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Research Article

The Liapunov Center Theorem for a Class of Equivariant Hamiltonian Systems

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We consider the existence of the periodic solutions in the neighbourhood of equilibria for C^∞ equivariant Hamiltonian vector fields. If the equivariant symmetry S acts antisymplectically and $S^2 = I$, we prove that generically purely imaginary eigenvalues are doubly degenerate and the equilibrium is contained in a local two-dimensional flow-invariant manifold, consisting of a one-parameter family of symmetric periodic solutions and two two-dimensional flow-invariant manifolds each containing a one-parameter family of nonsymmetric periodic solutions. The result is a version of Liapunov Center theorem for a class of equivariant Hamiltonian systems.

1. Introduction

We first give some definitions for our problem. A $2n \times 2n$ matrix T is called (anti)symplectic if $T^T JT = \pm J$. Consider a C^∞ vector field $f : O \subset \mathcal{R}^N \rightarrow \mathcal{R}^N$ and the system

$$\frac{d}{dt}x = f(x). \tag{1.1}$$

Let S be a diffeomorphism of $\mathcal{R}^N$ into itself. If $Sf = fS$, we call that the system (1.1) is S-equivariant or the vector field f is S-equivariant. Denote by I the identity matrix. In this paper, S satisfies $S^2 = I$. When the system (1.1) is S-equivariant, if $x(t)$ is a solution, then $Sx(t)$ is also a solution. An orbit $x(t)$ is called symmetric if it is S-invariant; that is, $Sx(t) = x(t)$.

Problems. Consider a system $\dot{x} = Ax+g(x)$, $x \in R^n$ with A being an $n\times n$ matrix and $g = O(x^2)$ a C^∞ vector function. Suppose that the system has a nondegenerate integral. Suppose that A has a pair of purely imaginary eigenvalues $\pm i$ and no other eigenvalues of the form $\pm ki$, $k \in Z$. That is, the eigenvalues $\pm i$ are nonresonant with the other ones. Then, the well-known Liapunov Center theorem tells us that there exists a one-parameter family of periodic orbits emanating from the equilibrium point with the period being close to 2π as they approach to the equilibrium. We call such families the Liapunov Center families.

This result can be used easily to Hamiltonian systems and obtain existence of periodic solutions. Later, many mathematicians were dedicated to study periodic solutions of Hamiltonian systems and tried to generalize the result. Gordon [1] obtained an infinite number of periodic solutions in arbitrarily small neighborhoods of the origin for Hamiltonian systems with convex potential. Weinstein [2] obtained the Liapunov Center families with no eigenvalue assumptions when the equilibrium point is a nondegenerate minimum. In [3], Moser proved that the integral manifold contains at least one periodic solution whose period is close to that of a periodic solution of the linearized system near the equilibrium point. In [4], Weinstein proved that a Hamiltonian system possesses at least one periodic solution on each energy surface, provided that this energy surface is compact, convex, and contains no stationary point of the vector field.

Bifurcation theory describes how the dynamics of systems change as parameters varied. The study of the Hamiltonian Hopf bifurcation has a long history. The Hamiltonian-Hopf bifurcation involves the loss of linear stability of a fixed point by the collision of two pairs of imaginary eigenvalues of the linearized flow and their subsequent departure off the imaginary axis. van der Meer [5] studied this bifurcation and classified the periodic solutions that are spawned by this resonance. Using Z_2 singularity theory with a distinguished parameter developed in [6], Bridges [7] obtained the periodic solutions in a Hamiltonian-Hopf bifurcation.

Moreover, reversible systems have been studied for many years. Devaney [8] first proved a Liapunov Center theorem for reversible vectors fields. Vanderbauwhede [9] and Sevryuk [10] also studied reversible vectors fields. In [11], Golubitsky et al. studied families of periodic solutions near generic elliptic equilibria for reversible equivariant systems. In [12, 13], Montaldi et al. considered families of periodic solutions near generic elliptic equilibria for reversible equivariant Hamiltonian systems that are both Hamiltonian and reversible at the same time. Since Lamb and Roberts [14] obtained the group theoretical classification of linear reversible equivariant systems, there has been an increasing interest for reversible equivariant systems. Later, Hoveijn et al. [15] obtained the linear normal form and unfolding theory of reversible equivariant linear systems.

