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Research Article

An Implicit Algorithm for Maximal Monotone Operators and Pseudocontractive Mappings

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The purpose of this paper is to construct an implicit algorithm for finding the common solution of maximal monotone operators and strictly pseudocontractive mappings in Hilbert spaces. Some applications are also included.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

Recall that S is said to be a *strictly pseudo contractive mapping* if there exists a constant $0 \leq \rho < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \rho\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

For such case, we also say that S is a ρ-strictly pseudo-contractive mapping. When $\rho = 0$, T is said to be nonexpansive. It is clear that (1.1) is equivalent to

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-\rho}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

We denote by $F(S)$ the set of fixed points of S.

A mapping $A : C \to H$ is said to be α-inverse strongly monotone if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \tag{1.3}$$

for some $\alpha > 0$ and for all $x, y \in C$. It is known that if A is an α-inverse strongly monotone, then $\|Ax - Ay\| \leq 1/\alpha \|x - y\|$ for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi valued mapping B is said to be a monotone operator on H iff

$$\langle x - y, u - v \rangle \geq 0, \tag{1.4}$$

for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to \mathrm{dom}(B)$, which is called the resolvent of B for λ. It is known that the resolvent J_λ^B is firmly nonexpansive, that is,

$$\left\| J_\lambda^B x - J_\lambda^B y \right\|^2 \leq \left\langle J_\lambda^B x - J_\lambda^B y, x - y \right\rangle, \tag{1.5}$$

for all $x, y \in C$ and $B^{-1}0 = F(J_\lambda^B)$ for all $\lambda > 0$.

Algorithms for finding the fixed points of nonlinear mappings or for finding the zero points of maximal monotone operators have been studied by many authors. The reader can refer to [1–24]. Especially, Takahashi et al. [6] recently gave the following convergence result.

Theorem 1.1. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$, and let S be a nonexpansive mapping of C into itself, such that $F(S) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$, be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S\left(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n)\right), \tag{1.6}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$\begin{gathered} 0 < a \leq \lambda_n \leq b < 2\alpha, \qquad < c \leq \beta_n \leq d < 1, \\ \lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = 0, \qquad \lim_{n \to \infty} \alpha_n = 0, \qquad \sum_n \alpha_n = \infty, \end{gathered} \tag{1.7}$$

then $\{x_n\}$ generated by (1.6) *converges strongly to a point of $F(S) \cap (A + B)^{-1}0$.*

Motivated and inspired by the works in this field, the purpose of this paper is to construct an implicit algorithm for finding the common solution of maximal monotone operators and strictly-pseudocontractive mappings in Hilbert spaces. Some applications are also included.

2. Preliminaries

The following resolvent identity is well known: for $\lambda > 0$ and $\mu > 0$, there holds the identity

$$J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^B x\right), \quad x \in H. \tag{2.1}$$

We use the following notation:

(i) $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;

(ii) $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

We need the following lemmas for the next section.

Lemma 2.1 (see[14]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to H$ be a ρ-strict pseudo contraction. Define $T : C \to H$ by $Tx = \alpha x + (1-\alpha)Tx$ for each $x \in C$, then, as $\alpha \in [\rho, 1)$, T is nonexpansive such that $F(S) = F(T)$.*

Lemma 2.2 (see[15]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $\lambda > 0$ a constant, then one has*

$$\left\|(I - \lambda A)x - (I - \lambda A)y\right\|^2 \leq \left\|x - y\right\|^2 + \lambda(\lambda - 2\alpha)\left\|Ax - Ay\right\|^2, \quad \forall x, y \in C. \tag{2.2}$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.3 (see[14]). *Let C be a nonempty, closed and convex of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping, then $I - T$ is demi closed at 0, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4 (see[16]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.5 (see[17]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n, \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$, *then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we will prove our main results.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H, and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for λ which satisfies $a \le \lambda \le b$ where $[a,b] \subset (0,2\alpha)$. Let $\kappa \in (0,1)$ be a constant and $S : C \to C$ a ρ-strict pseudocontraction with $\rho \in [0,1)$ such that $F(S)\cap(A+B)^{-1}0 \neq \emptyset$. For $t \in (0,1-\lambda/2\alpha)$, let $\{x_t\} \subset C$ be a net defined by*

