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Research Article

Quenching for a Non-Newtonian Filtration Equation with a Singular Boundary Condition

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This paper deals with a nonlinear p-Laplacian equation with singular boundary conditions. Under proper conditions, the solution of this equation quenches in finite time and the only quenching point that is $x = 1$ are obtained. Moreover, the quenching rate of this equation is established. Finally, we give an example of an application of our results.

1. Introduction

In this paper, we consider the following problem:

$$\begin{gathered}(\psi(u))_t = (|u_x|^{p-2}u_x)_x, \quad 0 < x < 1,\ t > 0,\\ u_x(0,t) = 0, \quad u_x(1,t) = -g(u(1,t)), \quad t > 0,\\ u(x,0) = u_0(x), \quad 0 \le x \le 1,\end{gathered} \tag{1.1}$$

where $\psi(u)$ is a monotone increasing function with $\psi(0) = 0$, $p > 1$, $g(u) > 0$, $g'(u) < 0$ for $u > 0$, and $\lim_{u\to 0^+} g(u) = +\infty$. The initial value $u_0(x)$ is positive and satisfying some compatibility conditions.

If $\psi(u) = u^{1/m}$ with $m > 0$, (1.1) becomes the well-known non-Newtonian filtration equation, which is used to describe the non-stationary flow in a porous medium of fluids with a power dependence of the tangential stress on the velocity of the displacement under polytropic conditions (see [1, 2]).

Many papers have been devoted to the study of critical exponents of non-Newtonian filtration equation, see [3–5]. There are many results on the quenching phenomenon, see, for instance [6–12]. By the quenching phenomenon we mean that the solution approaches a constant but its derivative with respect to time variable t tends to infinity as (x,t) tends to some point in the spatial-time space. The study of the quenching phenomenon began with the work of Kawarada through the famous initial boundary problem for the reaction-diffusion equation: $u_t = u_{xx} + 1/(1-u)$ (see [13]).

As an example of the type of results, we wish to obtain, let us recall results for a closely related problem

$$\begin{gathered} u_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{gathered} \tag{1.2}$$

where $q > 0$. In [14], it was shown that u quenches in finite time for all u_0, and the only quenching point is $x = 1$. Furthermore, the behavior of u near quenching was described there. It is easily seen that (1.2) is a special case of (1.1).

If $p = 2$, then (1.1) reduces to the following equation:

$$\begin{gathered} (\psi(u))_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -g(u(1,t)), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1. \end{gathered} \tag{1.3}$$

Deng and Xu proved in [15] the finite time quenching for the solution and established results on quenching set and rate for (1.3). If $\psi(u) = u$, then (1.1) reduces to the following equation, see [11],

$$\begin{gathered} u_t = (|u_x|^{p-2}u_x)_x, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -g(u(1,t)), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{gathered} \tag{1.4}$$

and they obtained that the bounds for the quenching rate, and the quenching occurs only at $x = 1$.

In this paper, we extend the equation $u_t = (|u_x|^{p-2}u_x)_x$, see [11], to a more general form $(\psi(u))_t = (|u_x|^{p-2}u_x)_x$. We prove that quenching occurs only at $x = 1$. We determine the bounds for the quenching rate, and present an example which shows the applicability of our results.

The main results are stated as follows.

Theorem 1.1. *Suppose that the initial data satisfies $u_0'(x) \le 0$ and $u_0''(x) \le 0$ for $0 \le x \le 1$, and one of the following conditions holds:*

(i) $\psi''(u) > 0$ *for* $u > 0$,

(ii) $\psi''(u) < 0$ *for* $u > 0$, $\limsup_{u\to 0^+}(g'(u)/\psi'(u)) < 0$.

Then every solution of (1.1) *quenches in finite time, and the only quenching point is* $x = 1$.

Next, we deal with the quenching rate. Before we establish upper bounds for the quenching rate, we introduce the following hypothesis:

(H_1) $\psi''(u)g(u) \geq 2(p-1)\psi'(u)g'(u)$, $\psi'(u)[(p-2)g'^2(u) + g(u)g''(u)] \geq \psi''(u)g(u)g'(u)$.

