal{F}_k(m - 1/(n\lambda))$ satisfying

$$\max_{z\in S^{k-1}} I^+_{\lambda,m,n}(T_n g_n(z)) < \inf\left\{ I_{\lambda,m,n}(u);\ u \in \Upsilon\left(\mu_{k+1}\left(m - \frac{1}{(n\lambda)}\right), m - \frac{1}{(n\lambda)}\right)\right\} \tag{3.36}$$

by applying Lemmas 3.5 and 3.9 to $I^+_{\lambda,m,n} = I_{\lambda,m-1/(n\lambda)}$ (note (3.35)). Set

$$\begin{gathered}\Sigma_n := \left\{ g \in C\left(S^k_+, W^{1,p}(\Omega)\right);\ g|_{S^{k-1}} = T_n g_n \right\}, \\ c_n := \inf_{g\in\Sigma_n} \max_{z\in S^k_+} I^+_{\lambda,m,n}(g(z))\end{gathered} \tag{3.37}$$

for each $n \geq n_1$. Then, for each $n \geq n_1$, we can obtain u_n satisfying

$$\left| I^+_{\lambda,m,n}(u_n) - c_n \right| < \frac{1}{n}, \qquad \left\| \left(I^+_{\lambda,m,n}\right)'(u_n) \right\|_{W^{1,p}(\Omega)} < \frac{1}{n} \tag{3.38}$$

by applying Ekeland's variational principle to each $I^+_{\lambda,m,n}$ (refer to [25, Theorem 4.3]). In addition, we can see that $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Indeed, if there exists a subsequence $\{u_{n_l}\}_l$ satisfying $\|u_{n_l}\|_p \to \infty$ as $l \to \infty$, then we can show that λ is an eigenvalue of $(EV; m)$ by the same argument as in Proposition 3.3. This contradicts to our assumption that λ is not an eigenvalue of $(EV; m)$. Moreover, the boundedness of $\|\nabla u_n\|_p$ follows from a similar inequality to (3.6) as in Proposition 3.3 under the boundedness of $\|u_n\|_p$.

Therefore, we may assume, by choosing a subsequence that $\{u_n\}$ is a Palais-Smale sequence of $I_{\lambda,m}$ since $I_{\lambda,m}$ is bounded on a bounded set and according to the following inequality:

$$\left\|I'_{\lambda,m}(u_n)\right\|_{(W^{1,p}(\Omega))^*} \le \left\|I'_{\lambda,m}(u_n) - \left(I^+_{\lambda,m,n}\right)'(u_n)\right\|_{(W^{1,p}(\Omega))^*} + \frac{1}{n} \le \frac{1}{n}\|u_n\|_p^{p-1} + \frac{1}{n}. \tag{3.39}$$

Therefore, because $I_{\lambda,m}$ satisfies the Palais-Smale condition by Proposition 3.3, $I_{\lambda,m}$ has a critical point, whence $(P;\lambda,m,h)$ has at least one solution. □

4. Proof of Theorem 1.2

First, we will prove the following result concerning the Palais-Smale condition under the additional hypothesis $(H\pm)$ or $(HF\pm)$.

Proposition 4.1. *Assume that one of the following conditions hold:*

(i) $\lambda = 0$ *and* $(HF+)$ *or* $(HF-)$*;*

(ii) $\lambda \neq 0$ *and one of* $(H+)$, $(H-)$, $(HF+)$ *and* $(HF-)$.

Then, $I_{\lambda,m}$ *satisfies the Palais-Smale condition.*

Proof. As the same reason in Proposition 3.3, it suffices to prove the boundedness of a Palais-Smale sequence $\{u_n\}$ such that $I_{\lambda,m}(u_n) \to c$ (for some $c \in \mathbb{R}$) and $\|I'_{\lambda,m}(u_n)\|_{W^*} \to 0$ as $n \to \infty$. By way of contradiction, we may assume that $\|u_n\|_p \to \infty$ as $n \to \infty$ by choosing a subsequence. Set $v_n := u_n/\|u_n\|_p$. Then, by the same argument as in Proposition 3.3, $\{v_n\}$ has a subsequence strongly convergent to v_0 being a nontrivial solution of

$$-\operatorname{div}\left(a_\infty(x)|\nabla u|^{p-2}\nabla u\right) = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

To simplify the notation, we denote the above subsequence strongly convergent to v_0 by $\{v_n\}$, again. Thus, $|u_n(x)| \to \infty$ as $n \to \infty$ for a.e. $x \in \Omega_0 := \{x' \in \Omega;\ v_0(x') \neq 0\}$ (note $\|v_0\|_p = 1$).

