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Research Article

A Fixed Point Approach to the Stability of an n-Dimensional Mixed-Type Additive and Quadratic Functional Equation

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We investigate the stability problems for a functional equation $2f(\sum_{j=1}^{n} x_j)+\sum_{1\le i,j\le n,\ i\ne j} f(x_i-x_j) = (n+1)\sum_{j=1}^{n} f(x_j) + (n-1)\sum_{j=1}^{n} f(-x_j)$ by using the fixed point method.

1. Introduction

In 1940, Ulam [1] gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. Hyers [2] was the first mathematician to present the result concerning the stability of functional equations. He answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. This result of Hyers is stated as follows.

Let $f : E_1 \to E_2$ be a function between Banach spaces such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \delta \tag{1.1}$$

for some $\delta > 0$ *and for all* $x, y \in E_1$. *Then the limit* $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ *exists for each* $x \in E_1$, *and* $A : E_1 \to E_2$ *is the unique additive function such that* $\|f(x) - A(x)\| \le \delta$ *for every* $x \in E_1$. *Moreover, if* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$, *then the function* A *is linear.*

We remark that the additive function A is directly constructed from the given function f and this method is called the *direct method*. The direct method is a very powerful method for studying the stability problems of various functional equations. Taking this famous result into consideration, the additive Cauchy equation $f(x+y) = f(x) + f(y)$ is said to have the *Hyers-Ulam stability* on (E_1, E_2) if, for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\delta \ge 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

In 1950, Aoki [3] generalized the theorem of Hyers for additive functions, and in the following year, Bourgin [4] extended the theorem without proof. Unfortunately, it seems that their results failed to receive attention from mathematicians at that time. No one has made use of these results for a long time.

In 1978, Rassias [5] actually rediscovered the result of Aoki; he proved the following theorem of Hyers for linear functions.

Let $f : E_1 \to E_2$ *be a function between Banach spaces. If* f *satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(\|x\|^p + \|y\|^p\right) \tag{1.2}$$

for some $\theta \ge 0$, p *with* $0 \le p < 1$ *and for all* $x, y \in E_1$, *then there exists a unique additive function* $A : E_1 \to E_2$ *such that* $\|f(x) - A(x)\| \le (2\theta/(2 - 2^p))\|x\|^p$ *for each* $x \in E_1$. *If, in addition,* $f(tx)$ *is continuous in* t *for each fixed* $x \in E_1$, *then the function* A *is linear.*

This result of Rassias attracted a number of mathematicians who began to be stimulated to investigate the stability problems of functional equations. By regarding a large influence of Ulam, Hyers, and Rassias on the study of stability problems of functional equations, the stability phenomenon proved by Aoki and Rassias is called the *Hyers-Ulam-Rassias stability*. For the last thirty years many results concerning the Hyers-Ulam-Rassias stability of various functional equations have been obtained (see [6–16]). For some discussion of possible definitions of stability for functional equations, see [17].

The following functional equation

$$2f\left(\sum_{j=1}^{n} x_j\right) + \sum_{1\le i,j\le n,\ i\ne j} f\left(x_i - x_j\right) = (n+1)\sum_{j=1}^{n} f\left(x_j\right) + (n-1)\sum_{j=1}^{n} f\left(-x_j\right) \tag{1.3}$$

is called an *n-dimensional mixed-type additive and quadratic functional equation*, and each solution of (1.3) is called a *quadratic-additive function*. Recently, Towanlong and Nakmahachalasint [18] solved the n-dimensional mixed-type additive and quadratic functional equation.

Let E_1 *and* E_2 *be vector spaces. A function* $f : E_1 \to E_2$ *satisfies* (1.3) *for all* $x_1, x_2, \ldots, x_n \in E_1$ *if and only if there exist an additive function* $a : E_1 \to E_2$ *and a quadratic function* $q : E_1 \to E_2$ *such that* $f(x) = a(x) + q(x)$ *for all* $x \in E_1$.