Theorem 1.2. *Suppose that the conditions of Theorem 1.1 and the hypothesis* (H_1) *hold. Then there exists a positive constant* C_1 *such that*

$$\int_0^{u(1,t)} \frac{\psi'(s)ds}{-g^{p-1}(s)g'(s)} \leq C_1(T-t). \tag{1.5}$$

Next, we will give the lower bound on the quenching rate, the derivation of which is in the spirit of [15]. We need the following additional hypotheses: there exists a constant $\sigma(-\infty < \sigma \leq \sigma_0 = \min\{1, 2-1/(p-1)\})$ such that

(H_2) $(g^{(p-1)(\sigma-1)}(u)g'(u)/\psi'(u))'' < 0$,

(H_3) $(5p - 2\sigma p - 2\sigma - 6)(g^{(p-1)(\sigma-1)}(u)g'(u))'\psi'(u) \geq (p-1)(3-\sigma)g^{(p-1)(\sigma-1)}(u)g'(u)\psi''(u)$,

(H_4) $(g^{(p-1)(\sigma-1)-1}(u)g'(u)/\psi'(u))' < 0$.

Theorem 1.3. *Suppose that the hypotheses of Theorem 1.1 hold. Furthermore, suppose that the hypotheses* (H_2)*–*(H_4) *hold. Then there exists a positive constant* C_2 *such that*

$$\int_0^{u(1,t)} \frac{\psi'(s)ds}{-g^{p-1}(s)g'(s)} \geq C_2(T-t). \tag{1.6}$$

Furthermore, if (H_1) *holds, then the quenching rates are*

$$C_1(T-t) \geq \int_0^{u(1,t)} \frac{\psi'(s)ds}{-g^{p-1}(s)g'(s)} \geq C_2(T-t). \tag{1.7}$$

Next, as an application of the main results of this paper, we study the following concrete example:

$$\begin{gathered}
(u^m)_t = (|u_x|^{p-2}u_x)_x, \quad 0 < x < 1,\ t > 0, \\
u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \\
u(x,0) = u_0(x), \quad 0 \leq x \leq 1,
\end{gathered} \tag{1.8}$$

where $q > 0$, $p > 1$, and $m > 0$. We will verify that (1.8) satisfies the hypotheses (H_1)–(H_4), and we give the following theorem.

Theorem 1.4. *Suppose that* $u_0'(x) \leq 0$ *and* $u_0''(x) \leq 0$ *for* $0 \leq x \leq 1$*. Then the solution of* (1.8) *satisfies*

$$C_4 \leq u(1,t)(T-t)^{-1/(m+pq+1)} \leq C_3, \tag{1.9}$$

where C_3 *and* C_4 *are positive constants.*

The plan of this paper is as follows. In Section 2, we prove that quenching occurs only at $x = 1$, that is the proof of Theorem 1.1. In Section 3, we derive the estimates for the quenching rate, that is the proof of Theorems 1.2 and 1.3. In Section 4, we present results for certain $\psi(u)$ and $g(u)$, that is the proof of the Theorem 1.4.

2. Quenching on the Boundary

In this section, we prove finite time quenching. We rewrite problem (1.1) into the following form:

$$\begin{gathered} u_t = a(u)(|u_x|^{p-2}u_x)_x, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -g(u(1,t)), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{gathered} \tag{2.1}$$

where $a(u) = 1/(\psi'(u))$. Clearly, $\psi'(u) \neq 0$ for $u > 0$.

Lemma 2.1. *Assume the solution u of problem* (2.1) *exists in* $(0, T_0)$ *for some* $T_0 > 0$*, and* $u_0'(x) \le 0$, $u_0''(x) \le 0$ *for* $0 \le x \le 1$*. Then* $u_x(x,t) < 0$ *and* $u_t(x,t) < 0$ *in* $(0,1] \times (0,T_0)$.

Proof. Let $v(x,t) = u_x(x,t)$. Then $v(x,t)$ satisfies

$$\begin{gathered} v_t = a(u)(|v|^{p-2}v)_{xx} + a'(u)v(|v|^{p-2}v)_x, \quad 0 < x < 1,\ 0 < t < T_0, \\ v(0,t) = 0, \quad v(1,t) = -g(u(1,t)), \quad 0 < t < T_0, \\ v(x,0) = u'_0(x), \quad 0 \le x \le 1. \end{gathered} \tag{2.2}$$

The maximum principle leads to $v(x,t) < 0$, and thus $u_x(x,t) < 0$ in $(0,1] \times (0,T_0)$. Then it is easy to see that the problem (2.2) is nondegenerate in $(0,1] \times (0,T_0)$. So $u_x(x,t)$ is a classical solution of (2.2). Similarly, letting $w(x,t) = u_t(x,t)$, we have

$$\begin{gathered} w_t = a'(u)(|u_x|^{p-2}u_x)_x w + (p-1)a(u)(|u_x|^{p-2}w_x)_x, \quad 0 < x < 1,\ 0 < t < T_0, \\ w_x(0,t) = 0, \quad w_x(1,t) = -g'(u(1,t))w(1,t), \quad 0 < t < T_0, \\ w(x,0) = (p-1)a(u_0(x))\left|u'_0(x)\right|^{p-2}u''_0(x), \quad 0 \le x \le 1. \end{gathered} \tag{2.3}$$