In [16], Buzzi and Lamb obtained a Liapunov Center theorem for purely reversible Hamiltonian vector fields that are both Hamiltonian and reversible at the same time. They obtained the existence of periodic solutions in the neighbourhood of elliptic equilibria when the reversing symmetry R acts symplectically or antisymplectically. Previously, the symmetric property of periodic solutions is not considered, and the existence of additional periodic solutions is not ruled out. But these problems were considered in [16]. The results in [16] are as follows. If R acts antisymplectically, generically purely imaginary eigenvalues are isolated, and the equilibrium is contained in a local two-dimensional invariant manifold containing a one-parameter family of symmetric periodic solutions. If R acts symplectically, generically purely imaginary eigenvalues are doubly degenerate, and the equilibrium is contained in two two-dimensional invariant manifolds, each containing a one-parameter family of nonsymmetric periodic solutions, and a three-dimensional invariant manifold containing a two-parameter family of symmetric periodic solutions. In [17], Sternberg theorem for equivariant Hamiltonian vector fields was considered.

Motivated by [16], in this paper, we consider a Liapunov Center theorem for equivariant Hamiltonian vector fields that are both Hamiltonian and equivariant at the same time.

Here, we assume that the equivariant symmetry S acts antisymplectically and $S^2 = I$. Now, we also consider the symmetric property of periodic solutions. This property was not studied for Hamiltonian vector fields without the other structure previously.

2. Main Results

Theorem 2.1. *Consider an equilibrium 0 of a C^∞ equivariant Hamiltonian vector field f, with the equivariant symmetry S acting antisymplectically and $S^2 = I$. Assume that the Jacobian matrix $Df(0)$ has two pairs of purely imaginary eigenvalues $\pm i$ and no other eigenvalues of the form $\pm ki$, $k \in Z$. Then, the equilibrium is contained in a two-dimensional flow-invariant surface that consists of a one-parameter family of symmetric periodic solutions whose period tends to 2π as they approach the equilibrium. Moreover, the equilibrium is also contained in two smooth two-dimensional flow-invariant manifolds, each containing a one-parameter family of nonsymmetric periodic solutions whose period tends to 2π as they approach the equilibrium. Furthermore, there are no other periodic solutions with period close to 2π in the neighbourhood of 0.*

Remark 2.2. Here, the existence and the symmetric property of periodic solutions near the equilibrium point are all considered. The main idea is similar to [16].

3. Linear Equivariant Hamiltonian Vector Field with Purely Imaginary Eigenvalues

We now consider the persistent occurrence of purely imaginary eigenvalues in equivariant Hamiltonian vector fields.

Let A_0 be a linear Hamiltonian vector field. Then, it follows that $A_0 J = -J A_0^T$. If A_0 is S-equivariant, we have $A_0 S = S A_0$. If S is (anti)symplectic, we get $SJ = \pm JS$.

Since we are interested in (partially) elliptic equilibria, we assume that A_0 has a pair of purely imaginary eigenvalues λ and $-\lambda$. Moreover, if the eigenvector e_1 of A_0 has λ, then $\overline{e}_1$ is an eigenvector for $\overline{\lambda}$.

Since A_0 is both Hamiltonian and S-equivariant, this implies that if the eigenvector e of A_0 has the eigenvalue λ, then Se is also an eigenvector for the eigenvalue λ.

Hoveijn et al. [15] considered the linear normal form theory which is based on the construction of minimal $\langle J, S\rangle$-invariant subspaces. By [15], we are only interested in minimal invariant subspaces on which A_0 is semisimple; that is, A_0 is diagonalizable over $\mathcal{C}$. Here, the type of minimal invariant subspace depends on whether S acts symplectically or antisymplectically.

Lemma 3.1. *Consider a linear S-equivariant Hamiltonian vector field A_0 with Sacting (anti)symplectically and $S^2 = I$. Let V be a minimal (A_0, J, S)-invariant subspace on which A_0 has purely imaginary eigenvalues. Then $A_0|_V$, $J|_V$ and $S|_V$ have the following normal forms.*

(1) *If S acts symplectically, it follows that* $\dim V = 2$, *and*

$$S|_V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J|_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_0|_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

(2) *If S acts antisymplectically, it follows that* $\dim V = 4$ *and*

$$S|_V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J|_V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad A_0|_V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.2}$$

Proof. Let W be a 2-dimensional symplectic subspace on which A_0 has purely imaginary eigenvalues. By standard Hamiltonian theory and multiplication of time by a scalar, A_0 and J can take the same normal form on W. If the equivariant symmetry S acts symplectically, we have $SA_0 = A_0S$ and $SJ = JS$. Let e_1 and $\overline{e}_1$ be the eigenvectors of A_0. By $SA_0 = A_0S$, a minimal invariant subspace is obtained by choosing $Se_1 = e_1$. Since $J = A_0$ on W, we have $SJ = JS$ on W.