Moreover, they also investigated the Hyers-Ulam-Rassias stability of (1.3) by using the direct method (see [18]). Indeed, they tried to approximate the even and odd parts of each solution of a perturbed inequality by the even and odd parts of an "exact" solution of (1.3), respectively.

In Theorems 3.1 and 3.3 of this paper, we will apply the fixed point method and prove the Hyers-Ulam-Rassias stability of the n-dimensional mixed-type additive and quadratic functional equation. The advantage of this paper, in comparison with [18], is to approximate each solution of a perturbed inequality by an "exact" solution of (1.3), and we obtain sharper estimations in consequence of this advantage.

Throughout this paper, let V be a (real or complex) vector space, Y a Banach space, and n an integer larger than 1.

2. Preliminaries

Let X be a nonempty set. A function $d : X^2 \to [0, \infty]$ is called a *generalized metric* on X if and only if d satisfies the following:

(M_1) $d(x, y) = 0$ if and only if $x = y$;

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [19].

Theorem 2.1. *Let (X, d) be a complete generalized metric space. Assume that $\Lambda : X \to X$ is a strict contraction with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}x, \Lambda^{n_0}x) < \infty$ for some $x \in X$, then the following statements are true.*

(i) *The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ.*

(ii) *x^* is the unique fixed point of Λ in $X^* = \{y \in X \mid d(\Lambda^{n_0}x, y) < \infty\}$.*

(iii) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y). \tag{2.1}$$

In 1991, Baker applied the fixed point method to prove the Hyers-Ulam stability of a nonlinear functional equation (see [20]). Thereafter, Radu noticed that many theorems concerning the Hyers-Ulam stability of various functional equations follow from the fixed point alternative (Theorem 2.1). Indeed, he applied the fixed point method to prove the existence of a solution of the inequality (1.1) and investigated the Hyers-Ulam stability of the additive Cauchy equation (see [21] and also [22–26]). For a somewhat different fixed point approach to stability of functional equations, see [27, 28].

3. Hyers-Ulam-Rassias Stability

Let V be a (real or complex) vector space and let Y be a Banach space. For a given function $f : V \to Y$, we use the following abbreviation:

$$Df(x_1, x_2, \ldots, x_n) := 2f\left(\sum_{j=1}^{n} x_j\right) + \sum_{1 \le i, j \le n,\ i \ne j} f(x_i - x_j) - (n+1)\sum_{j=1}^{n} f(x_j) - (n-1)\sum_{j=1}^{n} f(-x_j) \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n \in V$.

In the following theorem, we prove the stability of the functional equation (1.3) by using the fixed point method.

Theorem 3.1. *Let $\varphi : (V \setminus \{0\})^n \to [0, \infty)$ be a given function. Assume that a function $f : V \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varphi(x_1, x_2, \ldots, x_n) \tag{3.2}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(nx_1, nx_2, \ldots, nx_n) \le nL\varphi(x_1, x_2, \ldots, x_n) \tag{3.3}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, then there exists a unique function $F : V \to Y$ such that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$ and

$$\|f(x) - F(x)\| \le \frac{n+1}{4n^2(1-L)}\psi(x) \tag{3.4}$$

for all $x \in V \setminus \{0\}$, where the function $\psi : V \setminus \{0\} \to Y$ is defined by

$$\psi(x) := \varphi(x, x, \ldots, x) + \varphi(-x, -x, \ldots, -x) \tag{3.5}$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$F(x) = \lim_{m \to \infty}\left(\frac{f(n^m x) - f(-n^m x)}{2n^m} + \frac{f(n^m x) + f(-n^m x)}{2n^{2m}}\right) \tag{3.6}$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g, h) := \inf\{K \in [0, \infty] \mid \|g(x) - h(x)\| \le K\psi(x)\ \forall x \in V \setminus \{0\}\}, \tag{3.7}$$

where ψ is given in (3.5). It is not difficult to show that (S, d) is a complete generalized metric space (see [29] or [30, 31]).