Making use of the maximum principle, we obtain $u_t(x,t) < 0$ in $(0,1] \times (0,T_0)$. Hence, the solutions of problem (2.1) $u \in C^{2,1}((0,1] \times (0,T_0))$ with $u_x(x,t) < 0$ and $u_t(x,t) < 0$ in $(0,1] \times (0,T_0)$. □

The Proof of Theorem 1.1

By the maximum principle, we know that $0 < u(\cdot,t) \le M$ for all t in the existence interval, where $M = \max_{0 \le x \le 1} u_0(x)$. Define $F(t) = \int_0^1 \psi(u(x,t))dx$. Then $F(t)$ satisfies

$$F'(t) = \int_0^1 (\psi(u(x,t)))_t dx = \int_0^1 \left(|u_x|^{p-2}u_x\right)_x dx = -g(u(1,t))\left|g(u(1,t))\right|^{p-2} \le -g^{p-1}(M). \tag{2.4}$$

Thus $F(t) \le F(0) - g^{p-1}(M)t$, which means that $F(t_0) = 0$ for some $t_0 > 0$. From the fact that $\varphi(u) > 0$ for $u > 0$ and $u_x(x,t) < 0$ for $0 < x \le 1$, we find that there exists a $T(0 < t \le T_0)$ such that $\lim_{t\to T^-} u(1,t) = 0$. By virtue of the singular nonlinearity in the boundary condition, u must quench at $x = 1$. In what follows, we only need to prove that quenching cannot occur in $((1/2), 1) \times (\eta, T)$ for some $\eta(0 < \eta < T)$. Consider two cases.

Case 1. $\varphi''(u) > 0$ for $u > 0$. Let $h(x,t) = |u_x|^{p-2}u_x + \varepsilon(x - (1/4))g^{p-1}(M)$ in $((1/4), 1) \times (\eta, T)$, where ε is a positive constant. Then $h(x,t)$ satisfies

$$\begin{aligned} h_t &= \left(|u_x|^{p-2}u_x\right)_t = (p-1)|u_x|^{p-2}u_{xt} = (p-1)|u_x|^{p-2}\left[a(u)\left(|u_x|^{p-2}u_x\right)_x\right]_x \\ &= (p-1)|u_x|^{p-2}\left(a'(u)u_x\frac{u_t}{a(u)} + a(u)h_{xx}\right) \le (p-1)a(u)|u_x|^{p-2}h_{xx}, \end{aligned} \tag{2.5}$$

for $(x,t) \in ((1/4), 1) \times (\eta, T)$, since $(a'(u))/(a(u)) = -(\varphi''(u))/(\varphi'(u)) \le 0$. On the parabolic boundary, $h((1/4), t) = |u_x|^{p-2}u_x((1/4), t) < 0$ for $\eta \le t < T$; if ε is sufficiently small, $h(1,t) \le g^{p-1}(M)((3\varepsilon/4) - 1) < 0$ for $\eta \le t < T$, and $h(x,\eta) \le -|u_x((1/4),\eta)|^{p-1} + (3\varepsilon/4)g^{p-1}(M) < 0$ for $(1/4) \le x \le 1$. Thus by the maximum principle, we have $h(x,t) \le 0$ in $((1/4), 1) \times (\eta, T)$, which leads to

$$|u_x|^{p-2}u_x + \varepsilon\left(x - \frac{1}{4}\right)g^{p-1}(M) \le 0 \quad \text{in } \left(\frac{1}{4}, 1\right) \times (\eta, T). \tag{2.6}$$

So we have

$$\left[\varepsilon\left(x - \frac{1}{4}\right)g^{p-1}(M)\right]^{1/(p-1)} \le -u_x. \tag{2.7}$$

Integrating (2.7) from x to 1, we obtain

$$u(x,t) \ge u(1,t) + \int_x^1 \left[\varepsilon\left(x - \frac{1}{4}\right)g^{p-1}(M)\right]^{1/(p-1)} dx > \int_x^1 \left[\varepsilon\left(x - \frac{1}{4}\right)g^{p-1}(M)\right]^{1/(p-1)} dx > 0. \tag{2.8}$$

It then follows that $u(x,t) > 0$ if $x < 1$.