$$x_t = \frac{\kappa(1-\rho)}{1-\kappa\rho}Sx_t + \frac{1-\kappa}{1-\kappa\rho}J_\lambda^B((1-t)x_t - \lambda Ax_t), \tag{3.1}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$, where P is the metric projection.

Remark 3.2. Now, we show that the net $\{x_t\}$ defined by (3.1) is well defined. For any $t \in (0,1-\lambda/2\alpha)$, we define a mapping $W := \kappa(\rho I + (1-\rho)S) + (1-\kappa)J_\lambda^B((1-t)I - \lambda A)$. Note that $\rho I + (1-\rho)S$ (by Lemma 2.1), J_λ^B, and $I - \lambda/(1-t)\ A$ (by Lemma 2.2) are nonexpansive. For any $x, y \in C$, we have

$$\begin{aligned}\|Wx - Wy\| &= \Big\|\kappa(\rho x + (1-\rho)Sx) + (1-\kappa)\ J_\lambda^B((1-t)x - \lambda Ax) \\ &\qquad -\kappa(\rho y + (1-\rho)Sy) - (1-\kappa)J_\lambda^B((1-t)y - \lambda Ay)\Big\| \\ &\le \kappa\|\rho(x-y) + (1-\rho)(Sx - Sy)\| \\ &\quad + (1-\kappa)\left\|(1-t)\left(x - \frac{\lambda}{1-t}Ax\right) - (1-t)\left(y - \frac{\lambda}{1-t}Ay\right)\right\| \\ &\le [1-(1-\kappa)t]\|x-y\|,\end{aligned} \tag{3.2}$$

which implies the mapping T is a contraction on C. We use x_t to denote the unique fixed point of W in C. Therefore, $\{x_t\}$ is well defined. We can rewrite (3.1) as

$$x_t = \kappa(\rho x_t + (1-\rho)Sx_t) + (1-\kappa)J_\lambda^B((1-t)x_t - \lambda Ax_t). \tag{3.3}$$

In order to prove Theorem 3.1, we need the following propositions.

Proposition 3.3. *Under the assumptions of Theorem 3.1, the net $\{x_t\}$ defined by* (3.1) *and hence* (3.3) *is bounded.*

Proof. Let $z \in F(S)\cap(A+B)^{-1}0$. It follows that

$$z = Sz = \rho z + (1-\rho)Sz = J_\lambda^B(z - \lambda Az), \tag{3.4}$$

for all $\lambda > 0$. We can write $J_\lambda^B(z - \lambda Az)$ as $J_\lambda^B(tz + (1-t)(z - \lambda Az/(1-t)))$, for all $t \in (0,1)$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned} &\left\| J_\lambda^B((1-t)x_t - \lambda Ax_t) - z \right\|^2 \\ &\quad = \left\| J_\lambda^B((1-t)(x_t - \lambda Ax_t/(1-t))) - J_\lambda^B(tz + (1-t)(z - \lambda Az/(1-t))) \right\|^2 \\ &\quad \le \|((1-t)(x_t - \lambda Ax_t/(1-t))) - (tz + (1-t)(z - \lambda Az/(1-t)))\|^2 \\ &\quad = \|(1-t)((x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))) + t(-z)\|^2. \end{aligned} \tag{3.5}$$