Now we consider the operator $J : S \to S$ defined by

$$Jg(x) := \frac{g(nx) - g(-nx)}{2n} + \frac{g(nx) + g(-nx)}{2n^2} \tag{3.8}$$

for all $x \in V$. We can apply induction on m to prove

$$J^m g(x) = \frac{g(n^m x) - g(-n^m x)}{2n^m} + \frac{g(n^m x) + g(-n^m x)}{2n^{2m}} \tag{3.9}$$

for all $x \in V$ and $m \in \mathbb{N}$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. By the definition of d, (3.3), and (3.5), we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &\leq \frac{(n+1)\|g(nx) - h(nx)\|}{2n^2} + \frac{(n-1)\|g(-nx) - h(-nx)\|}{2n^2} \leq K\frac{\psi(nx)}{n} \\ &\leq KL\psi(x) \end{aligned} \tag{3.10}$$

for all $x \in V \setminus \{0\}$, which implies that

$$d(Jg, Jh) \leq Ld(g, h) \tag{3.11}$$

for any $g, h \in S$. That is, J is a strict contraction with the Lipschitz constant L.

Moreover, by (3.2), we see that

$$\|f(x) - Jf(x)\| = \left\| \frac{n-1}{4n^2} Df(-x, -x, \ldots, -x) - \frac{n+1}{4n^2} Df(x, x, \ldots, x) \right\| \leq \frac{n+1}{4n^2}\psi(x) \tag{3.12}$$

for all $x \in V \setminus \{0\}$, that is, $d(f, Jf) \leq (n+1)/4n^2 < \infty$ (see the definition of d). Therefore, according to Theorem 2.1, the sequence $\{J^m f\}$ converges to the unique "fixed point" $F : V \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$ and F is represented by (3.6) for all $x \in V$. Notice that $F(0) = 0$. By Theorem 2.1, we have

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{n+1}{4n^2(1-L)}, \tag{3.13}$$

which implies the validity of (3.4).

By a somewhat tedious manipulation, it follows from (3.2), (3.3), and (3.6) that

$$\begin{aligned}\|DF(x_1, x_2, \ldots, x_n)\| &= \lim_{m\to\infty}\left\| \frac{Df(n^m x_1, n^m x_2, \ldots, n^m x_n) - Df(-n^m x_1, -n^m x_2, \ldots, -n^m x_n)}{2n^m} \right. \\ &\qquad \left. + \frac{Df(n^m x_1, n^m x_2, \ldots, n^m x_n) + Df(-n^m x_1, -n^m x_2, \ldots, -n^m x_n)}{2n^{2m}} \right\| \\ &\le \lim_{m\to\infty} \frac{n^m + 1}{2n^{2m}} \left[\varphi(n^m x_1, \ldots, n^m x_n) + \varphi(-n^m x_1, \ldots, -n^m x_n)\right] \\ &= 0 \end{aligned} \tag{3.14}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. □

We can prove the following corollary by a way similar to that presented in the proof of the preceding theorem.

Corollary 3.2. *Let $\varphi : V^n \to [0, \infty)$ be a given function such that $\varphi(x_1, x_2, \ldots, x_n) = \varphi(-x_1, -x_2, \ldots, -x_n)$ for all $x_1, x_2, \ldots, x_n \in V$. Assume that a function $f : V \to Y$ satisfies $f(0) = 0$ and the inequality* (3.2) *for all $x_1, x_2, \ldots, x_n \in V$. If there exists a constant $0 < L < 1$ such that φ has the property* (3.3) *for all $x_1, x_2, \ldots, x_n \in V$, then there exists a unique quadratic-additive function $F : V \to Y$ such that*

$$\|f(x) - F(x)\| \le \frac{\varphi(x, x, \ldots, x)}{2n(1 - L)} \tag{3.15}$$

for all $x \in V$.