If S acts antisymplectically, the dimension of the minimal invariant subspace is not two. A 2-dimensional subspace W is defined as above. Assume that $S(W) = W$. If $S(W) = W$, by $(SA_0)|_W = (A_0S)|_W$, it follows that $(SJ)|_W = (JS)|_W$. This is converse that S acts antisymplectically. So, we have $S(W) = W' \neq W$ and a minimal invariant subspace is given by $V = W \oplus W'$. So, $\dim V = 4$. Moreover, we get $J|_{W'} = (S^{-1}JS)|_W = -(S^{-1}SJ)|_W = -J|_W$ and $A_0|_{W'} = (S^{-1}A_0S)|_W = (S^{-1}SA_0)|_W = A_0|_W$. Since $A_0|_W = J|_W$, it follows that $J|_{W'} = -A_0|_{W'}$. □

Remark 3.2. Now, we give the examples for the system (1.1) whether S acts symplectically or antisymplectically, where J and S here are defined as $J|_V$ and $S|_V$ in Lemma 3.1. If S acts symplectically, the system (1.1) can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_{x_1} \\ H_{y_1} \end{pmatrix} = \begin{pmatrix} -y_1 - 3y_1^2 \\ x_1 + 3x_1^2 \end{pmatrix} = f(x), \tag{3.3}$$

where the Hamiltonian function is $H(x_1, y_1) = (1/2)(x_1^2 + y_1^2) + x_1^3 + y_1^3$, f satisfies $fS = Sf$, and $A_0 = df(0)$ is calculated the same as $A_0|_V$ in Lemma 3.1. If S acts antisymplectically, the system (1.1) can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} H_{x_1} \\ H_{y_1} \\ H_{y_2} \\ H_{x_2} \end{pmatrix} = \begin{pmatrix} -y_1 - x_1y_2 \\ x_1 + y_1y_2 - x_2y_2 \\ -x_2 - x_1y_2 \\ y_2 - x_1y_1 + x_1x_2 \end{pmatrix} = f(x), \tag{3.4}$$

where the Hamiltonian function is $H(x_1, y_1, y_2, x_2) = (1/2)(x_1^2 + y_1^2) - (1/2)(x_2^2 + y_2^2) + x_1y_1y_2 - x_1x_2y_2$, f satisfies $fS = Sf$, and $A_0 = df(0)$ is calculated the same as $A_0|_V$ in Lemma 3.1.

Remark 3.3. When S acts antisymplectically, under the base (e_1, e_2, Se_1, Se_2), we obtain S, J and A_0 have the forms of $S|_V$, $J|_V$ and $A_0|_V$ in Lemma 3.1, respectively. However, when S acts antisymplectically, under the base (e_1, Se_2, e_2, Se_1), we have

$$S|_V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad J|_V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_0|_V = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

In this case, $J|_V$ is the standard form. However, for convenience, we use the forms of Lemma 3.1 in this paper.

4. Liapunov-Schmidt Reduction

In this paper, when S acts antisymplectically or symplectically, by Lemma 3.1, A_0 has a single pair or double pairs of purely imaginary eigenvalues $\pm i$. Moreover, these purely imaginary eigenvalues of A_0 are nonresonant; that is, A_0 has no other eigenvalues of the form $\pm ki$ with $k \in Z$. This condition is clearly generic (codimension zero). We want to find the families of periodic solutions in the neighbourhood of the equilibrium point.

In this section, we introduce the main technique which is a Liapunov-Schmidt reduction. The Liapunov-Schmidt reduction here is similar to the one in [16, 18, 19].