Assume $(HF+)$ or $(HF-)$. Then, we can obtain

$$(I) := \int_\Omega \frac{f(x,u_n)u_n - pF(x,u_n)}{\|u_n\|_p^{1+q}}dx \longrightarrow \pm\infty \quad \text{if } (HF\pm), \text{ respectively.} \tag{4.2}$$

Indeed, it follows from $(HF+)$ that there exist $R > 0$ and $C > 0$ independent of n such that $f(x,t)t - pF(x,t) \ge 0$ if $|t| \ge R$ and a.e. $x \in \Omega$, and $|f(x,t)t - pF(x,t)| \le C$ for every $|t| \le R$ and a.e. $x \in \Omega$. Therefore, since $|u_n(x)| \to \infty$ a.e. $x \in \Omega_0$ and $|\Omega_0| > 0$ (note $\|v_0\|_p = 1$), we have (4.2) if $(HF+)$ holds, by applying Fatou's lemma to the following inequality:

$$(I) \ge \int_{\Omega_0} \frac{f(x,u_n)u_n - pF(x,u_n)}{|u_n|^{1+q}}|v_n|^{1+q}dx - \frac{C|\Omega \setminus \Omega_0|}{\|u_n\|_p^{1+q}}. \tag{4.3}$$

In the case of $(HF-)$, by considering $-f$ instead of f as in the above argument, we can show our claim (4.2).

Furthermore, by Hölder's inequality, we have

$$\begin{aligned}(II) &:= \int_\Omega \frac{p\widetilde{G}(x,\nabla u_n) - \widetilde{a}(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^{1+q}}dx \\ &\le H_0 \int_\Omega \left(|\nabla v_n|^{1+q} + \frac{1}{\|u_n\|_p^{1+q}} \right) dx \le H_0 \|\nabla v_n\|_p^{1+q} |\Omega|^{(p-1-q)/p} + o(1) \\ &\le H_0 \|\nabla v_0\|_p^{1+q} |\Omega|^{(p-1-q)/p} + o(1)\end{aligned} \tag{4.4}$$

in the case of $(HF-)$ because $v_n \to v_0$ in $W^{1,p}(\Omega)$, where $q \in [0, p-1]$ and $H_0 > 0$ are constants as in $(HF-)$. Similarly, we obtain

$$(II) \ge -H_0 \|\nabla v_0\|_p^{1+q} |\Omega|^{(p-1-q)/p} + o(1) \tag{4.5}$$

in the case of $(HF+)$.

Hence, we have a contradiction because of (4.2), (4.4), or (4.5) by taking a limit inferior or superior in the following equality:

$$o(1) = \frac{pI_{\lambda,m}(u_n) - \left\langle I'_{\lambda,m}(u_n), u_n \right\rangle}{\|u_n\|_p^{1+q}} = (II) + (I) + (1-p) \int_\Omega \frac{hv_n}{\|u_n\|_p^q} dx, \tag{4.6}$$

where we use the fact that $\|u_n\|/\|u_n\|_p^{1+q} = \|v_n\|/\|u_n\|_p^q$ is bounded because of $q \ge 0$.

Assume $\lambda \neq 0$ and $(H+)$ or $(H-)$: because v_0 is a nontrivial solution of (4.1) with $\lambda \neq 0$, v_0 is not a constant function, that is, $\|\nabla v_0\|_p > 0$. Therefore, we have $|\nabla u_n(x)| \to \infty$ as $n \to \infty$ for a.e. $x \in \widetilde{\Omega}_0 := \{x' \in \Omega;\ |\nabla v_0(x')| \neq 0\}$. Because of $|\widetilde{\Omega}_0| > 0$, we can show

$$\int_\Omega \frac{p\widetilde{G}(x,\nabla u_n) - \widetilde{a}(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^{1+q}} dx \longrightarrow \pm\infty \quad \text{if } (H\pm), \text{ respectively}, \tag{4.7}$$

by a similar argument to one for f in the above. In addition, we can easily obtain the following inequality:

$$\pm \int_\Omega \frac{f(x,u_n)u_n - pF(x,u_n)}{\|u_n\|_p^{1+q}} dx \ge -H_0 \|v_n\|_{1+q}^{1+q} + \mathrm{o}(1) = -H_0 \|v_0\|_{1+q}^{1+q} + o(1) \tag{4.8}$$

in the case of $(H\pm)$, respectively. Hence, we have a contradiction by considering $o(1) = (pI_{\lambda,m}(u_n) - \langle I'_{\lambda,m}(u_n), u_n \rangle)/\|u_n\|_p^{1+q}$. □

By a similar way to the case $\int_\Omega m\,dx = 0$, we introduce the following approximate functionals on $W^{1,p}(\Omega)$:

$$I^{\pm}_{\lambda,m,n}(u) := I_{\lambda,m}(u) \pm \frac{1}{pn}\|u\|_p^p \quad \text{for } u \in W^{1,p}(\Omega). \tag{4.9}$$

Note $I^{\pm}_{\lambda,m,n}(u) = I_{\lambda,m\mp 1/(\lambda n)}(u)$ on $W^{1,p}(\Omega)$ provided $\lambda \neq 0$.