Case 2. $\varphi''(u) < 0$ for $u > 0$. Let $k(x,t) = u_t - \varepsilon(x - (1/2))u_x$ in $((1/2), 1) \times (\eta, T)$. Then $k(x,t)$ satisfies

$$\begin{aligned} k_t &= u_{tt} - \varepsilon\left(x - \frac{1}{2}\right)u_{tx} \\ &= (p-1)a(u)|u_x|^{p-2}k_{xx} + (p-1)(p-2)a(u)|u_x|^{p-2}(-u_{xx})k_x + a'(u)\left(|u_x|^{p-2}u_x\right)_x k \\ &\quad + \varepsilon p(p-1)a(u)|u_x|^{p-2}u_{xx} \\ &\le (p-1)a(u)|u_x|^{p-2}k_{xx} + (p-1)(p-2)a(u)|u_x|^{p-2}(-u_{xx})k_x + a'(u)\left(|u_x|^{p-2}u_x\right)_x k, \end{aligned} \tag{2.9}$$

for $(x,t) \in ((1/2),1) \times (\eta,T)$. On the boundary, $k((1/2),t) = u_t((1/2),t) < 0$ for $\eta \le t < T$. Since $\lim_{t\to T^-} u(1,t) = 0$ and $\limsup_{u\to 0^+}((g'(u))/(\varphi'(u))) < 0$, if η is close to T and ε is small enough, $g'(u(1,t)) + (\varepsilon\varphi(u(1,t))'/2(p-1)g^{(p-2)}u(1,t)) + 2 \le 0$ for $\eta \le t < T$. Thus

$$\begin{aligned} k_x(1,t) &= u_{tx}(1,t) - \frac{\varepsilon}{2}u_{xx}(1,t) - \varepsilon u_x(1,t) \\ &= -\left[g'(u(1,t)) + \frac{\varepsilon\varphi'(u(1,t))}{2(p-1)g^{p-2}(u(1,t))} + 2\right]u_t(1,t) + 2k(1,t) \\ &\le 2k(1,t). \end{aligned} \tag{2.10}$$

It is easily seen that $k(x,\eta) \le 0$ for $(1/2) \le x \le 1$. Hence, the maximum principle yields that $k(x,t) \le 0$ in $[(1/2),1] \times [\eta,T)$. In particular,

$$u_t(1,t) + \frac{\varepsilon}{2}g(u(1,t)) \le 0 \quad \text{for } \eta \le t < T. \tag{2.11}$$

Integrating (2.11) from t to T, we obtain

$$\int_0^{u(1,t)} \frac{ds}{g(s)} \ge \frac{\varepsilon}{2}(T-t) \quad \text{for } \eta \le t < T. \tag{2.12}$$

Define $G(u) = \int_0^u (1/g(s))ds$. Since $G'(u) = 1/g(u) > 0$ for $u > 0$, the inverse G^{-1} exists. In view of (2.12), we can see

$$u(1,t) = G^{-1}(G(u(1,t))) \ge G^{-1}\left(\frac{\varepsilon}{2}(T-t)\right) \quad \text{for } \eta \le t < T. \tag{2.13}$$

Let $H(x,t) = u(x,t) - c_1(1-x^2) - c_2G^{-1}((\varepsilon/2)(T-t))$, where c_1 and c_2 are positive constants. Since $g'(u) < 0$, $u_x(x,t) < 0$, and by (2.13), we have that in $(0,1) \times (\eta,T)$

$$\begin{aligned} H_t &= (p-1)a(u)|u_x|^{p-2}u_{xx} + \frac{\varepsilon c_2}{2}g\left(G^{-1}\left(\frac{\varepsilon}{2}(T-t)\right)\right) \\ &\ge (p-1)a(u)|u_x|^{p-2}H_{xx} - 2c_1(p-1)a(u)|u_x|^{p-2} + \frac{\varepsilon c_2}{2}g(u(1,t)) \\ &\ge (p-1)a(u)|u_x|^{p-2}H_{xx} - \frac{2c_1(p-1)|u_x|^{p-2}}{\varphi'(M)} + \frac{\varepsilon c_2}{2}g(M) \\ &\ge (p-1)a(u)|u_x|^{p-2}H_{xx}, \end{aligned} \tag{2.14}$$

provided $\varepsilon c_2 g(M)\varphi'(M) \ge 4c_1(p-1)|u_x|^{p-2}$, which is true since $0 \le |u_x(x,t)| \le g(u(1,t)) = g(M_0)$, where $0 < M_0 \le M$. On the other hand, $H_x(0,t) = 0$; $H(1,t) = u(1,t) - c_2G^{-1}((\varepsilon/2)(T-t)) \ge 0$ if $c_2 \le 1$ and $H(x,\eta) \ge 0$ if c_1 and c_2 are small enough. Thus by the maximum principle, we find that $H(x,t) \ge 0$ in $(0,1)\times(\eta,T)$, which implies that $u(x,t) \ge c_1(1-x^2) > 0$ if $x < 1$.