Assume that a C^∞ vector field $f : O \subset \mathcal{R}^N \to \mathcal{R}^N$ has an equivariant symmetry group G, which implies the existence of representations $\rho : G \to O(N)$ such that $f\rho(\gamma) = \rho(\gamma)f$, for all $\gamma \in G$. Define $F : C^1_{2\pi} \times \mathcal{R} \to C^0_{2\pi}$ by

$$F(u, \tau) = (1 + \tau)\frac{du}{ds} - f(u), \tag{4.1}$$

where $C^1_{2\pi}$ is the space of C^1 maps $u : S^1 \to \mathcal{R}^N$ and $C^0_{2\pi}$ is the space of C^0 maps $v : S^1 \to \mathcal{R}^N$. The map F is C^∞ by the "Ω-lemma," that is, in Section 2.4 of [20]. Clearly, the solutions of $F(u, \tau) = 0$ correspond to $2\pi/(1 + \tau)$-periodic solutions of (1.1).

Now, define an action $T : \tilde{G} \times C^0_{2\pi} \to C^0_{2\pi}$ or in $C^1_{2\pi}$ by

$$(T_g u)(t) = \rho(\gamma)(u(t + \theta)), \tag{4.2}$$

where $g = \gamma\theta$ is an element of $\tilde{G}$, $\tilde{G} = G \times S^1$, $\gamma \in G$ and $\theta \in S^1$.

By the G-equivariance of f, we have that F is $\tilde{G}$-equivariant

$$F(T_g u, \tau) = T_g F(u, \tau), \qquad \forall g = \gamma\theta \in \tilde{G}. \tag{4.3}$$

Assume that $f(0) = 0$. The derivative of F at $u = 0$ is L, where

$$Lv = dF(0, 0) \cdot v = v' - A_0 v, \tag{4.4}$$

with $A_0 = Df(0)$. Moreover, $\ker L = \operatorname{span}\{\operatorname{Re}(e^{is}v_0), \operatorname{Im}(e^{is}v_0)\} = \{\operatorname{Re}(ze^{is}v_0) \mid z \in \mathcal{C}\}$, where $A_0 v_0 = iv_0$.

By (4.3), L is also $\widetilde{G}$-equivariant such that $LT_g = T_g L$. Then, T_g preserves $\ker L$ and Range L.

Below, we will obtain that $(\ker L)^{\perp}$ and $(\text{Range } L)^{\perp}$ are also T_g-invariant. We have

$$C^1_{2\pi} = \ker L \oplus (\ker L)^{\perp}, \qquad C^0_{2\pi} = \text{ Range } L \oplus (\text{Range } L)^{\perp}. \tag{4.5}$$

Here, the orthogonal complement is taken in $C^0_{2\pi}$ and $C^1_{2\pi}$ by

$$[u, v] = \int_{\widetilde{G}} \langle T_g u, T_g v \rangle d\mu, \tag{4.6}$$

where $\langle u, v \rangle = \int_0^{2\pi} [u(t)]^t v(t) dt$ and μ is a normalized Haar measure for $\widetilde{G}$. Note that $[T_g u, T_g v] = [u, v]$ for all $g \in \widetilde{G}$. So, $(\ker L)^{\perp}$ and $(\text{Range } L)^{\perp}$ are T_g-invariant.

By Range L and $(\text{Range } L)^{\perp}$ are T_g-invariant, we can obtain that the projections

$$P : C^0_{2\pi} \longrightarrow \text{ Range } L, \qquad (I - P) : C^0_{2\pi} \longrightarrow (\text{Range } L)^{\perp} \tag{4.7}$$

commute with T_g. If $u \in C^0_{2\pi}$, let $u = v + w$, where $v \in (\text{Range } L)^{\perp}$ and $w \in$ Range L. Then, $T_g(P(u)) = T_g w = P(T_g w) = P(T_g v + T_g w) = P(T_g u)$ and $T_g((I - P)(u)) = T_g v = (I - P) T_g v = (I - P)(T_g v + T_g w) = (I - P)(T_g u)$.

Next, define a C^{∞} map $\omega : \ker L \times \mathcal{R} \to (\ker L)^{\perp}$ with $\omega(0, 0) = 0$ by solving

$$PF(k + \omega, \tau) = 0, \tag{4.8}$$

for $\omega = \omega(k, \tau)$ using the implicit function theorem.

