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Research Article

A New Blow-Up Criterion for the DGH Equation

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We investigate the DGH equation. Analogous to the Camassa-Holm equation, this equation possesses the blow-up phenomenon. We establish a new blow up criterion on the initial data to guarantee the formulation of singularities in finite time.

1. Introduction

In 2001, Dullin et al. [1] derived the following equation to describe the unidirectional propagation of surface waves in a shallow water regime:

$$y_t + c_0 u_x + u y_x + 2 y u_x + \gamma u_{xxx} = 0, \tag{1.1}$$

where $y = u - \alpha^2 u_{xx}$, the constants α^2 and γ / c_0 are squares of length scales, and the constant $c_0 = \sqrt{gh} > 0$ is the critical shallow water speed for undisturbed water at rest at spatial infinity, where h is the mean fluid depth and g is the gravitational constant, $g = 9.8\,\mathrm{m/s^2}$.

In [2], local well-posedness of strong solutions to (1.1) was established by applying Katos theory [3] and some sufficient conditions were found to guarantee the finite blowup of the corresponding solutions for the spatially nonperiodic case. In [4], the author finds best constants for two convolution problems on the unit circle via a variational method. Then, by using the best constants on the DGH equation, sufficient conditions on the initial data are given to guarantee finite time singularity formation for the corresponding solutions.

When $\gamma = 0$ and $\alpha = 1$, system (1.1) reduces to the Camassa-Holm equation, which was derived physically by Camassa and Holm in [5] (found earlier by Fokas and Fuchssteiner [6] as a bi-Hamiltonian generalization of the KdV equation) by approximating directly the Hamiltonian for Eulers equation in the shallow water region with $u(x, t)$ representing the free surface above a flat bottom. For the hydrodynamic derivation one should also refer to the discussion in the papers [7–9]. The Camassa-Holm equation is completely integrable and has infinitely many conservation laws. The local well-posedness for Camassa-Holm equation in the Sobolev space H^s with $s > 3/2$ was proved in [10, 11]. One of the remarkable features of Camassa-Holm equation is the presence of breaking waves. Wave breaking for a large class of initial data has been established in [10–16]. Here we would like to mention that in [16] we give a new and direct proof for McKeans theorem. The wave breaking and propagation speed for more general family of one-dimensional shallow water equations are studied in [17]. In [18] Guo and Zhu established sufficient conditions on the initial data to guarantee blow-up phenomenon for the modified two component Camassa-Holm (MCH2) system. The existence of global solutions was also explored in [10, 12]. In addition to [10, 12], it is worth pointing out that the solution can be continued uniquely after wave breaking either as a conservative global solution or as a dissipative global solution, see the discussions in the papers [19, 20]. The solitary waves of Camassa-Holm equation are peaked solitons. The peaked solitons replicate a feature that is characteristic for the waves of great height-waves of largest amplitude which are exact solutions of the governing equations for water waves compare the papers [21–23]. The orbital stability of the peakons was shown by Constantin and Strauss in [24], here the orbital stability means that the shape of the waves is stable under perturbations, so that these patterns can be detected. It is worthy of being mentioned here the property of propagation speed of solutions to the Camassa-Holm equation; the first results in this direction were provided in the papers [25–27], which was also presented by Zhou and his collaborators in [28]. In [29], it is proved that strong solution to the Camassa-Holm equation decays algebraically with the same exponent as that of the initial datum.

In this paper, we will establish a new blow-up criterion on the profile of the initial data. Now, we give our main result as follows.

Theorem 1.1. *Suppose that* $u_0 \in H^s(\mathbb{R})$, $s \geq 3/2$, $y_0 = u_0 - \alpha^2 u_{0xx}$ *satisfies* $y_0(x_0) + (1/2)(c_0 + \gamma/\alpha^2) = 0$,

$$\int_{-\infty}^{x_0} e^{\xi/\alpha}\left[y_0(\xi) + \frac{1}{2}\left(c_0 + \frac{\gamma}{\alpha^2}\right)\right] d\xi > 0\,, \qquad \int_{x_0}^{\infty} e^{-\xi/\alpha}\left[y_0(\xi) + \frac{1}{2}\left(c_0 + \frac{\gamma}{\alpha^2}\right)\right] d\xi < 0, \tag{1.2}$$

for some point $x_0 \in \mathbb{R}$. *Then the solution* $u(x,t)$ *to* (1.1) *with initial datum* $u_0(x)$ *blows up in finite time.*