Proposition 4.2. *If either $\lambda \neq 0$ and (H+) or (HF+) (resp., either $\lambda \neq 0$ and (H−) or (HF−)) and $\{u_n\}$ satisfies*

$$\sup_{n\in\mathbb{N}} I^{+}_{\lambda,m,n}(u_n) < +\infty, \qquad \lim_{n\to\infty}\left\|\left(I^{+}_{\lambda,m,n}\right)'(u_n)\right\|_{W^{1,p}(\Omega)^*} = 0, \tag{4.10}$$

$$\left(\text{resp. } \inf_{n\in\mathbb{N}} I^{-}_{\lambda,m,n}(u_n) > -\infty, \lim_{n\to\infty}\left\|\left(I^{-}_{\lambda,m,n}\right)'(u_n)\right\|_{W^{1,p}(\Omega)^*} = 0\right), \tag{4.11}$$

then $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$.

Proof. First, we note that the boundedness of $\|u_n\|_p$ guarantees that $\|u_n\|$ is bounded by $\lim_{n\to\infty}\|(I^{\pm}_{\lambda,m,n})'(u_n)\|_{W^{1,p}(\Omega)^*} = 0$ (refer to (3.6) as in the proof of Proposition 3.3). Moreover, because of the following equality:

$$\begin{aligned}\frac{pI^{\pm}_{\lambda,m,n}(u_n) - \left\langle \left(I^{\pm}_{\lambda,m,n}\right)'(u_n), u_n\right\rangle}{\|u_n\|_p^{1+q}} &= (1-p)\int_\Omega \frac{hv_n}{\|u_n\|_p^q}dx,\\ &+\int_\Omega \frac{p\widetilde{G}(x,\nabla u_n) - \widetilde{a}(x,|\nabla u_n|)|\nabla u_n|^2}{\|u_n\|_p^{1+q}}dx + \int_\Omega \frac{f(x,u_n)u_n - pF(x,u_n)}{\|u_n\|_p^{1+q}}dx,\end{aligned} \tag{4.12}$$

we can prove the boundedness of $\|u_n\|_p$ by the same argument as in Proposition 4.1. □

Proof of Theorem 1.2. Because of $\lambda m = (-\lambda)(-m)$, we may assume $\lambda \geq 0$. In the case where $\int_\Omega m\,dx \neq 0$ and $\mu_k(m) < \lambda < \mu_{k+1}(m)$ for some $k \in \mathbb{N}$, the proof of Theorem 1.1 implies the existence of a critical point of $I_{\lambda,m}$ because $I_{\lambda,m}$ satisfies the Palais-Smale condition by Proposition 4.1. Concerning other cases, in the next section, we will prove the existence of a bounded sequence $\{u_n\}$ satisfying $(I^{+}_{\lambda,m,n})'(u_n) \to 0$ or $(I^{-}_{\lambda,m,n})'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to \infty$. Because $I_{\lambda,m}$ is bounded on a bounded set, we may assume that $I_{\lambda,m}(u_n)$ converges to some $c \in \mathbb{R}$ by choosing a subsequence. In addition, by noting the inequality $\|I'_{\lambda,m}(u_n)\|_{W^{1,p}(\Omega)^*} \leq \|(I^{\pm}_{\lambda,m,n})'(u_n)\|_{W^{1,p}(\Omega)^*} + \|u_n\|_p^{p-1}/n$, we easily see that $\{u_n\}$ is a bounded Palais-Smale sequence of $I_{\lambda,m}$. Therefore, $I_{\lambda,m}$ has a critical point since $I_{\lambda,m}$ satisfies the Palais-Smale condition by Proposition 4.1. □

5. Construction of a Bounded Palais-Smale Sequence

In this section, due to the reason stated in the proof of Theorem 1.2, we will construct a bounded sequence $\{u_n\}$ satisfying $(I^+_{\lambda,m,n})'(u_n) \to 0$ or $(I^-_{\lambda,m,n})'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to \infty$. It implies the existence of a bounded Palais-Smale sequence of $I_{\lambda,m}$.