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, we derive

$$\begin{aligned} &\|(1-t)((x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))) + t(-z)\|^2 \\ &\quad \le (1-t)\|(x_t - \lambda Ax_t/(1-t)) - (z - \lambda Az/(1-t))\|^2 + t\|z\|^2 \\ &\quad = (1-t)\|(x_t - z) - \lambda(Ax_t - Az)/(1-t)\|^2 + t\|z\|^2 \\ &\quad = (1-t)\left(\|x_t - z\|^2 - \frac{2\lambda}{1-t}\langle Ax_t - Az, x_t - z\rangle + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2 \right) + t\|z\|^2 \\ &\quad \le (1-t)\left(\|x_t - z\|^2 - \frac{2\alpha\lambda}{1-t}\|Ax_t - Az\|^2 + \frac{\lambda^2}{(1-t)^2}\|Ax_t - Az\|^2 \right) + t\|z\|^2 \\ &\quad = (1-t)\left(\|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2 \right) + t\|z\|^2. \end{aligned} \tag{3.6}$$

By the assumption, we have $\lambda - 2(1-t)\alpha \le 0$, for all $t \in (0, 1 - \lambda/2\alpha)$. Then, from (3.5) and (3.6), we obtain

$$\begin{aligned} &\left\| J_\lambda^B((1-t)x_t - \lambda Ax_t) - z \right\|^2 \\ &\quad \le (1-t)\left(\|x_t - z\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|Ax_t - Az\|^2 \right) + t\|z\|^2 \\ &\quad \le (1-t)\|x_t - z\|^2 + t\|z\|^2. \end{aligned} \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$\begin{aligned} \|x_t - z\|^2 &\le \kappa \left\| (\rho I + (1-\rho)S)x_t - z \right\|^2 + (1-\kappa)\left\| J_\lambda^B((1-t)x_t - \lambda Ax_t) - z \right\|^2 \\ &\le \kappa\|x_t - z\|^2 + (1-\kappa)\left\| J_\lambda^B((1-t)x_t - \lambda Ax_t) - z \right\|^2 \\ &\le \kappa\|x_t - z\|^2 + (1-\kappa)\left[(1-t)\|x_t - z\|^2 + t\|z\|^2 \right]. \end{aligned} \tag{3.8}$$

It follows that

$$\|x_t - z\| \le \|z\|. \tag{3.9}$$

Therefore, $\{x_t\}$ is bounded. $\square$

Remark 3.4. Since A is α-inverse strongly monotone, it is $1/\alpha$-Lipschitz continuous. At the same time, S is nonexpansive. So, from the boundedness, we deduce immediately that $\{Ax_t\}$, $J_\lambda^B((1-t)x_t-\lambda Ax_t)$, and $\{Sx_t\}$ are also bounded.

Proposition 3.5. *Assume that all conditions in Theorem 3.1 hold. Let $\{x_t\}$ be the net defined by* (3.1), *then one has* $\lim_{t\to 0+}\|x_t-Sx_t\|=0$ *and* $\lim_{t\to 0+}\|x_t-J_\lambda^B((1-t)x_t-\lambda Ax_t)\|=0$.

Proof. By (3.7) and (3.8), we obtain

$$\begin{aligned}\|x_t-z\|^2 \le{}& [1-(1-\kappa)t]\|x_t-z\|^2\\ &+\frac{\lambda(1-\kappa)}{(1-t)}(\lambda-2(1-t)\alpha)\|Ax_t-Az\|^2+(1-\kappa)t\|z\|^2.\end{aligned}\tag{3.10}$$

So,

$$\frac{\lambda}{(1-t)}(2(1-t)\alpha-\lambda)\|Ax_t-Az\|^2 \le t\|z\|^2-t\|x_t-z\|^2 \longrightarrow 0.\tag{3.11}$$