2. Proof of Theorem 1.1

As direct calculation, we can rewrite (1.1) as

$$y_t + u y_x + 2 y u_x - \frac{\gamma}{\alpha^2} y_x + \left(c_0 + \frac{\gamma}{\alpha^2}\right) u_x = 0. \tag{2.1}$$

Letting $\lambda = -\gamma/\alpha^2$ and $\kappa = (c_0 + (\gamma/\alpha^2))/2$, the above equation can reduce to

$$y_t + u y_x + 2 y u_x + \lambda y_x + 2\kappa u_x = 0. \tag{2.2}$$

Set $\Lambda = (1-\alpha^2\partial_x^2)^{1/2}$, then the operator Λ^{-2} can be expressed by its associated Green's function $G = (1/2\alpha)e^{-|x|/\alpha}$ as $\Lambda^{-2}f(x) = G * f(x) = (1/2\alpha)\int_{\mathbb{R}} e^{-|x-y|/\alpha}f(y)dy$. Then, (2.2) is equivalent to the following equation:

$$u_t + (u+\lambda)u_x + \partial_x G * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + 2\kappa u\right) = 0. \tag{2.3}$$

Motivated by Mckean's deep observation for the Camassa-Holm equation [13], we can do similar particle trajectory as

$$\begin{gathered} q_t = u(q,t) + \lambda, \quad 0 < t < T, \ x \in \mathbb{R}, \\ q(x,0) = x, \quad x \in \mathbb{R}, \end{gathered} \tag{2.4}$$

where T is the life span of the solution, then q is a diffeomorphism of the line; here it is worthwhile pointing out that this corresponds to the Lagrangian viewpoint in hydrodynamics (see the discussion in [30]). Differentiating the first equation in (2.4) with respect to x, one has

$$\frac{dq_t}{dx} = q_{xt} = u_x(q,t)q_x, \quad t \in (0,T). \tag{2.5}$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t u_x(q,s)ds\right\}, \quad q_x(x,0) = 1. \tag{2.6}$$

From (2.2) and (2.5) we can obtain

$$\frac{d}{dt}\left((y(q)+\kappa)q_x^2\right) = \left[y_t(q) + (u(q,t)+\lambda)y_x(q) + 2u_x(q,t)(y(q)+k)\right]q_x^2 = 0, \tag{2.7}$$

then it follows that

$$(y(q)+\kappa)q_x^2 = y_0(x) + \kappa. \tag{2.8}$$

In the case of the CH-equation (with $\kappa = 0$), the invariance of (2.8) has a geometric interpretation, compare [31, 32]. Differentiating (2.2), it follows from the definition of $q(x,t)$ that

$$\begin{aligned}
\frac{d}{dt}u_x(q(x_0,t),t) &= u_{tx}(q(x_0,t),t) + u_{xx}(q(x_0,t),t)(u(q(x_0,t),t)+\lambda) \\
&= -\frac{1}{2}u_x^2(q(x_0,t),t) + \frac{1}{\alpha^2}u^2(q(x_0,t),t) + \frac{1}{\alpha^2}2\kappa u(q(x_0,t),t) \\
&\quad - \frac{1}{\alpha^2}G*\left(u^2 + \frac{\alpha^2}{2}u_x^2 + 2\kappa u\right)(q(x_0,t),t) \\
&= -\frac{1}{2}u_x^2(q(x_0,t),t) + \frac{1}{\alpha^2}(u+\kappa)^2(q(x_0,t),t) \\
&\quad - \frac{1}{\alpha^2}G*\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_x^2\right](q(x_0,t),t) \\
&\le -\frac{1}{2}u_x^2(q(x_0,t),t) + \frac{1}{2\alpha^2}(u+\kappa)^2(q(x_0,t),t),
\end{aligned} \tag{2.9}$$

here we have used the inequality

$$G*\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_x^2\right] \ge \frac{1}{2}(u+\kappa)^2. \tag{2.10}$$