5.1. The Case $\lambda = 0$

Assume (HF+)

In this c ase, we can show that for each $n \in \mathbb{N}$, $I^+_{\lambda,m,n}$ has a global minimizer u_n. Indeed, for $0 < \varepsilon < 1/(pn)$, there exists $C_\varepsilon > 0$ such that $I^+_{\lambda,m,n}(u) \geq C_0\|\nabla u\|_p^p/(p(p-1)) + (1/(pn) - \varepsilon)\|u\|_p^p - \|h\|_\infty\|u\|_1 - C_\varepsilon$ for every $u \in W^{1,p}(\Omega)$ by (1.1), (1.16) and $\lambda = 0$ (refer to the inequality as in the proof of Lemma 3.5). This means that $I^+_{\lambda,m,n}$ is coercive and bounded from below on $W^{1,p}(\Omega)$. Therefore, $I^+_{\lambda,m,n}$ has a global minimizer u_n since $I^+_{\lambda,m,\mathrm{n}}$ is w.l.s.c. on $W^{1,p}(\Omega)$ as the same reason in Lemma 3.6.

Furthermore, because of $(I^+_{\lambda,m,n})'(u_n) = 0$ in $W^{1,p}(\Omega)^*$ and $I^+_{\lambda,m,n}(u_n) = \min_{W^{1,p}(\Omega)} I^+_{\lambda,m,n} \leq I^+_{\lambda,m,n}(0) = 0$, it follows from Proposition 4.2 that $\{u_n\}$ is bounded.

Assume (HF−)

Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < c(1) = \mu_2(1)$, where $c(1)$ is the second eigenvalue of $(EV;1)$ (so the weight function $m \equiv 1$ and see (2.22) for the definition). Then, by noting that $I^-_{0,m,n_0} = I_{1/n_0,1}$, we have

$$\alpha := \inf\left\{I^-_{0,m,n_0}(u);\ u \in Y(c(1),1)\right\} > -\infty \tag{5.1}$$

by Lemma 3.8, where $Y(c(1),1)$ is a subset defined by (3.14) with the weight $m \equiv 1$, that is,

$$Y(c(1),1) := \left\{u \in W^{1,p}(\Omega);\ \int_\Omega a_\infty|\nabla u|^p dx \geq c(1)\|u\|_p^p\right\}. \tag{5.2}$$

Moreover, $\inf\{I^-_{0,m,n}(u); u \in Y(c(1),1)\} \geq \alpha$ for every $n \geq n_0$ holds because $I^-_{0,m,n}(u) \geq I^-_{0,m,n_0}(u)$ for every $u \in W^{1,p}(\Omega)$. Since $\int_\Omega F(x,u)dx = o(1)\|u\|_p^p$ as $\|u\|_p \to \infty$ by (1.1), there exists $T_n > 0$ such that $I^-_{0,m,n}(\pm T_n) = -T_n^p(|\Omega|/(np) - o(1)) < \alpha - 2$.

Define

$$\begin{gathered}\Sigma_n := \left\{g \in C\Big([0,1], W^{1,p}(\Omega)\Big);\ g(0) = T_n,\ g(1) = -T_n\right\},\\ c_n := \inf_{g\in\Sigma_n}\max_{t\in[0,1]} I^-_{0,m,n}(g(t))\end{gathered} \tag{5.3}$$

for $n \geq n_0$. By the definition of $c(1)$, we easily see that $g([0,1]) \cap Y(c(1),1) \neq \emptyset$ for every $g \in \Sigma_n$ (refer to [6] or Lemma 3.4). Hence,

$$c_n \geq \inf\left\{I^-_{0,m,n}(u); u \in Y(c(1),1)\right\} \geq \alpha > I_{0,m,n}(\pm T_n) \tag{5.4}$$

holds, whence c_n is bounded from below. Moreover, by applying Ekeland's variational principle to each $I^-_{0,m,n}$, we can obtain a sequence $\{u_n\}$ satisfying $|I^-_{0,m,n}(u_n) - c_n| < 1/n$ and $\|(I^-_{0,m,n})'(u_n)\|_{W^{1,p}(\Omega)^*} < 1/n$. Since c_n is bounded from below, it follows from Proposition 4.2 that $\{u_n\}$ is bounded. As a result, we can construct a *bounded* sequence $\{u_n\}$ satisfying $(I^-_{0,m,n})'(u_n) \to 0$ as $n \to \infty$ in $W^{1,p}(\Omega)^*$.

5.2. The Case $\lambda = \lambda^*(m) = \mu_1(m)$ with $\int_\Omega m dx < 0$

Assume (H+) or (HF+)

Since we see that $I^+_{\lambda,m,n} = I_{\lambda,m-1/(n\lambda)}$ and $\lambda^*(m - 1/(n\lambda)) > \lambda^*(m) = \lambda > 0$ (according to Lemma 2.5), $I^+_{\lambda,m,n}$ is coercive, bounded from below and w.l.s.c. on $W^{1,p}(\Omega)$ by Lemma 3.6. Thus, we obtain a global minimizer u_n of $I^+_{\lambda,m,n}$ for sufficiently large n such that $|\{m-1/(n\lambda) > 0\}| > 0$. Because of $I^+_{\lambda,m,n}(u_n) \le I^+_{\lambda,m,n}(0) = 0$ for every n, Proposition 4.2 guarantees that $\{u_n\}$ is bounded.