Proof. Notice that

$$\|f(x) - Jf(x)\| = \left\| \frac{n-1}{4n^2} Df(-x, -x, \ldots, -x) - \frac{n+1}{4n^2} Df(x, x, \ldots, x) \right\| \le \frac{\varphi(x, x, \ldots, x)}{2n} \tag{3.16}$$

for all $x \in V$ (see the proof of Theorem 3.1). By a similar method used in the proof of Theorem 3.1, we can show that there exists a unique quadratic-additive function $F : V \to Y$ satisfying (3.15). □

In the following theorem, we prove the Hyers-Ulam-Rassias stability of (1.3) under the condition (3.17) instead of (3.3).

Theorem 3.3. *Given $\varphi : (V \setminus \{0\})^n \to [0, \infty)$, assume that a function $f : V \to Y$ satisfies the condition $f(0) = 0$ and the inequality* (3.2) *for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. If there exists a constant $0 < L < 1$ such that*

$$L\varphi(nx_1, nx_2, \ldots, nx_n) \ge n^2 \varphi(x_1, x_2, \ldots, x_n) \tag{3.17}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, *then there exists a unique function* $F : V \to Y$ *such that* $DF(x_1, x_2, \ldots, x_n) = 0$ *for all* $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$ *and*

$$\|f(x) - F(x)\| \le \frac{L}{2n^2(1-L)}\varphi(x) \tag{3.18}$$

for all $x \in V \setminus \{0\}$, *where* φ *is defined by* (3.5). *In particular,* F *is represented by*

$$F(x) = \lim_{m\to\infty}\left(\frac{n^m}{2}\left(f\left(\frac{x}{n^m}\right) - f\left(-\frac{x}{n^m}\right)\right) + \frac{n^{2m}}{2}\left(f\left(\frac{x}{n^m}\right) + f\left(-\frac{x}{n^m}\right)\right)\right) \tag{3.19}$$

for all $x \in V$.

Proof. Let S and d be defined as in the proof of Theorem 3.1. Then (S, d) is a complete generalized metric space. We now consider the operator $J : S \to S$ defined by

$$Jg(x) := \frac{n}{2}\left(g\left(\frac{x}{n}\right) - g\left(-\frac{x}{n}\right)\right) + \frac{n^2}{2}\left(g\left(\frac{x}{n}\right) + g\left(-\frac{x}{n}\right)\right) \tag{3.20}$$

for all $g \in S$ and $x \in V$.

Notice that

$$J^m g(x) = \frac{n^m}{2}\left(g\left(\frac{x}{n^m}\right) - g\left(-\frac{x}{n^m}\right)\right) + \frac{n^{2m}}{2}\left(g\left(\frac{x}{n^m}\right) + g\left(-\frac{x}{n^m}\right)\right) \tag{3.21}$$

for all $x \in V$ and $m \in \mathbb{N}$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. By the definition of d, together with (3.5) and (3.17), we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &\le \frac{n^2+n}{2}\left\|g\left(\frac{x}{n}\right) - h\left(\frac{x}{n}\right)\right\| + \frac{n^2-n}{2}\left\|g\left(-\frac{x}{n}\right) - h\left(-\frac{x}{n}\right)\right\| \\ &\le n^2 K\varphi\left(\frac{x}{n}\right) \\ &\le LK\varphi(x) \end{aligned} \tag{3.22}$$

for all $x \in V \setminus \{0\}$. Thus, we get

$$d(Jg, Jh) \le Ld(g, h) \tag{3.23}$$

for any $g, h \in S$. That is, J is a strict contraction with the Lipschitz constant L.

Moreover, it follows from (3.2), (3.5), and (3.17) that

$$\begin{aligned}\|f(x)-Jf(x)\| &= \left\|\frac{1}{2}Df\left(\frac{x}{n},\frac{x}{n},\ldots,\frac{x}{n}\right)\right\| \\ &\le \frac{1}{2}\varphi\left(\frac{x}{n},\frac{x}{n},\ldots,\frac{x}{n}\right) \\ &\le \frac{L}{2n^2}\varphi(x,x,\ldots,x) \\ &\le \frac{L}{2n^2}\varphi(x)\end{aligned} \tag{3.24}$$

for all $x \in V \setminus \{0\}$, which implies that $d(f,Jf) \le L/2n^2 < \infty$. Therefore, according to Theorem 2.1, the sequence $\{J^m f\}$ converges to the unique "fixed point" F of J in the set $T = \{g \in S \mid d(f,g) < \infty\}$ and F is represented by (3.19). In view of Theorem 2.1, we have

$$d(f,F) \le \frac{1}{1-L}d(f,Jf) \le \frac{L}{2n^2(1-L)}. \tag{3.25}$$

Hence, the inequality (3.18) is true.