In fact,

$$\begin{aligned}
G*\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_x^2\right] &= \frac{1}{2\alpha}\int_{\mathbb{R}} e^{-|x-\xi|/\alpha}\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_\xi^2\right](\xi,t)d\xi \\
&= \frac{1}{2\alpha}e^{-x/\alpha}\int_{-\infty}^{x} e^{\xi/\alpha}\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_\xi^2\right](\xi,t)d\xi \\
&\quad + \frac{1}{2\alpha}e^{x/\alpha}\int_{x}^{\infty} e^{-\xi/\alpha}\left[(u+\kappa)^2 + \frac{\alpha^2}{2}u_\xi^2\right](\xi,t)d\xi.
\end{aligned} \tag{2.11}$$

From

$$\begin{aligned}
\int_{-\infty}^{x} e^{\xi/\alpha}\left[(u+\kappa)^2 + \alpha^2 u_\xi^2\right](\xi,t)d\xi &\ge 2\alpha\int_{-\infty}^{x} e^{\xi/\alpha}(u+\kappa)u_\xi d\xi \\
&= \alpha\int_{-\infty}^{x} e^{\xi/\alpha}(u+\kappa)_\xi^2 d\xi \\
&= \alpha e^{x/\alpha}(u+\kappa)^2 - \int_{-\infty}^{x} e^{\xi/\alpha}(u+\kappa)^2 d\xi,
\end{aligned} \tag{2.12}$$

we can deduce

$$\int_{-\infty}^{x} e^{\xi/\alpha}\left[(u+\kappa)^2+\frac{\alpha^2}{2}u_\xi^2\right](\xi,t)d\xi \geq \frac{\alpha}{2}e^{x/\alpha}(u+\kappa)^2. \tag{2.13}$$

Similarly we can obtain

$$\int_{x}^{\infty} e^{-\xi/\alpha}\left[(u+\kappa)^2+\frac{\alpha^2}{2}u_\xi^2\right](\xi,t)d\xi \geq \frac{\alpha}{2}e^{-x/\alpha}(u+\kappa)^2. \tag{2.14}$$

Combining (2.11), (2.13), and (2.14) we deduce our inequality (2.10).

Claim.

$u_x(q(x_0),t) < 0$ is strictly decreasing and $(u+\kappa)^2(q(x_0,t),t) < \alpha^2 u_x^2(q(x_0,t),t)$ for all $t \geq 0$.

Suppose there exists a t_0 such that $(u+\kappa)^2(q(x_0,t),t) > \alpha^2 u_x^2(q(x_0,t),t)$ on $[0,t_0)$ and $(u+\kappa)^2(q(x_0,t_0),t_0) = \alpha^2 u_x^2(q(x_0,t_0),t_0)$. From the expression of $u(x,t)$ in terms of $y(x,t)$, we can rewrite $u(x,t)+\kappa$ and $u_x(x,t)$ as follows:

$$\begin{aligned} u(x,t)+\kappa &= \frac{1}{2\alpha}e^{-x/\alpha}\int_{-\infty}^{x} e^{\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi + \frac{1}{2\alpha}e^{x/\alpha}\int_{x}^{\infty} e^{-\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi, \\ u_x(x,t) &= -\frac{1}{2\alpha^2}e^{-x/\alpha}\int_{-\infty}^{x} e^{\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi + \frac{1}{2\alpha^2}e^{x/\alpha}\int_{x}^{\infty} e^{-\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi. \end{aligned} \tag{2.15}$$

Now, let

$$\begin{aligned} I(t) &:= e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi, \\ II(t) &:= e^{q(x_0,t)/\alpha}\int_{q(x_0,t)}^{\infty} e^{-\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi. \end{aligned} \tag{2.16}$$

Then,

$$\begin{aligned} \frac{dI(t)}{dt} &= -\frac{1}{\alpha}\left(u(q(x_0,t))+\lambda\right)e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi \\ &\quad + e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}y_t(\xi,t)d\xi. \end{aligned} \tag{2.17}$$

From (2.2), the equation for $y(x,t)$ can be written as

$$y_t + \left((y+\kappa)u\right)_x + \frac{1}{2}\left(u^2-\alpha^2u_x^2\right)_x + \lambda\left(y+\kappa\right)_x + \kappa u_x = 0. \tag{2.18}$$