Since $\lim\inf_{t\to 0+}(\lambda/(1-t))(2(1-t)\alpha-\lambda)>0$, we obtain

$$\lim_{t\to 0+}\|Ax_t-Az\|=0.\tag{3.12}$$

Next, we show $\|x_t-Sx_t\| \to 0$. By using the firm nonexpansivity of J_λ^B, we have

$$\begin{aligned}&\left\|J_\lambda^B((1-t)x_t-\lambda Ax_t)-z\right\|^2\\ &\quad=\left\|J_\lambda^B((1-t)x_t-\lambda Ax_t)-J_\lambda^B(z-\lambda Az)\right\|^2\\ &\quad\le\left\langle(1-t)x_t-\lambda Ax_t-(z-\lambda Az),\ J_\lambda^B((1-t)x_t-\lambda Ax_t)-z\right\rangle\\ &\quad=\frac{1}{2}\left(\|(1-t)x_t-\lambda Ax_t-(z-\lambda Az)\|^2+\left\|J_\lambda^B((1-t)x_t-\lambda Ax_t)-z\right\|^2\right.\\ &\qquad\left.-\left\|(1-t)x_t-\lambda(Ax_t-\lambda Az)-J_\lambda^B((1-t)x_t-\lambda Ax_t)\right\|^2\right).\end{aligned}\tag{3.13}$$

By the nonexpansivity of $I-\lambda A/(1-t)$, we have

$$\begin{aligned}&\|(1-t)x_t-\lambda Ax_t-(z-\lambda Az)\|^2\\ &\quad=\|(1-t)(x_t-\lambda Ax_t/(1-t)-(z-\lambda Az/(1-t)))+t(-z)\|^2\\ &\quad\le(1-t)\|(x_t-\lambda Ax_t/(1-t)\ \ -(z-\lambda Az/(1-t)))\|^2+t\|z\|^2\\ &\quad\le(1-t)\|x_t-z\|^2+t\|z\|^2.\end{aligned}\tag{3.14}$$

It follows that

$$
\begin{aligned}
&\left\| J_\lambda^B((1-t)x_t - \lambda A x_t) - z \right\|^2 \\
&\quad \le \frac{1}{2}\left((1-t)\|x_t - z\|^2 + t\|z\|^2 + \left\| J_\lambda^B((1-t)x_t - \lambda A x_t) - z \right\|^2 \right. \\
&\qquad \left. - \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) - \lambda(Ax_t - Az) \right\|^2 \right).
\end{aligned}
\tag{3.15}
$$

Thus,

$$
\begin{aligned}
&\left\| J_\lambda^B((1-t)x_t - \lambda A x_t) - z \right\|^2 \\
&\quad \le (1-t)\|x_t - z\|^2 + t\|z\|^2 \\
&\quad - \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) - \lambda(Ax_t - Az) \right\|^2 \\
&\quad = (1-t)\|x_t - z\|^2 + t\|z\|^2 - \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\|^2 \\
&\quad + 2\lambda\left\langle (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t), Ax_t - Az \right\rangle - \lambda^2\|Ax_t - Az\|^2 \\
&\quad \le (1-t)\|x_t - z\|^2 + t\|z\|^2 - \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\|^2 \\
&\quad + 2\lambda\left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\| \|Ax_t - Az\|.
\end{aligned}
\tag{3.16}
$$

This together with (3.8) implies that

$$
\begin{aligned}
\|x_t - z\|^2 &\le \left\| J_\lambda^B((1-t)x_t - \lambda A x_t) - z \right\|^2 \\
&\le (1-t)\|x_t - z\|^2 + t\|z\|^2 - \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\|^2 \\
&\quad + 2\lambda\left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\| \|Ax_t - Az\|.
\end{aligned}
\tag{3.17}
$$

Hence,

$$
\begin{aligned}
&\left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\|^2 \\
&\quad \le t\left(\|z\|^2 - \|x_t - z\|^2 \right) + 2\lambda\left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\| \|Ax_t - Az\|.
\end{aligned}
\tag{3.18}
$$

Since $\|Ax_t - Az\| \to 0$ (by (3.12)), we deduce

$$
\lim_{t \to 0+} \left\| (1-t)x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\| = 0. \tag{3.19}
$$

Therefore,

$$
\lim_{t \to 0+} \left\| x_t - J_\lambda^B((1-t)x_t - \lambda A x_t) \right\| = 0. \tag{3.20}
$$

Hence,

$$\lim_{t\to 0+}\|x_t - Sx_t\| = \lim_{t\to 0+}\left\|x_t - J_\lambda^B((1-t)x_t - \lambda A x_t)\right\| = 0. \tag{3.21}$$

□

Finally, we prove Theorem 3.1.