Moreover, we can prove that ω commutes with T_g. Define $\omega_g : \ker L \times \mathcal{R} \to (\ker L)^{\perp}$ by $\omega_g(k, \tau) = T_{g^{-1}} \omega(T_g k, \tau)$. Note that $T_g k \in \ker L$. We have $PF(k + \omega_g(k, \tau), \tau) = PF(T_{g^{-1}}(T_g k + \omega(T_g k, \tau)), \tau) = T_{g^{-1}} PF(T_g k + \omega(T_g k, \tau), \tau) = 0$. Then, ω_g is also the solution of (4.8). Moreover, $\omega_g(0, 0) = \omega(0, 0) = 0$. By uniqueness, we get $\omega_g(k, \tau) = \omega(k, \tau)$. So, ω commutes with T_g. Then,

$$T_g \omega(k, \tau) = \omega(T_g k, \tau). \tag{4.9}$$

Then, solutions of the equation $F(u, \tau) = 0$ are given by $u = k + \omega(k, \tau)$, where k is a solution of the bifurcation equation

$$\varphi(k, \tau) = (I - P) F(k + \omega(k, \tau), \tau) = 0. \tag{4.10}$$

Now, we will obtain that $\varphi(k, \tau)$ is also $\widetilde{G}$-equivariant.

Lemma 4.1. *If f is G-equivariant, then the bifurcation map φ is $\widetilde{G}$-equivariant*

$$\varphi(T_g k, \tau) = T_g \varphi(k, \tau), \quad \forall g \in G \times S^1. \tag{4.11}$$

Proof. Since $\widetilde{G}$-equivariance of $I - P$, F and $\omega(k, \tau)$, it is easy to obtain this result. □

Lemma 4.1 indicates that how the symmetry enters the bifurcation equation φ. Below, we will also consider the relation between the symmetry and the Hamiltonian function of φ.

Since the vector field $f = X_H$ is Hamiltonian, it follows that

$$\omega(X_H(u), v) = \langle dH(u), v\rangle, \tag{4.12}$$

holds for all $v \in \mathcal{R}^{2n}$. Define the map

$$\Phi : C^1_{2\pi} \times \mathcal{R} \times C^0_{2\pi} \longrightarrow \left(C^1_{2\pi}\right)^*, \tag{4.13}$$

by

$$\Phi(u, \tau, v) \cdot U = \int_0^{2\pi} \left\{\omega\left(v - (1+\tau)\frac{du}{ds}, U\right) + dH(u) \cdot U\right\} ds. \tag{4.14}$$

Since the vector field f is Hamiltonian with the Hamiltonian function H, we have that the implicit constraint Φ satisfies

$$\Phi(u, \tau, F(u, \tau)) = 0. \tag{4.15}$$

This condition is rephrased as saying that the map F, regarded as a (parameter-dependent) vector field on $C^1_{2\pi}$ and is Hamiltonian with respect to the weak symplectic form

$$\Omega(u, v) = \frac{1}{2\pi} \int_0^{2\pi} \omega(u(s), v(s)) ds, \tag{4.16}$$

and with the Hamiltonian function

$$\mathcal{H}(u, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \left\{\frac{1}{2}\omega\left((1+\tau)\frac{du}{ds}, u\right) - H(u)\right\} ds. \tag{4.17}$$

If the actions of G are antisymplectic, we have

$$\omega(\gamma x, \gamma y) = -\omega(x, y). \tag{4.18}$$

In this case, Ω is $\widetilde{G} = G \times S^1$ anti-invariant; that is,

$$\Omega(gu, gv) = -\Omega(u, v), \quad g = \gamma\theta,\ \gamma \in G,\ \theta \in S^1, \tag{4.19}$$

and the Hamiltonian function $\mathcal{H}$ satisfies

$$\mathcal{H}(gu, \tau) = -\mathcal{H}(u, \tau). \tag{4.20}$$

By Theorem 6.2 in [18], it follows that if

$$\ker L = \ker L^*, \tag{4.21}$$

then the bifurcation map φ is also a Hamiltonian vector field with Hamiltonian $h(k) = \mathscr{H}(k + \omega(k))$, and φ and the function h have the same invariance properties as the given Hamiltonian $\mathscr{H}$.