Putting (2.18) into the second term on the right-hand side of (2.17) and using (2.8), we have

$$
\begin{aligned}
&e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha} y_t(\xi,t)\,d\xi \\
&\quad= -e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left(\left((y+\kappa)u\right)_x + \frac{1}{2}\left(u^2-\alpha^2u_x^2\right)_x + \lambda(y+\kappa)_x + \kappa u_x\right)(\xi,t)\,d\xi \\
&\quad= \frac{1}{2}\left(\alpha^2u_x^2-u^2\right)(q(x_0,t),t) - \kappa u \\
&\qquad+ \frac{1}{\alpha}e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left((y+\kappa)u + \frac{1}{2}\left(u^2-\alpha^2u_x^2\right) + \lambda(y+\kappa) + \kappa u\right)(\xi,t)\,d\xi \\
&\quad= \frac{1}{2}\left(\alpha^2u_x^2-u^2\right)(q(x_0,t),t) - \kappa u \\
&\qquad+ \frac{1}{\alpha}e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left(\frac{3}{2}u^2 - \frac{\alpha^2}{2}u_x^2 - \alpha^2uu_{xx} + \lambda(y+\kappa) + 2\kappa u\right)(\xi,t)\,d\xi \\
&\quad= \frac{\alpha^2}{2}u_x^2 - \alpha uu_x - \kappa u \\
&\qquad+ \frac{1}{\alpha}e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left(u^2 + \frac{\alpha^2}{2}u_x^2 + \lambda(y+\kappa) + 2\kappa u\right)(\xi,t)\,d\xi.
\end{aligned} \tag{2.19}
$$

Here we have used

$$
\begin{aligned}
-\alpha e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}(uu_{xx})(\xi,t)\,d\xi &= -\alpha(uu_x)(q(x_0,t),t) + \frac{1}{2}u^2(q(x_0,t),t) \\
&\quad+ \alpha e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left(u_x^2(\xi,t) - \frac{1}{2\alpha^2}u^2\right)(\xi,t)\,d\xi.
\end{aligned} \tag{2.20}
$$

On the other hand, after integration by parts, the first term on the right-hand side of (2.17) yields

$$
\begin{aligned}
&-\frac{1}{\alpha}u(q(x_0,t))e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left[y(\xi,t)+\kappa\right]d\xi \\
&\quad= \alpha(uu_x)(q(x_0,t),t) - u^2(q(x_0,t),t) - \kappa u.
\end{aligned} \tag{2.21}
$$

Hence after combining the above terms and inequalities together, (2.17) reads as

$$\begin{aligned}
\frac{dI(t)}{dt} &= \left[\frac{\alpha^2}{2}u_x^2 - u^2 - 2\kappa u\right](q(x_0,t),t) \\
&\quad + \frac{1}{\alpha}e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left(u^2 + 2\kappa u + \frac{\alpha^2}{2}u_x^2\right)(\xi,t)d\xi \\
&= \left[\frac{\alpha^2}{2}u_x^2 - (u+\kappa)^2\right](q(x_0,t),t) \\
&\quad + \frac{1}{\alpha}e^{-q(x_0,t)/\alpha}\int_{-\infty}^{q(x_0,t)} e^{\xi/\alpha}\left((u+\kappa)^2 + \frac{\alpha^2}{2}u_x^2\right)(\xi,t)d\xi \\
&\geq \frac{1}{2}\left[\alpha^2 u_x^2 - (u+\kappa)^2\right](q(x_0,t),t) > 0, \quad \text{on } [0,t_0),
\end{aligned} \tag{2.22}$$

here (2.13) is used again. From the continuity property we have

$$e^{-q(x_0,t_0)/\alpha}\int_{-\infty}^{q(x_0,t_0)} e^{\xi/\alpha}\left[y(\xi,t_0)+\kappa\right]d\xi > e^{-x_0/\alpha}\int_{-\infty}^{x_0} e^{\xi/\alpha}\left[y_0(\xi)+\kappa\right]d\xi > 0. \tag{2.23}$$

Similarly,

$$\frac{dII(t)}{dt} \leq \frac{1}{2}\left[(u+\kappa)^2 - \alpha^2 u_x^2\right](q(x_0,t),t) < 0, \quad \text{on } [0,t_0). \tag{2.24}$$