Assume (H−) or (HF−)

First, we note that $I^-_{\lambda,m,n} = I_{\lambda,m+1/(n\lambda)}$ and $0 < \lambda^*(m + 1/(n\lambda)) < \lambda^*(m) = \lambda$ by Lemma 2.5 for sufficiently large n such that $\int_\Omega (m+1/(n\lambda))dx < 0$. Moreover, it follows from Proposition 2.10 and $\mu_1(m) < \mu_2(m)$ that there exists an $n_0 \in \mathbb{N}$ satisfying $\int_\Omega m + 1/(n_0\lambda)dx < 0$ and

$$\lambda^*\left(m + \frac{1}{n\lambda}\right) < \lambda = \mu_1(m) < \mu_2\left(m + \frac{1}{n_0\lambda}\right) \le \mu_2\left(m + \frac{1}{n\lambda}\right) \le \mu_2(m) \tag{5.5}$$

for every $n \ge n_0$. By applying Theorem 1.1 to each case of a weight $m + 1/(n\lambda)$ (note that λ is not an eigenvalue of $(EV; m + 1/(n\lambda))$ by (5.5), there exists u_n satisfying $(I^-_{\lambda,m,n})'(u_n) = 0$ (note $I^-_{\lambda,m,n} = I_{\lambda,m+1/(n\lambda)}$) and

$$I^-_{\lambda,m,n}(u_n) = c_n \ge \inf\left\{I^-_{\lambda,m,n}(u);\ u \in Y\left(\mu_2(m_{n_0}), m_{n_0}\right)\right\}, \tag{5.6}$$

where the last inequality follows from Lemma 3.4 with $m_{n_0} := m + 1/(n_0\lambda)$. On the other hand, because $I^-_{\lambda,m,n}(u) \ge I^-_{\lambda,m,n_0}(u) = I_{\lambda,m_{n_0}}(u)$ for every $u \in W^{1,p}(\Omega)$ and $n \ge n_0$, we have

$$c_n \ge \inf\left\{I_{\lambda,m_{n_0}}(u);\ u \in Y\left(\mu_2(m_{n_0}), m_{n_0}\right)\right\} > -\infty \tag{5.7}$$

for every $n \ge n_0$, where the last inequality follows from Lemma 3.8. Thus, c_n is bounded from below. Hence, Proposition 4.2 guarantees the boundedness of $\{u_n\}$.

5.3. The Case $\lambda = \mu_{k+1}(m)$ *with* $\int_\Omega m\,dx \neq 0$

Assume (H+) or (HF+)

We may assume $\mu_k(m) < \mu_{k+1}(m) = \lambda$ by taking k anew if necessary (note that we have already proved the case of $\mu_k(m) < \lambda < \mu_{k+1}(m)$ in Section 4). Here, we can choose an $n_0 \in \mathbb{N}$ such that $\int_\Omega (m - 1/(n\lambda))dx \neq 0$, $|\{m - 1/(n\lambda) > 0\}| > 0$ and

$$\mu_k\left(m - \frac{1}{n\lambda}\right) \leq \mu_k\left(m - \frac{1}{n_0\lambda}\right) < \lambda - \frac{1}{n\|m\|_\infty} < \lambda = \mu_{k+1}(m) \leq \mu_{k+1}\left(m - \frac{1}{n\lambda}\right) \tag{5.8}$$

for every $n \geq n_0$ by $\int_\Omega m\,dx \neq 0$ and Proposition 2.10 (i), (iii). Note the following inequality:

$$I^+_{\lambda,m,n_0}(u) \geq I^+_{\lambda,m,n}(u) \geq I_{\lambda - 1/(n\|m\|_\infty),m}(u) \tag{5.9}$$

for every $u \in W^{1,p}(\Omega)$ and $n \geq n_0$, where the last inequality is obtained by $\|u\|_p^p \geq \int_\Omega m|u|^p dx/\|m\|_\infty$. Let $n \geq n_0$. It follows from Lemma 3.8 and (5.8) that $I_{\lambda-1/(n\|m\|_\infty),m}$ is bounded from below on $Y(\lambda, m)$. Hence, (5.9) yields that $I^+_{\lambda,m,n}$ is also bounded from below on $Y(\lambda, m)$, namely,

$$\alpha_n := \inf\left\{ I^+_{\lambda,m,n}(u);\ u \in Y(\lambda, m)\right\} > -\infty. \tag{5.10}$$