In a similar way presented in the proof of Theorem 3.1, it follows from (3.2), (3.17), and (3.19) that

$$\begin{aligned}\|DF(x_1,x_2,\ldots,x_n)\| &= \lim_{m\to\infty}\left\|\frac{n^m}{2}\left(Df\left(\frac{x_1}{n^m},\frac{x_2}{n^m},\ldots,\frac{x_n}{n^m}\right)-Df\left(-\frac{x_1}{n^m},-\frac{x_2}{n^m},\ldots,-\frac{x_n}{n^m}\right)\right)\right. \\ &\qquad \left. + \frac{n^{2m}}{2}\left(Df\left(\frac{x_1}{n^m},\frac{x_2}{n^m},\ldots,\frac{x_n}{n^m}\right)+Df\left(-\frac{x_1}{n^m},-\frac{x_2}{n^m},\ldots,-\frac{x_n}{n^m}\right)\right)\right\| \\ &\le \lim_{m\to\infty}\frac{n^{2m}+n^m}{2}\left(\varphi\left(\frac{x_1}{n^m},\frac{x_2}{n^m},\ldots,\frac{x_n}{n^m}\right)+\varphi\left(-\frac{x_1}{n^m},-\frac{x_2}{n^m},\ldots,-\frac{x_n}{n^m}\right)\right) \\ &= 0\end{aligned} \tag{3.26}$$

for all $x_1,x_2,\ldots,x_n \in V \setminus \{0\}$. □

We can prove the following corollary by a similar way as we did in the proof of Corollary 3.2. We omit the proof.

Corollary 3.4. *Let $\varphi : V^n \to [0,\infty)$ be a given function such that $\varphi(x_1,x_2,\ldots,x_n) = \varphi(-x_1,-x_2,\ldots,-x_n)$ for all $x_1,x_2,\ldots,x_n \in V$. Assume that a function $f : V \to Y$ satisfies $f(0) = 0$ and the inequality* (3.2) *for all $x_1,x_2,\ldots,x_n \in V$. If there exists a constant $0 < L < 1$ such that φ has* *the property* (3.17) *for all* $x_1, x_2, \ldots, x_n \in V$, *then there exists a unique quadratic-additive function* $F : V \to Y$ *such that*

$$\|f(x) - F(x)\| \le \frac{L}{2n^2(1-L)}\varphi(x, x, \ldots, x) \tag{3.27}$$

for all $x \in V$.

4. Applications

For a given function $f : V \to Y$, we will use the following abbreviation:

$$Af(x, y) := f(x+y) - f(x) - f(y) \tag{4.1}$$

for all $x, y \in V$.

Corollary 4.1. *Let* $f_k : V \to Y$ *be functions for which* $f_k(0) = 0$ *and there exist functions* $\phi_k : V^2 \to [0, \infty)$ *such that*

$$\|Af_k(x, y)\| \le \phi_k(x, y) \tag{4.2}$$

for all $x, y \in V$ *and* $k \in \{1, 2\}$. *If there exists a constant* $0 < L < 1$ *such that*

$$\phi_1(nx, ny) \le nL\phi_1(x, y), \tag{4.3}$$

$$n^2\phi_2(x, y) \le L\phi_2(nx, ny), \tag{4.4}$$

and $\phi_k(x, y) = \phi_k(-x, -y)$ *for all* $x, y \in V$ *and* $k \in \{1, 2\}$, *then there exist additive functions* $F_k : V \to Y$ *such that*