3. Bounds for Quenching Rate

In this section, we establish bounds on the quenching rate. We first present the upper bound.

The Proof of Theorem 1.2

We define a function $\Phi(x,t) = |u_x|^{p-2}u_x + \varphi^{p-1}(x)g^{p-1}(u(x,t))$ in $(0,1) \times (\eta, T)$, where $\varphi(x)$ is given as follows:

$$\varphi(x) = \begin{cases} 0, & x \in [0, x_0], \\ \dfrac{(x-x_0)^l}{(1-x_0)^l}, & x \in (x_0, 1], \end{cases} \tag{3.1}$$

with some $x_0 < 1$ and $l \geq \max\{3, (1/(p-1))\}$ is chosen so large that $\varphi(x) \leq -(u_0'(x)/g(u_0(x))$ for $x_0 < x \leq 1$. It is easy to see that $\Phi(0,t) = \Phi(1,t) = 0$, and $\Phi(x,0) \leq 0$. On the other hand, in $(0,1) \times (\eta, T)$, Φ satisfies

$$\Phi_t = (p-1)a(u)|u_x|^{p-2}\Phi_{xx} + (p-1)a'(u)u_x|u_x|^{p-2}\Phi_x - R_1(x,t), \tag{3.2}$$

where

$$\begin{aligned} R_1(x,t) = {} & (p-1)^2 a(u)|u_x|^{p-2}\varphi^{p-3}(x)g^{p-1}(u)\left[(p-2)\varphi'^2(x) + \varphi(x)\varphi''(x)\right] \\ & + (p-1)^2 a^2(u)|u_x|^{p-2}u_x\varphi^{p-2}(x)\varphi'(x)g^{p-2}(u)\left[-g(u)\psi''(u) + 2(p-1)g'(u)\psi'(u)\right] \\ & + (p-1)^2 a(u)|u_x|^p\varphi^{p-1}(x)g^{p-3}(u)\left[-\frac{\psi''(u)}{\psi'(u)}g(u)g'(u) + (p-2)g'^2(u) + g(u)g''(u)\right]. \end{aligned} \tag{3.3}$$

By (H_1) and the definition of $\varphi(x)$, it follows that

$$\Phi_t \leq (p-1)a(u)|u_x|^{p-2}\Phi_{xx} + (p-1)a'(u)u_x|u_x|^{p-2}\Phi_x. \tag{3.4}$$

Thus, the maximum principle yields $\Phi(x,t) \leq 0$, that is

$$\varphi(x)g(u(x,t)) \leq -u_x(x,t), \quad \text{for } [0,1] \times [\eta, T). \tag{3.5}$$

Moreover, by the definition of the limit, we see that $\Phi_x(1,t) \geq 0$ since $\Phi(x,t) \leq 0$. In fact,

$$\Phi_x(1,t) = \lim_{x \to 1^-} \frac{\Phi(x,t) - \Phi(1,t)}{x-1} \geq 0, \tag{3.6}$$

which means

$$u_t(1,t) \geq (p-1)a(u(1,t))g^{p-1}(u(1,t))\left(g'(u(1,t)) - \varphi'(1)\right)$$
$$\geq c_3(p-1)a(u(1,t))g^{p-1}(u(1,t))g'(u(1,t)). \tag{3.7}$$

Integrating (3.7) from t to T, we get

$$\int_0^{u(1,t)} \frac{\varphi'(s)ds}{-g^{p-1}(s)g'(s)} \leq C_1(T-t). \tag{3.8}$$

We then give the lower bound.