On the other hand, because of $\mu_k(m - 1/(n_0\lambda)) < \lambda$ (see (5.8)), Lemma 3.5 guarantees the existence of $g_0 \in \mathcal{F}_k(m - 1/(n_0\lambda))$ satisfying

$$\max_{z \in S^{k-1}} I^+_{\lambda,m,n_0}(Tg_0(z)) = \max_{z \in S^{k-1}} I_{\lambda,m-1/(n_0\lambda)}(Tg_0(z)) \longrightarrow -\infty \quad \text{as } |T| \longrightarrow \infty. \tag{5.11}$$

Thus, for each $n \geq n_0$, we can take $T_n > 0$ such that

$$\max_{z \in S^{k-1}} I^+_{\lambda,m,n}(T_n g_0(z)) \leq \max_{z \in S^{k-1}} I^+_{\lambda,m,n_0}(T_n g_0(z)) \leq \alpha_n - 1, \tag{5.12}$$

(note (5.9) for the first inequality). Set

$$\begin{gathered} \Sigma_n := \left\{ g \in C\left(S^k_+, W^{1,p}(\Omega)\right);\ g|_{S^{k-1}} = T_n g_0 \right\}, \\ c_n^+ := \inf_{g \in \Sigma_n} \max_{z \in S^k_+} I^+_{\lambda,m,n}(g(z)) \end{gathered} \tag{5.13}$$

for $n \geq n_0$. Since $g(S^k_+) \cap Y(\lambda, m) \neq \emptyset$ for every $g \in \Sigma_n$ by Lemma 3.4 and $\lambda = \mu_{k+1}(m)$, we have $c_n^+ \geq \alpha_n > \max_{z \in S^{k-1}} I^+_{\lambda,m,n}(T_n g_0(z))$. Therefore, Ekeland's variational principle (refer to [25, Theorem 4.3]) guarantees the existence of u_n satisfying $|I^+_{\lambda,m,n}(u_n) - c_n| < 1/n$ and $\|(I^+_{\lambda,m,n})'(u_n)\|_{W^{1,p}(\Omega)^*} < 1/n$.

Finally, to show the boundedness of $\{u_n\}$ due to Proposition 4.2, we will prove that c_n^+ is bounded from above. For each $n \geq n_0$, we define a continuous map g_n from S_+^k to $W^{1,p}(\Omega)$ by

$$g_n(z) := \begin{cases} (1 - z_{k+1})T_n g_0\left(\dfrac{z'}{\sqrt{1 - z_{k+1}^2}}\right) & \text{for } z = (z', z_{k+1}) \in S_+^k \text{ with } 0 \leq z_{k+1} < 1, \\ 0 & \text{for } z = (z', z_{k+1}) \in S_+^k \text{ with } z_{k+1} = 1. \end{cases} \tag{5.14}$$

Then, $g_n \in \Sigma_n$ holds. This leads to

$$c_n^+ \leq \sup_{t \geq 0, z \in S^{k-1}} I_{\lambda,m,n}^+(tg_0(z)) \leq \sup_{t \geq 0, z \in S^{k-1}} I_{\lambda,m,n_0}^+(tg_0(z)) < +\infty \tag{5.15}$$

because of (5.9), (5.11), and the compactness of $g_0(S^{k-1})$.

Assume (H−) or (HF−)

Because the case of $\mu_1(m) = \lambda^*(m)$ is already shown (see Sections 5.1 and 5.2), We may assume $(0 <)\mu_k(m) = \lambda < \mu_{k+1}(m)$ for some $k \geq 2$ by taking k anew if necessary. Here, we can choose an $n_0 \in \mathbb{N}$ such that $\int_\Omega (m + 1/(n\lambda))dx \neq 0$ and

$$\mu_k\left(m + \frac{1}{n\lambda}\right) \leq \mu_k(m) = \lambda < \mu_{k+1}\left(m + \frac{1}{n_0\lambda}\right) \leq \mu_{k+1}\left(m + \frac{1}{n\lambda}\right) \leq \mu_{k+1}(m) \tag{5.16}$$

for every $n \geq n_0$ by $\int_\Omega m\,dx \neq 0$ and Proposition 2.10 (i), (iii). Moreover, we note the following inequality:

$$I_{\lambda,m,n_0}^-(u) \leq I_{\lambda,m,n}^-(u) = I_{\lambda,m+1/(n\lambda)}(u) \leq I_{\lambda+1/(n\|m\|_\infty),m}(u) \tag{5.17}$$