In this paper, $\ker L$ is finite dimensional and (4.21) holds. So, the bifurcation equation φ is a Hamiltonian vector field with Hamiltonian $h(k,\tau) = \mathscr{H}(k + \omega(k), \tau)$ and if the actions of G are antisymplectic, then

$$h(gk, \tau) = -h(k, \tau). \tag{4.22}$$

The corresponding symplectic form is the restriction of Ω to $\ker L$. Moreover, in this paper, we have $G = \mathcal{Z}_2$ and $\widetilde{G} = \mathcal{Z}_2 \times S^1 \cong O(2)$.

5. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. Theorem 2.1 can be proved by Lemmas 5.1 and 5.3. By Lemma 5.1, the symmetric periodic solutions are obtained. The existence of the nonsymmetric periodic solutions can be verified by Lemma 5.3.

Below, we consider the case that the equivariant symmetry S acts antisymplectically. By Lemma 3.1, in this case, there are two pairs of purely imaginary eigenvalues. Without loss of generality, $S|_V$, $J|_V$ and $A_0|_V$ take the (normal) forms of Lemma 3.1, where V is the four-dimensional eigenspace of $\pm i$ for A_0.

Note that $\dim\ker L = 4$ and (4.21) holds. By Theorem 6.2 in [18], the bifurcation equation φ is also a Hamiltonian vector field. We now proceed to apply the Liapunov-Schmidt reduction of Section 4.

Since $\ker L \cong \mathcal{R}^4 \cong \mathcal{C}^2$, it follows that

$$\mathcal{R}^4 \ni (x_1, y_1, x_2, y_2) \cong (x_1 + iy_1, x_2 + iy_2) = (z_1, z_2) \in \mathcal{C}^2. \tag{5.1}$$

So, the bifurcation equation is denoted by

$$\varphi : \mathcal{C}^2 \times \mathcal{R} \longrightarrow \mathcal{C}^2. \tag{5.2}$$

Since φ is Hamiltonian, let $\varphi = 2J\nabla_{\overline{z}} h$ with the Hamiltonian function

$$h : \mathcal{C}^2 \times \mathcal{R} \longrightarrow \mathcal{R}, \tag{5.3}$$

where J takes the (normal) form of Lemma 3.1. By (4.22), it follows that h is S^1-invariant, $h \circ \theta = h$ with $\theta \in S^1$ acting on $\mathcal{C}^2$ as

$$\theta(z_1, z_2) = \left(e^{-i\theta} z_1, e^{-i\theta} z_2\right), \tag{5.4}$$

and h is S-anti-invariant, $h \circ S = -h$ with

$$S(z_1, z_2) = (z_2, z_1). \tag{5.5}$$

By S^1-invariance, it follows that

$$h = h\left(|z_1|^2, |z_2|^2, z_1\overline{z}_2, z_2\overline{z}_1, \tau\right). \tag{5.6}$$

Then, by the S-anti-invariance, we have that

$$h = \left(|z_1|^2 - |z_2|^2\right)\psi\left(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1\overline{z}_2, z_2\overline{z}_1, \tau\right) \tag{5.7}$$

is equivalent to $(\partial^n \psi/\partial(z_1\overline{z}_2)^n)|_{z_1\overline{z}_2=0}$ is real for all n, z_1 and z_2 which satisfy $n \in \mathcal{Z}$ and $z_1\overline{z}_2 = 0$. If $(\partial^n \psi/\partial(z_1\overline{z}_2)^n)|_{z_1\overline{z}_2=0}$ is not real for some n, z_1 and z_2 which satisfy $n \in \mathcal{Z}$ and $z_1\overline{z}_2 = 0$, by the S-anti-invariance, we have $h = (|z_1|^2 - |z_2|^2)\psi(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, \tau)$. Thus, without generality, since h is real, we obtain

$$\begin{aligned} h &= \left(|z_1|^2 - |z_2|^2\right)\psi\left(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1\overline{z}_2, z_2\overline{z}_1, \tau\right) \\ &= \left(|z_1|^2 - |z_2|^2\right)\left(\psi^1\left(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, \tau\right)\right) \\ &\quad + 2\,\mathrm{Re}\left(z_1\overline{z}_2\psi^2\left(|z_1|^2 + |z_2|^2, |z_1 z_2|^2, z_1\overline{z}_2, \tau\right)\right), \end{aligned} \tag{5.8}$$

where $\psi = \psi^1 + 2\ \mathrm{Re}\,(z_1\overline{z}_2\psi^2)$.