Proof. From (3.5) and (3.8), we have

$$\begin{aligned}
\|x_t - z\|^2 &\le \left\|(1-t)\left(\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right) - tz\right\|^2 \\
&= (1-t)^2\left\|\left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right\|^2 \\
&\quad - 2t(1-t)\left\langle z, \left(x_t - \frac{\lambda}{1-t}Ax_t\right) - \left(z - \frac{\lambda}{1-t}Az\right)\right\rangle + t^2\|z\|^2 \\
&\le (1-t)^2\|x_t - z\|^2 - 2t(1-t)\left\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\rangle + t^2\|z\|^2 \\
&= (1-2t)\|x_t - z\|^2 + 2t\left\{-(1-t)\left\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\rangle \right. \\
&\qquad \left. + \frac{t}{2}\left(\|z\|^2 + \|x_t - z\|^2\right)\right\}.
\end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned}
\|x_t - z\|^2 &\le -\left\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\rangle + \frac{t}{2}\left(\|z\|^2 + \|x_t - z\|^2\right) \\
&\quad + t\|z\|\left\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\| \\
&\le -\left\langle z, x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\rangle + tM,
\end{aligned} \tag{3.23}$$

where M is some constant such that

$$\sup_{t\in(0,1-\lambda/2\alpha)}\left\{\|z\|^2 + \|x_t - z\|^2 + \|z\|\left\|x_t - \frac{\lambda}{1-t}(Ax_t - Az) - z\right\|\right\} \le M. \tag{3.24}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0+$. Assume that $\{t_n\} \subset (0, 1-\lambda/2\alpha)$ is such that $t_n \to 0+$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.23), we have

$$\|x_n - z\|^2 \le -\left\langle z, x_n - \frac{\lambda}{1-t_n}(Ax_n - Az) - z\right\rangle + t_n M, \qquad z \in F(S)\cap(A+B)^{-1}0. \tag{3.25}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \rightharpoonup \tilde{x} \in C$. From (3.21), we have

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{3.26}$$

We can apply Lemma 2.3 to (3.26) to deduce $\tilde{x} \in F(S)$. Further, we show that $\tilde{x}$ is also in $(A+B)^{-1}0$. Let $v \in Bu$. Set $z_n = J_\lambda^B((1-t_n)x_n - \lambda A x_n)$, for all n, then we have

$$(1-t_n)x_n - \lambda A x_n \in (I+\lambda B)z_n \Longrightarrow \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{z_n}{\lambda} \in Bz_n. \tag{3.27}$$

Since B is monotone, we have, for $(u,v) \in B$,

$$\begin{aligned}
&\left\langle \frac{1-t_n}{\lambda}x_n - Ax_n - \frac{z_n}{\lambda} - v, z_n - u \right\rangle \geq 0 \\
&\Longrightarrow \langle (1-t_n)x_n - \lambda A x_n - z_n - \lambda v, z_n - u\rangle \geq 0 \\
&\Longrightarrow \langle Ax_n + v, z_n - u\rangle \leq \frac{1}{\lambda}\langle x_n - z_n, z_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, z_n - u\rangle \\
&\Longrightarrow \langle A\tilde{x} + v, z_n - u\rangle \leq \frac{1}{\lambda}\langle x_n - z_n, z_n - u\rangle - \frac{t_n}{\lambda}\langle x_n, z_n - u\rangle + \langle A\tilde{x} - Ax_n, z_n - u\rangle \\
&\Longrightarrow \langle A\tilde{x} + v, z_n - u\rangle \leq \frac{1}{\lambda}\|x_n - z_n\|\|z_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|z_n - u\| + \|A\tilde{x} - Ax_n\|\|z_n - u\|.
\end{aligned} \tag{3.28}$$