for every $u \in W^{1,p}(\Omega)$ and $n \geq n_0$. It follows from Lemma 3.8 and (5.16) (note (5.17) also) that $I_{\lambda,m,n_0}^- = I_{\lambda,m_0}$ is bounded from below on $Y(\mu_{k+1}(m_0), m_0)$ with $m_0 := m + 1/(n_0\lambda)$. Hence, (5.17) implies

$$\begin{aligned} &\inf\left\{I_{\lambda,m,n}^-(u);\, u \in Y(\mu_{k+1}(m_0), m_0)\right\} \\ &\quad \geq \inf\left\{I_{\lambda,m,n_0}^-(u);\, u \in Y(\mu_{k+1}(m_0), m_0)\right\} =: \alpha_0 > -\infty \end{aligned} \tag{5.18}$$

for every $n \geq n_0$. Because of $\lambda + 1/(n\|m\|_\infty) > \lambda = \mu_k(m)$, there exist $g_n \in \mathcal{F}_k(m)$ and $T_n > 0$ such that

$$\max_{z \in S^{k-1}} I_{\lambda,m,n}^-(T_n g_n(z)) \leq \max_{z \in S^{k-1}} I_{\lambda+1/(n\|m\|_\infty),m}(T_n g_n(z)) < \alpha_0 - 1 \tag{5.19}$$

by Lemma 3.5. Define

$$\Sigma_n := \left\{ g \in C\left(S^k_+, W^{1,p}(\Omega)\right);\ g|_{S^{k-1}} = T_n g_n \right\},$$
$$c^-_n := \inf_{g\in\Sigma_n} \max_{z\in S^k_+} I^-_{\lambda,m,n}(g(z)) \tag{5.20}$$

for $n \ge n_0$. Then, $c^-_n \ge \alpha_0$ occurs (see (5.18)) since $g(S^k_+) \cap Y(\mu_{k+1}(m_0), m_0) \neq \emptyset$ for every $g \in \Sigma_n$ by Lemma 3.4. This means that c^-_n is bounded from below. Consequently, we can obtain a desired bounded sequence by the same argument as in Sections 5.1 and 5.2.

5.4. The Case (iii) as in Theorem 1.2

First, note that we are assuming the hypothesis $(H+)$ or $(HF+)$ in this case (iii). In addition, as the reason in the proof of Theorem 1.2, it suffices to handle with $\lambda > 0$.

Let $k \in \mathbb{N}$ satisfy $\mu_k(m) < \lambda \le \mu_{k+1}(m)$. According to Proposition 2.10 (i) and (ii), we can take an $n_0 \in \mathbb{N}$ such that $|\{m - 1/(n\lambda) > 0\}| > 0$ and

$$\mu_k\left(m - \frac{1}{2n\lambda}\right) \le \mu_k\left(m - \frac{1}{n_0\lambda}\right) < \lambda - \frac{1}{2n\|m\|_\infty} < \lambda \le \mu_{k+1}(m) \le \mu_{k+1}\left(m - \frac{1}{2n\lambda}\right) \tag{5.21}$$

for every $n \ge n_0$. The following inequality follows from the easy estimates:

$$I^+_{\lambda,m,n_0}(u) \ge I^+_{\lambda,m,n}(u) = I_{\lambda,m-1/(n\lambda)}(u) \ge I_{\lambda-1/(2n\|m\|_\infty),m-1/(2n\lambda)}(u) \tag{5.22}$$

for every $u \in W^{1,p}(\Omega)$ and $n \ge n_0$. Let $n \ge n_0$ and set $m_n := m - 1/(2n\lambda)$. Because of (5.21), Lemma 3.8 implies that $I_{\lambda-1/(2n\|m\|_\infty),m_n}$ is bounded from below on $Y(\mu_{k+1}(m_n), m_n)$ with (note $\int_\Omega m_n dx \neq 0$). Hence, (5.22) yields that

$$\alpha_n := \inf\left\{ I^+_{\lambda,m,n}(u);\ u \in Y(\mu_{k+1}(m_n), m_n) \right\} > -\infty \tag{5.23}$$

for each $n \ge n_0$. On the other hand, because of $\mu_k(m - 1/(n_0\lambda)) < \lambda$ (see (5.21)), Lemma 3.5 guarantees the existence of $g_0 \in \mathcal{F}_k(m - 1/(n_0\lambda))$ satisfying

$$\max_{z\in S^{k-1}} I^+_{\lambda,m,n_0}(Tg_0(z)) = \max_{z\in S^{k-1}} I_{\lambda,m-1/(n_0\lambda)}(Tg_0(z)) \longrightarrow -\infty \quad \text{as } T \longrightarrow \infty. \tag{5.24}$$