The Proof of Theorem 1.3

Let $d(u) = a(u)g^{(p-1)(\sigma-1)}(u)g'(u)$. Notice that the hypotheses (H_2)–(H_4) are equivalent to

$(\widetilde{H}_2)$ $d''(u) \leq 0$,

$(\widetilde{H}_3)$ $d'(u)(5p-2\sigma p+2\sigma-6) \geq (d(u)/a(u))a'(u)(2p-\sigma p+\sigma-3)$,

$(\widetilde{H}_4)$ $d(u)g'(u) > d'(u)g(u)$,

respectively. Letting τ be close to T, we consider $\Psi(x,t) = u_t - \varepsilon d(u)(-u_x)^{(p-1)(2-\sigma)}$ in $(1-T+\tau,1)\times(\tau,T)$, where ε is a positive constant. Through a fairly complicated calculation, we find that

$$\Psi_t = (p-1)a(u)|u_x|^{p-2}\Psi_{xx} + (p-1)(p-2)a(u)|-u_x|^{p-3}(-u_{xx})\Psi_x + C(x,t)\Psi + R_2(x,t), \tag{3.9}$$

where

$$\begin{aligned} C(x,t) = {} & \left[\frac{a'(u)}{a(u)} + \varepsilon(2-\sigma)\frac{d(u)}{a(u)}(2p-\sigma p+\sigma-3)(-u_x)^{p-\sigma p+\sigma-2}\right]u_t \\ & + \varepsilon^2(2-\sigma)\frac{d^2(u)}{a(u)}(2p-\sigma p+\sigma-3)(-u_x)^{3p-2\sigma p+2\sigma-4} \\ & + \varepsilon\left[d'(u)(5p-2\sigma p+2\sigma-6) - \frac{d(u)}{a(u)}a'(u)(2p-\sigma p+\sigma-3)\right](-u_x)^{2(p-1)(2-\sigma)}, \\ R_2(x,t) = {} & \varepsilon^3(2-\sigma)\frac{d^3(u)}{a(u)}(2p-\sigma p+\sigma-3)(-u_x)^{5p-3\sigma p+3\sigma-6} \\ & + \varepsilon^2 d(u)\left[d'(u)(5p-2\sigma p+2\sigma-6) - \frac{d(u)}{a(u)}a'(u)(2p-\sigma p+\sigma-3)\right](-u_x)^{2(p-1)(2-\sigma)} \\ & + \varepsilon a(u)(p-1)d''(u)(-u_x)^{(p-1)(3-\sigma)+1}. \end{aligned} \tag{3.10}$$

Since the hypotheses $(\widetilde{H}_2)$–$(\widetilde{H}_3)$ hold, and $d(u) < 0$, $a(u) > 0$, we see that $R_2(x,t) < 0$. Thus, we have

$$\Psi_t \leq (p-1)a(u)|u_x|^{p-2}\Psi_{xx} + (p-1)(p-2)a(u)(u_x)^{p-3}(-u_{xx})\Psi_x + C(x,t)\Psi, \tag{3.11}$$

for $(x,t) \in (1-T+\tau, 1) \times (\tau, T)$. On the parabolic boundary, since $x = 1$ is the only quenching point, if ε is small enough, then both $\Psi(1-T+\tau,t)$ and $\Psi(x,\tau)$ are negative. At $x = 1$, in view of $(\widetilde{H}_4)$, we have

$$\begin{aligned}\Psi_x(1,t) &= -[1-\varepsilon(2-\sigma)]g'(u(1,t))\Psi(1,t) \\ &\quad - \varepsilon\{[1-\varepsilon(2-\sigma)]g'(u(1,t))d(u(1,t)) - d'(u(1,t))g(u(1,t))\}g^{(p-1)(2-\sigma)}(u(1,t)) \\ &\leq -[1-\varepsilon(2-\sigma)]g'(u(1,t))\Psi(1,t),\end{aligned} \tag{3.12}$$

provided ε is sufficiently small. Hence, by the maximum principle, we have $\Psi(x,t) \leq 0$ on $[1-T+\tau, 1] \times [\tau, T)$. In particular, $\Psi(1,t) \leq 0$, that is,

$$u_t(1,t) \leq \varepsilon d(u(1,t))g^{(p-1)(2-\sigma)}(u(1,t)) = \varepsilon a(u(1,t))g^{p-1}(u(1,t))g'(u(1,t)). \tag{3.13}$$

Integration of (3.13) over (t,T), then leads to

$$\int_0^{u(1,t)} \frac{\varphi'(s)ds}{-g^{p-1}(s)g'(s)} \geq C_2(T-t). \tag{3.14}$$

4. Results for Certain Nonlinearities

In this section, we give the concrete quenching rate of solutions for (1.8).

The Proof of Theorem 1.4

We first present the upper bound. Consider two cases.