Thus by continuity property

$$e^{q(x_0,t_0)/\alpha}\int_{q(x_0,t_0)}^{\infty} e^{-\xi/\alpha}\left[y(\xi,t_0)+\kappa\right]d\xi < e^{x_0/\alpha}\int_{x_0}^{\infty} e^{-\xi/\alpha}\left[y_0(\xi)+\kappa\right]d\xi < 0. \tag{2.25}$$

Summarizing (2.23) and (2.25), we obtain

$$\begin{aligned}
&\alpha^2 u_x^2(q(x_0,t_0),t_0) - (u+\kappa)^2(q(x_0,t_0),t_0) \\
&\quad = -\frac{1}{\alpha^2}\int_{-\infty}^{q(x_0,t_0)} e^{\xi/\alpha}\left[y(\xi,t_0)+\kappa\right]d\xi \int_{q(x_0,t_0)}^{\infty} e^{-\xi/\alpha}\left[y(\xi,t_0)+\kappa\right]d\xi \\
&\quad > -\frac{1}{\alpha^2}\int_{-\infty}^{x_0} e^{\xi/\alpha}\left[y_0(\xi)+\kappa\right]d\xi \int_{x_0}^{\infty} e^{-\xi/\alpha}\left[y_0(\xi)+\kappa\right]d\xi \\
&\quad = \alpha^2 u_{0x}^2(x_0) - (u_0+\kappa)^2(x_0) > 0.
\end{aligned} \tag{2.26}$$

This is a contradiction. So the claim is true.

Moreover, due to (2.22) and (2.24), we have the following for $(\alpha^2 u_x^2 - (u+\kappa)^2)(q(x_0,t),t)$:

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t) \\
&\quad = -\frac{1}{\alpha^2}\frac{d}{dt}\left(\int_{-\infty}^{q(x_0,t)} e^{\xi}(y+\kappa)(\xi,t)d\xi \int_{q(x_0,t)}^{\infty} e^{-\xi}(y+\kappa)(\xi,t)d\xi\right) \\
&\quad = -\frac{1}{\alpha^2}\frac{d}{dt}\left(e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(y+\kappa)(\xi,t)d\xi\right)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}(y+\kappa)(\xi,t)d\xi \\
&\qquad - \frac{1}{\alpha^2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(y+\kappa)(\xi,t)d\xi\frac{d}{dt}\left(e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}(y+\kappa)(\xi,t)d\xi\right) \\
&\quad \geq -\frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}(y+\kappa)(\xi,t)d\xi \\
&\qquad + \frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(y+\kappa)(\xi,t)d\xi \\
&\quad = -u_x(q(x_0,t),t)\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t).
\end{aligned} \tag{2.27}$$

Now, substituting (2.9) into (2.27), it yields

$$\begin{aligned}
\frac{d}{dt}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t) &\geq \frac{1}{2\alpha^2}\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,t),t) \\
&\times\left(\int_0^t\left(\alpha^2 u_x^2 - (u+\kappa)^2\right)(q(x_0,\tau),\tau)d\tau - 2\alpha^2 u_{0x}(x_0)\right).
\end{aligned} \tag{2.28}$$

Before completing the proof, we need the following technical lemma.

Lemma 2.1 (see [33]). *Suppose that $\Psi(t)$ is twice continuously differential satisfying*

$$\begin{gathered}
\Psi''(t) \geq C_0\Psi'(t)\Psi(t), \quad t>0,\ C_0>0, \\
\Psi(t)>0, \quad \Psi'(t)>0.
\end{gathered} \tag{2.29}$$

Then $\Psi(t)$ blows up in finite time. Moreover, the blow-up time can be estimated in terms of the initial datum as

$$T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}. \tag{2.30}$$

Let $\Psi(t) = (\int_0^t(\alpha^2 u_x^2 - (u+\kappa)^2)(q(x_0,\tau),\tau)d\tau - 2\alpha^2 u_{0x}(x_0))$, then (2.28) is an equation of type (2.29) with $C_0 = 1/2\alpha^2$. The proof is complete by applying Lemma 2.1.

Remark 2.2. The special case $c_0 + \gamma/\alpha^2 = 0$ was studied in [4]. We can regard the special case as the CH($\kappa = 0$) equation with a strong dispersive term $c_0 y_x$. Result here without the restriction is an improvement.

Remark 2.3. Recall, (2.2), we can easily find that the DGH equation is the CH($\kappa \neq 0$) equation with a strong dispersive term.

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