Therefore, for each $n \ge n_0$, we can choose $T_n > 0$ such that

$$\max_{z\in S^{k-1}} I^+_{\lambda,m,n}(T_n g_0(z)) \le \max_{z\in S^{k-1}} I^+_{\lambda,m,n_0}(T_n g_0(z)) \le \alpha_n - 1, \tag{5.25}$$

(note (5.22) for the first inequality). Set

$$\Sigma_n := \left\{g \in C\left(S_+^k, W^{1,p}(\Omega)\right); g|_{S^{k-1}} = T_n g_0\right\},$$
$$c_n^+ := \inf_{g\in\Sigma_n} \max_{z\in S_+^k} I_{\lambda,m,n}^+\left(g(z)\right) \tag{5.26}$$

for $n \geq n_0$. Since $g(S_+^k) \cap Y(\mu_{k+1}(m_n), m_n) \neq \emptyset$ for every $g \in \Sigma_n$ by Lemma 3.4, we have $c_n^+ \geq \alpha_n > \max_{z\in S^{k-1}} I_{\lambda,m,n}^+(T_n g_0(z))$. Moreover, by the same argument as in Section 5.3 (note (5.24)), we have

$$c_n^+ \leq \sup_{t\geq 0, z\in S^{k-1}} I_{\lambda,m,n}^+\left(tg_0(z)\right) \leq \sup_{t\geq 0, z\in S^{k-1}} I_{\lambda,m,n_0}^+\left(tg_0(z)\right) < +\infty, \tag{5.27}$$

and hence our conclusion is shown.

Remark 5.1. If $\int_\Omega m\,dx = 0$ holds, then we can not show the continuity of $\mu_k(m)$ with respect to m (refer to Proposition 2.10). Hence, we are not able to construct a bounded Palais-Smale sequence under $(H-)$ or $(HF-)$. However, if we have the additional information about the existence of a suitable $m' \in L^\infty(\Omega)$ such that $m' \geq m$ in Ω, $\int_\Omega m' dx \neq 0$ and $\mu_k(m) \leq \lambda < \mu_{k+1}(m')$ when $\mu_k(m) \leq \lambda < \mu_{k+1}(m)$ occurs, then we can still easily prove that equation $(P; \lambda, m, h)$ has a solution in the case also where $\lambda \neq 0$, $\int_\Omega m\,dx = 0$ and $(H-)$ or $(HF-)$. In fact, let $0 < \mu_k(m) \leq \lambda < \mu_{k+1}(m')$ for some $k \geq 2$. Note the following inequality:

$$I_{\lambda+1/(n\|m\|_\infty),m}(u) \geq I_{\lambda,m,n}^-(u) \geq I_{\lambda,m'}(u) - \frac{1}{np}\|u\|_p^p = I_{\lambda,m'-1/(n\lambda)}(u) \tag{5.28}$$

for every $u \in W^{1,p}(\Omega)$ and n. Fix $n_0 \in \mathbb{N}$ such that $\int_\Omega m' - 1/(n_0\lambda)dx > 0$ and $|\{m' - 1/(n_0\lambda) > 0\}| > 0$. Set $m_0' := m' - 1/(n_0\lambda)$. Because of $\lambda < \mu_{k+1}(m') \leq \mu_{k+1}(m_0')$ (the last inequality follows from Proposition 2.10 (i)), Lemma 3.8 implies that $I_{\lambda,m_0'}$ is bounded from below on $Y(\mu_{k+1}(m_0'), m_0')$ (note $\int_\Omega m_0'\,dx > 0$). By combining this fact and (5.28), we have

$$\inf_{n\geq n_0} \inf\left\{I_{\lambda,m,n}^-(u);\ u \in Y\left(\mu_{k+1}(m_0'), m_0'\right)\right\}$$
$$\geq \inf\left\{I_{\lambda,m_0'}(u);\ u \in Y\left(\mu_{k+1}(m_0'), m_0'\right)\right\} > -\infty. \tag{5.29}$$

Because of $\lambda + 1/(n\|m\|_\infty) > \lambda \geq \mu_k(m)$, for each $n \geq n_0$, we can take a $g_n \in \mathcal{F}_k(m)$ satisfying

$$\max_{z\in S^{k-1}} I_{\lambda,m,n}^-\left(Tg_n(z)\right) \leq \max_{z\in S^{k-1}} I_{\lambda+1/(n\|m\|_\infty),m}\left(Tg_n(z)\right) \longrightarrow -\infty \tag{5.30}$$

as $T \to \infty$ by Lemma 3.5.

Since any extension $g \in C(S_+^k, W^{1,p}(\Omega))$ of Tg_n $(T > 0)$ links $Y(\mu_{k+1}(m_0'), m_0')$ by Lemma 3.4, we can construct a desired sequence by the same argument as in Section 5.3 under $(H-)$ or $(HF-)$.

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