We have that $\ker L$ is generated by

$$\begin{aligned} v_1 &= (\sin s, -\cos s, 0, 0)^T, \qquad v_2 = (\cos s, \sin s, 0, 0)^T, \\ v_3 &= (0, 0, \sin s, -\cos s)^T, \qquad v_4 = (0, 0, \cos s, \sin s)^T. \end{aligned} \tag{5.9}$$

Then, the symplectic form Ω for the reduced bifurcation equation satisfies

$$\Omega(v_n, v_m) = \frac{1}{2\pi}\int_0^{2\pi} \langle v_n, J v_m\rangle ds, \tag{5.10}$$

where J takes the (normal) form of Lemma 3.1. So,

$$\Omega(k_1, k_2) = \langle k_1, J k_2\rangle, \tag{5.11}$$

for all $k_1, k_2 \in \ker L$.

Using the similar calculation and by (4.17), it follows that the τ-dependence of the lowest (quadratic) order of h has the form

$$h = \left(|z_1|^2 - |z_2|^2\right)\frac{\tau}{2} + l(z_1, z_2, \tau). \tag{5.12}$$

Using (5.8) and (5.12), we have $\psi^1(0, 0, \tau) = \tau/2$.

We first give Lemma 5.1, which is used for finding the symmetric periodic orbits near the equilibrium point.

5.1. Symmetric Periodic Solutions

Since $h \circ S = -h$, then the symmetric periodic solutions of the bifurcation equation lie in the level set $h = 0$. In fact, if $h = 0$, we get $\varphi = 0$. Then, by (4.9), the symmetric solutions lying in $h = 0$ correspond to symmetric 2π-periodic solutions for (4.1). Below, we prove there exist the symmetric solutions for the equation $h = 0$.

Lemma 5.1. *In the neighbourhood of the equilibrium* $(z_1, z_2, \tau) = (0, 0, 0)$, *there exist the symmetric solutions for the equation* $h = 0$.

Proof. Since $S(z_1, z_2) = (z_2, z_1)$, the symmetric solutions lie in $\text{Fix}(S) = \{(z, z) \in \mathcal{C}^2\}$. Then, the equation $\varphi = 2J\nabla_{\overline{z}}h = 0$ is equivalent to $z = 0$ or $\psi(z, \tau) = 0$. By (5.8) and (5.12), we have $(\partial\psi/\partial\tau)(0, 0) = 1/2 \neq 0$. Using the implicit function theorem for $\psi = 0$, there exists a function $\tau = \tau(|z|^2)$ for all sufficiently small $|z|$ that corresponds to a family of symmetric periodic solutions of the system (1.1) with period $2\pi/(1 + \tau)$. □

Remark 5.2. Apparently, the symmetric periodic solutions should be twoparameters. But since the vector field f is the equivariant Hamiltonian vector field, then the symmetric periodic solutions are in fact one-parameter.

Next, we study the nonsymmetric periodic solutions in the neighbourhood of the equilibrium 0.

5.2. Nonsymmetric Periodic Solutions

Lemma 5.3. *Except the symmetric Liapunov Center family described in Lemma 5.1, there are two nonsymmetric Liapunov Center families of periodic solutions, each contained in a local two-dimensional smooth manifold with the period of the periodic solutions converging to* 2π *as the solutions tend to the equilibrium point.*

Proof. The proof can be divided into three cases: $z_1 \neq 0,\ z_2 \neq 0$ *and* $z_1 \neq z_2,\ z_1 = 0$ and $z_2 \neq 0$, and $z_1 \neq 0$ and $z_2 = 0$.

$z_1 \neq 0,\ z_2 \neq 0$ and $z_1 \neq z_2$: let $I_1 = |z_1|^2 - |z_2|^2$. It follows that $\varphi = 0$ is equivalent to $\nabla_{\overline{z}}h = 0$. By (5.8), $\nabla_{\overline{z}}h = 0$ can be written as

$$\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} \psi + I_1 \nabla_{\overline{z}}\psi = 0. \tag{5.13}$$