It follows that

$$\begin{aligned}
\langle A\tilde{x} + v, \tilde{x} - u\rangle \leq{}& \frac{1}{\lambda}\|x_n - z_n\|\|z_n - u\| + \frac{t_n}{\lambda}\|x_n\|\|z_n - u\| \\
&+ \|A\tilde{x} - Ax_n\|\|z_n - u\| + \langle A\tilde{x} + v, \tilde{x} - z_n\rangle.
\end{aligned} \tag{3.29}$$

Since

$$\langle x_n - \tilde{x}, Ax_n - A\tilde{x}\rangle \geq \alpha\|Ax_n - A\tilde{x}\|^2, \tag{3.30}$$

$Ax_n \to Az$, and $x_n \rightharpoonup \tilde{x}$, we have $Ax_n \to A\tilde{x}$. We also observe that $t_n \to 0$, $\|x_n - z_n\| \to 0$ and $z_n \rightharpoonup \tilde{x}$. Then, from (3.29), we derive

$$\langle -A\tilde{x} - v, \tilde{x} - u\rangle \geq 0. \tag{3.31}$$

Since B is maximal monotone, we have $-A\tilde{x} \in B\tilde{x}$. This shows that $0 \in (A+B)\tilde{x}$. So, we have $\tilde{x} \in F(S) \cap (A+B)^{-1}0$. Hence, $x_n - (\lambda/1-t_n)\ (Ax_n - Az) \rightharpoonup \tilde{x}$ because of $\|Ax_n - Az\| \to 0$. Therefore, we can substitute $\tilde{x}$ for z in (3.25) to get

$$\|x_n - \tilde{x}\|^2 \leq -\left\langle \tilde{x}, x_n - \frac{\lambda}{1-t_n}(Ax_n - A\tilde{x}) - \tilde{x} \right\rangle + t_n M. \tag{3.32}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0+$.

Now we return to (3.25) and take the limit as $n \to \infty$ to get

$$\|\tilde{x} - z\|^2 \leq -\langle z, \tilde{x} - z\rangle, \qquad z \in F(S) \cap (A+B)^{-1}0. \tag{3.33}$$

Equivalently,

$$\|\tilde{x}\|^2 \leq \langle \tilde{x}, z \rangle, \quad z \in F(S) \cap (A+B)^{-1}0. \tag{3.34}$$

This clearly implies that

$$\|\tilde{x}\| \leq \|z\|, \quad z \in F(S) \cap (A+B)^{-1}0. \tag{3.35}$$

Therefore, $\tilde{x}$ is the minimum norm element in $F(S) \cap (A+B)^{-1}0$. This completes the proof. □

Corollary 3.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H, and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for λ which satisfies $a \leq \lambda \leq b$ where $[a, b] \subset (0, 2\alpha)$. Let $\kappa \in (0, 1)$ be a constant and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap (A+B)^{-1}0 \neq \emptyset$. For $t \in (0, 1 - \lambda/2\alpha)$, let $\{x_t\} \subset C$ be a net defined by*

$$x_t = \frac{\kappa(1-\rho)}{1-\kappa\rho} S x_t + \frac{1-\kappa}{1-\kappa\rho} J_\lambda^B((1-t)x_t - \lambda A x_t), \tag{3.36}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S) \cap (A+B)^{-1}0}(0)$.