Case 1. $m + q(p-1) \geq 2$. We only need to verify the hypothesis (H_1). Since

$$\begin{aligned}\varphi''(u)g(u) - 2(p-1)\varphi'(u)g'(u) &= m[m+2q(p-1)-1]u^{m-q-2} \geq 0, \\ \varphi'(u)\left[(p-2)g'^2(u) + g(u)g''(u)\right] - \varphi''(u)g(u)g'(u) &= mq[pq-q+m-2]u^{m-2q-3} \geq 0,\end{aligned} \tag{4.1}$$

then we have

$$\begin{aligned}\int_0^{u(1,t)} \frac{\varphi'(s)ds}{-g^{p-1}(s)g'(s)} &= \int_0^{u(1,t)} \frac{ms^{m-1}}{-s^{-q(p-1)}(-q)s^{-q-1}}ds = \frac{m}{q}\int_0^{u(1,t)} s^{qp+m}ds \\ &= \frac{m}{q(m+qp+1)}u^{m+qp+1}(1,t) \leq C_1(T-t).\end{aligned} \tag{4.2}$$

Therefore, we get the upper bound as

$$u(1,t)(T-t)^{-(1/(m+qp+1))} \leq \left[C_1 \frac{m}{q(m+qp+1)}\right]^{1/(m+qp+1)} := C_3. \tag{4.3}$$

Case 2. $m+q(p-1)<2$. We use a modification of an argument from [16]. For $t \in [\eta, T)$ with some η such that $u(1,\eta)<1$, set

$$y(t) = u^{\gamma}(1,t)\int_{1-\xi(t)}^{1} u^m(x,t)dx, \tag{4.4}$$

with

$$\xi(t) = u^{q+1}(1,t), \tag{4.5}$$

where $-(m+q+1) < \gamma \leq -(m+1)(q+1)$.

A routine calculation shows

$$\begin{aligned} y'(t) &= \gamma u^{\gamma-1}(1,t)u_t(1,t)\int_{1-\xi(t)}^{1} u^m(x,t)dx + (q+1)u^{q+\gamma}(1,t)u^m(1-\xi(t),t)u_t(1,t) \\ &\quad + u^{\gamma}(1,t)\left[|u_x|^{p-2}u_x(1,t) - |u_x|^{p-2}u_x(1-\xi(t),t)\right] \\ &\geq u^{\gamma-1}(1,t)u_t(1,t)I(t) - u^{\gamma-q(p-1)}(1,t), \end{aligned} \tag{4.6}$$

where $I(t) = \gamma\int_{1-\xi(t)}^{1} u^m(x,t)dx + (q+1)u^{q+1}(1,t)u^m(1-\xi(t),t)$. Since $u_x \leq 0$ and $u_{xx} \leq 0$ in $[0,1]\times[\eta,T)$, we find

$$u(1,t) \leq u(x,t) \leq u(1-\xi(t),t) \leq 2u(1,t), \tag{4.7}$$

for any $x \in [1-\xi(t),1]$ and $t \in [\eta,T)$. By (4.4), (4.5) and (4.7), we have

$$u^{\gamma+m+q+1}(1,t) \leq y(t) \leq 2^m u^{\gamma+m+q+1}(1,t), \tag{4.8}$$

or equivalently,

$$u(1,t) \leq y^{1/(\gamma+m+q+1)}(t) \leq c_4 u(1,t) \quad \text{for } t \in [\eta, T). \tag{4.9}$$

We now claim that $I(t) \le 0$ on $[\eta, T)$. In fact

$$
\begin{aligned}
I(t) &= (\gamma+q+1)\int_{1-\xi(t)}^{1} u^m(x,t)dx - (q+1)\int_{1-\xi(t)}^{1} (u^m(x,t) - u^m(1-\xi(t),t))dx \\
&= (\gamma+q+1)\int_{1-\xi(t)}^{1} u^m(x,t)dx \\
&\quad - (q+1)\int_{1-\xi(t)}^{1} m u^{m-1}(\zeta(t),t)u_x(\zeta(t),t)(x-1+\xi(t))dx,
\end{aligned}
\tag{4.10}
$$

where $1-\xi(t) < \zeta(t) < 1$. When $0 < m < 1$, $\gamma \le -(((m+2)(q+1))/2)$, because $u_x, u_{xx} \le 0$ and $\int_{1-\xi(t)}^{1}(1-\xi(t),t))\ dx = (1/2)u^{2q+2}(1,t)$, we have

$$
\begin{aligned}
I(t) &\le (\gamma+q+1)\int_{1-\xi(t)}^{1} u^m(x,t)dx + m(q+1)\int_{1-\xi(t)}^{1} u^{m-1}(1,t)(-u_x(1,t))(x-1+\xi(t))dx \\
&= \left[\gamma+q+1+\frac{m(q+1)}{2}\right]u^{m+q+1}(1,t) \le 0;
\end{aligned}
\tag{4.11}
$$