$$\|f_1(x) - F_1(x)\| \le \frac{\Phi_1(x)}{2n(1-L)}, \tag{4.5}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L\Phi_2(x)}{2n^2(1-L)}, \tag{4.6}$$

for all $x \in V \setminus \{0\}$, *where* $\Phi_k : V \to Y$ *are defined by*

$$\Phi_k(x) := n(n-1)\phi_k(x, -x) + 2\sum_{j=1}^{n-1}\phi_k(jx, x) \tag{4.7}$$

for all $x \in V$ and $k \in \{1,2\}$. In particular, the functions F_1 and F_2 are represented by

$$F_1(x) = \lim_{m\to\infty} \frac{f_1(n^m x)}{n^m}, \tag{4.8}$$

$$F_2(x) = \lim_{m\to\infty} n^m f_2\left(\frac{x}{n^m}\right) \tag{4.9}$$

for all $x \in V$.

Proof. By a long manipulation, we obtain

$$Df_k(x_1, x_2, \dots, x_n) = \sum_{1\le i,j\le n,\ i\ne j} Af_k(x_i, -x_j) + 2\sum_{j=1}^{n-1} Af_k\left(\sum_{i=1}^{j} x_i, x_{j+1}\right) \tag{4.10}$$

for all $x_1, x_2, \dots, x_n \in V$ and $k \in \{1,2\}$. If we put

$$\varphi_k(x_1, x_2, \dots, x_n) := \sum_{1\le i,j\le n,\ i\ne j} \phi_k(x_i, -x_j) + 2\sum_{j=1}^{n-1} \phi_k\left(\sum_{i=1}^{j} x_i, x_{j+1}\right) \tag{4.11}$$

for all $x_1, x_2, \dots, x_n \in V$ and $k \in \{1,2\}$, then it follows from (4.2) and (4.10) that

$$\left\|Df_k(x_1, x_2, \dots, x_n)\right\| \le \varphi_k(x_1, x_2, \dots, x_n) \tag{4.12}$$

for all $x_1, x_2, \dots, x_n \in V$ and $k \in \{1,2\}$. By (4.3) and (4.4), we know that φ_1 and φ_2 satisfy (3.3) and (3.17), respectively. Furthermore, $\varphi_k(x_1, x_2, \dots, x_n) = \varphi_k(-x_1, -x_2, \dots, -x_n)$ for all $x_1, x_2, \dots, x_n \in V$ and $k \in \{1,2\}$.

Therefore, according to Corollary 3.2, there exists a unique quadratic-additive function $F_1 : V \to Y$ satisfying (4.5) and F_1 is represented by (3.6). By (4.2) and (4.3), we see

$$\begin{aligned}
\lim_{m\to\infty}\left\|\frac{f_1(n^m x) + f_1(-n^m x)}{2n^m}\right\| &= \lim_{m\to\infty}\left\|\frac{f_1(n^m x) + f_1(-n^m x) - f_1(0)}{2n^m}\right\| \\
&= \lim_{m\to\infty}\frac{1}{2n^m}\left\|Af_1(n^m x, -n^m x)\right\| \\
&\le \lim_{m\to\infty}\frac{1}{2n^m}\phi_1(n^m x, -n^m x) \\
&\le \lim_{m\to\infty}\frac{L^m}{2}\phi_1(x, -x) \\
&= 0, \\
\lim_{m\to\infty}\left\|\frac{f_1(n^m x) + f_1(-n^m x)}{2n^{2m}}\right\| &\le \lim_{m\to\infty}\frac{L^m}{2n^m}\phi_1(x, -x) = 0
\end{aligned} \tag{4.13}$$

for all $x \in V$. From these and (3.6), we get (4.8).

Moreover, we have

$$\left\| \frac{Af_1(n^m x, n^m y)}{n^m} \right\| \le \frac{\phi_1(n^m x, n^m y)}{n^m} \le L^m \phi_1(x,y) \tag{4.14}$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the aforementioned inequality, we get

$$AF_1(x,y) = 0 \tag{4.15}$$

for all $x, y \in V$.