Corollary 3.7. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H, and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ such that $(A+B)^{-1}0 \neq \emptyset$. Let λ be a constant satisfying $a \leq \lambda \leq b$ where $[a, b] \subset (0, 2\alpha)$. For $t \in (0, 1 - \lambda/2\alpha)$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = J_\lambda^B((1-t)x_t - \lambda A x_t), \tag{3.37}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{(A+B)^{-1}0}(0)$.

4. Applications

Next, we consider the problem for finding the minimum norm solution of a mathematical model related to equilibrium problems. Let C be a nonempty, closed, and convex subset of a Hilbert space, and let $G : C \times C \to R$ be a bifunction satisfying the following conditions:

(E1) $G(x, x) = 0$, for all $x \in C$,

(E2) G is monotone, that is, $G(x, y) + G(y, x) \leq 0$, for all $x, y \in C$,

(E3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$,

(E4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\tilde{x} \in C$ such that

$$G(\tilde{x}, y) \geq 0, \tag{4.1}$$

for all $y \in C$. The set of such solutions $\tilde{x}$ is denoted by $EP(G)$. The following lemma appears implicitly in Blum and Oettli [19].

Lemma 4.1. *Let C be a nonempty, closed, and convex subset of H, and let G be a bifunction of $C \times C$ into R satisfying (E1)–(E4). Let $r > 0$ and $x \in H$, then there exists $z \in C$ such that*

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{4.2}$$

The following lemma was given in Combettes and Hirstoaga [20].

Lemma 4.2. *Assume that $G : C \times C \to R$ satisfies (E1)–(E4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}, \tag{4.3}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single valued,*

(2) *T_r is a firmly nonexpansive mapping, that is, for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \tag{4.4}$$

(3) *$F(T_r) = EP(G)$,*

(4) *$EP(G)$ is closed and convex.*

We call such T_r the resolvent of G for $r > 0$. Using Lemmas 4.1 and 4.2, we have the following lemma. See [18] for a more general result.

Lemma 4.3. *Let H be a Hilbert space, and let C be a nonempty, closed, and convex subset of H. Let $G : C \times C \to R$ satisfy (E1)–(E4). Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x,y) \geq \langle y - x, z\rangle, \quad \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C, \end{cases} \tag{4.5}$$

then, $EP(G) = A_G^{-1}(0)$, and A_G is a maximal monotone operator with $\mathrm{dom}(A_G) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of A_G, that is,

$$T_r x = (I + rA_G)^{-1} x. \tag{4.6}$$

From Lemma 4.3, Theorem 3.1, and Lemma 4.2, one has the following results.

Corollary 4.4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying (E1)–(E4), and let T_r be the resolvent of G for $r > 0$. Let*

$\kappa \in (0,1)$ be a constant and $S : C \to C$ a ρ-strict pseudocontraction with $\rho \in [0,1)$ such that $F(S) \cap EP(G) \neq \emptyset$. For $t \in (0,1)$, let $\{x_t\} \subset C$ be a net defined by

$$x_t = \frac{\kappa(1-\rho)}{1-\kappa\rho} S x_t + \frac{1-\kappa}{1-\kappa\rho} T_r((1-t)x_t), \tag{4.7}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap EP(G)}(0)$.

Corollary 4.5. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying (E1)–(E4), and let T_r be the resolvent of G for $r > 0$. Let $\kappa \in (0,1)$ be a constant and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap EP(G) \neq \emptyset$. For $t \in (0,1)$, let $\{x_t\} \subset C$ be a net defined by*

$$x_t = \frac{\kappa(1-\rho)}{1-\kappa\rho} S x_t + \frac{1-\kappa}{1-\kappa\rho} T_r((1-t)x_t), \tag{4.8}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap EP(G)}(0)$.

Corollary 4.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying (E1)–(E4), and let T_r be the resolvent of G for $r > 0$. Suppose $EP(G) \neq \emptyset$. For $t \in (0,1)$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = T_r((1-t)x_t), \quad t \in (0,1), \tag{4.9}$$

then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{EP(G)}(0)$.

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