By (5.8), and multiplying the first equation of (5.13) by $\overline{z}_1$ and the second equation of (5.13) by $\overline{z}_2$, we have

$$|z_1|^2\left(\psi + I_1\left(\psi_1^1 + |z_2|^2\psi_2^1 + 2\,\text{Re}\left[z_1\overline{z}_2\left(\psi_1^2 + |z_2|^2\psi_2^2\right)\right]\right)\right) + 2I_1\left(z_2\overline{z}_1\left(\overline{\psi^2 + z_1\overline{z}_2\psi_3^2}\right)\right) = 0, \tag{5.14}$$

$$|z_2|^2\left(-\psi + I_1\left(\psi_1^1 + |z_1|^2\psi_2^1 + 2\,\text{Re}\left[z_1\overline{z}_2\left(\psi_1^2 + |z_1|^2\psi_2^2\right)\right]\right)\right) + 2I_1\,\text{Re}\left(z_1\overline{z}_2\left(\psi^2 + z_1\overline{z}_2\psi_3^2\right)\right) = 0, \tag{5.15}$$

where ψ_i^j is the partial derivative of ψ^j with respect to X_i, $X_1 = |z_1|^2 + |z_2|^2$, $X_2 = |z_1 z_2|^2$, and $X_3 = z_1\overline{z}_2$.

Now, we consider the real parts of (5.14) and (5.15). Subtracting the real part of (5.14) from (5.15), we obtain

$$X_1\varphi + I_1^2\varphi_1^1 + 2I_1^2 \operatorname{Re}\left(z_1\overline{z}_2\varphi_1^2\right) = 0. \tag{5.16}$$

So,

$$X_1\varphi + I_1^2\frac{\partial\varphi}{\partial X_1} = 0. \tag{5.17}$$

By (5.17), we have

$$\varphi = ce^{-X_1^2/2I_1^2}, \tag{5.18}$$

where c is a constant. Let us take (5.18) into $h = I_1\varphi$ and verify $\nabla_{\overline{z}}h = 0$. Then, the first equation $\nabla_{\overline{z}_1}h = 0$ becomes

$$ce^{-X_1^2/2I_1^2}z_1 - I_1ce^{-X_1^2/2I_1^2}\cdot\frac{4X_1z_1I_1^2 - 4I_1z_1X_1^2}{4I_1^4} = 0. \tag{5.19}$$

If $\varphi \neq 0$, it follows that

$$1 = \frac{I_1X_1 - X_1^2}{I_1^2}. \tag{5.20}$$

By (5.20), we get

$$|z_1|^4 + |z_2|^4 = -2|z_2|^4. \tag{5.21}$$

But (5.21) does not hold in this case. Therefore, we find no small solutions of the bifurcation equation for $z_1 \neq 0$, $z_2 \neq 0$ and $z_1 \neq z_2$.

$z_1 = 0$ and $z_2 \neq 0$: in this case, the equation $\nabla_{\overline{z}}h = 0$ is equivalent to

$$z_2\left(\varphi^1 + |z_2|^2\varphi_1^1\right) = 0 \Longleftrightarrow r\left(|z_2|^2, \tau\right) = \varphi^1 + |z_2|^2\varphi_1^1 = 0. \tag{5.22}$$

Moreover, $(\partial r/\partial\tau)|_{(|z_2|^2,\tau)=(0,0)} = 1/2$. Using the implicit function theorem for (5.22), there exists a function $\tau = \tau(|z_2|^2)$ with $\tau(0) = 0$ for all sufficiently small $|z_2|$. Correspondingly, we have a one-parameter family of nonsymmetric periodic solutions of the system (1.1) contained in a local smooth two-dimensional invariant manifold.

$z_1 \neq 0$ and $z_2 = 0$: similarly to the above case, for $\varphi = 0$, there exists a function $\tau = \tau(|z_1|^2)$ with $\tau(0) = 0$ for all sufficiently small $|z_1|$. Correspondingly, there is another one-parameter family of nonsymmetric periodic solutions of the system (1.1) filling out a local smooth two-dimensional invariant manifold. This family of nonsymmetric periodic solutions are the S-image of the family with $z_1 = 0$ and $z_2 \neq 0$. □

Remark 5.4. We now consider the case that S acts symplectically. In this case, purely imaginary eigenvalues pairs typically arise isolated. Since S on the two-dimensional eigenspace V of $\pm i$ for A_0 is the identity matrix as in Lemma 3.1, then the symmetric property of the periodic solutions is insignificant. However, we can prove the existence of a Liapunov Center family using the fact that the flow is Hamiltonian [18, 21].

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