when $1 \le m < 2$, $\gamma \le -(q+1)(m+1)$, we have

$$
\begin{aligned}
I(t) &\le (\gamma+q+1)\int_{1-\xi(t)}^{1} u^m(x,t)dx + m(q+1)\int_{1-\xi(t)}^{1} (2u(1,t))^{m-1}(-u_x(1,t))(x-1+\xi(t))dx \\
&= \left[\gamma+q+1+2^{m-2}m(q+1)\right]u^{m+q+1}(1,t) \\
&\le [\gamma+q+1+m(q+1)]u^{m+q+1}(1,t) \le 0.
\end{aligned}
\tag{4.12}
$$

In conclusion, when $0 < m < 2$, $\gamma \le -(q+1)(m+1)$, we have

$$
I(t) \le 0. \tag{4.13}
$$

From (4.6), (4.9), and (4.13), it then follows that

$$
y'(t) \ge -u^{\gamma-q(p-1)}(1,t) \ge -c_4^{-\gamma+q(p-1)} y^{(\gamma-q(p-1))/(\gamma+m+q+1)}(1,t). \tag{4.14}
$$

Integrating the above inequality from t to T, we obtain

$$
y^{(m+qp+1)/(\gamma+m+q+1)}(t) \le \frac{m+qp+1}{\gamma+m+q+1} c_4^{-\gamma+q(p-1)}(T-t), \tag{4.15}
$$

that is,

$$y^{1/(\gamma+m+q+1)}(1,t) \le C_3(T-t)^{1/(\gamma+m+q+1)}, \tag{4.16}$$

which in conjunction with (4.9) yields the desired upper bound.

We then give the lower bound. We examine the validity of hypotheses (H_2)–(H_4).

Firstly, for (H_2), we find

$$\begin{aligned}\left(\frac{g^{(p-1)(\sigma-1)}(u)g'(u)}{\psi'(u)}\right)'' &= -\frac{q}{m}\left[q(p\sigma-\sigma-p+2)+m\right] \\ &\quad\cdot\left[q(p\sigma-\sigma-p+2)+m+1\right]u^{-q(p\sigma-\sigma-p+2)-m-2} < 0,\end{aligned} \tag{4.17}$$

provided $\sigma < \sigma_1 = -((m+1)/(q(p-1))) + ((p-2)/(p-1))$ or $\sigma > \sigma_2 = -(m/(q(p-1))) + ((p-2)/(p-1))$.

Secondly, for (H_3), we have

$$\begin{aligned}&(5p-2\sigma p+2\sigma-6)\left(g^{(p-1)(\sigma-1)}(u)g'(u)\right)'\psi'(u)-(p-1)(3-\sigma)g^{(p-1)(\sigma-1)}(u)g'(u)\psi''(u) \\ &\quad = qm\Big[-2q(p-1)^2\sigma^2+(p-1)(7pq-10q-m-1)\sigma \\ &\qquad +\left(-5p^2q+16pq-12q+3pm+2p-3m-3\right)\Big]u^{-q(p-1)(\sigma-1)-q+m-3} \ge 0,\end{aligned} \tag{4.18}$$

provided $\sigma_3 \le \sigma \le \sigma_4$ with

$$\begin{aligned}\sigma_3 &= \frac{(p-1)(7pq-10q-m-1)-\sqrt{b^2-4ac}}{4q(p-1)^2}, \\ \sigma_4 &= \frac{(p-1)(7pq-10q-m-1)+\sqrt{b^2-4ac}}{4q(p-1)^2},\end{aligned} \tag{4.19}$$

where $a = -2q(p-1)^2$, $b = (p-1)(7pq-10q-m-1)$, and $c = -5p^2q+16pq-12q+3pm+2p-3m-3$.

Thirdly, for (H_4), we obtain

$$\left(\frac{g^{(p-1)(\sigma-1)-1}(u)g'(u)}{\psi'(u)}\right)' = -\frac{q}{m}\left[-q(p-1)(\sigma-1)-m\right]u^{-q(p-1)(\sigma-1)-m-1} < 0, \tag{4.20}$$

provided $\sigma < \sigma_5 = 1-(q/(m(p-1)))$. Since $\sigma_2 > \sigma_5$, we choose a σ such that $\sigma_3 < \sigma < \min\{\sigma_1,\sigma_4,\sigma_5\}$ and hypotheses (H_2)–(H_4) hold. Thus the proof is completed.

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