Now, according to Corollary 3.4, there exists a unique function $F_2 : V \to Y$ satisfying (4.6) and F_2 is represented by (3.19). Due to (4.2) and (4.4), we get

$$\begin{aligned} \lim_{m\to\infty} \frac{n^{2m}}{2} \left\| f_2\left(\frac{x}{n^m}\right) + f_2\left(-\frac{x}{n^m}\right) \right\| &= \lim_{m\to\infty} \frac{n^{2m}}{2} \left\| Af_2\left(\frac{x}{n^m}, -\frac{x}{n^m}\right) \right\| \\ &\le \lim_{m\to\infty} \frac{n^{2m}}{2} \phi_2\left(\frac{x}{n^m}, -\frac{x}{n^m}\right) \\ &\le \lim_{m\to\infty} \frac{L^m}{2} \phi_2(x,-x) \\ &= 0 \end{aligned} \tag{4.16}$$

as well as

$$\lim_{m\to\infty} \frac{n^m}{2} \left\| f_2\left(\frac{x}{n^m}\right) + f_2\left(-\frac{x}{n^m}\right) \right\| \le \lim_{m\to\infty} \frac{L^m}{2n^m} \phi_2(x,-x) = 0 \tag{4.17}$$

for all $x \in V$. From these and (3.19), we get (4.9).

Moreover, we have

$$\left\| n^m Af_2\left(\frac{x}{n^m}, \frac{y}{n^m}\right) \right\| \le n^m \phi_2\left(\frac{x}{n^m}, \frac{y}{n^m}\right) \le \frac{L^m}{n^m} \phi_2(x,y) \tag{4.18}$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the aforementioned inequality, we get

$$AF_2(x,y) = 0 \tag{4.19}$$

for all $x, y \in V$. □

Corollary 4.2. *Let X be a normed space and $p \in (-\infty, 1) \cup (2, \infty)$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\left\| Df(x_1, x_2, \ldots, x_n) \right\| \le \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \right) \tag{4.20}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and for some $\theta \geq 0$, then there exists a unique function $F : X \to Y$ such that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\|f(x) - F(x)\| \leq \begin{cases} \dfrac{(n+1)\theta\|x\|^p}{2(n-n^p)} & \text{if } p < 1, \\ \dfrac{n\theta\|x\|^p}{n^p - n^2} & \text{if } p > 2 \end{cases} \tag{4.21}$$

for all $x \in X \setminus \{0\}$.

Proof. If we put

$$\varphi(x_1, x_2, \ldots, x_n) := \theta\left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right) \tag{4.22}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$L := \begin{cases} n^{p-1} & \text{if } p < 1, \\ n^{2-p} & \text{if } p > 2, \end{cases} \tag{4.23}$$

then our assertions follow from Theorems 3.1 and 3.3. □

Corollary 4.3. *Let X be a normed space. If a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \theta \sum_{1 \leq i \leq n,\ x_i \neq 0} \|x_i\|^p \tag{4.24}$$

for all $x_1, x_2, \ldots, x_n \in X$ and for some $\theta \geq 0$ and $p \in (-\infty, 1) \cup (2, \infty)$, then there exists a unique quadratic-additive function $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \dfrac{n\theta\|x\|^p}{2(n-n^p)} & \text{if } p < 1, \\ \dfrac{n\theta\|x\|^p}{2(n^p - n^2)} & \text{if } p > 2 \end{cases} \tag{4.25}$$

for all $x \in X \setminus \{0\}$. In particular, if $p < 0$, then f is a quadratic-additive function.

Proof. This corollary follows from Corollaries 3.2 and 3.4 by putting

$$\varphi(x_1, x_2, \ldots, x_n) := \theta \sum_{1 \leq i \leq n,\ x_i \neq 0} \|x_i\|^p \tag{4.26}$$

for all $x_1, x_2, \ldots, x_n \in X$ with L defined in (4.23). □

Corollary 4.4. *Let X be a normed space and let $p_1, p_2, \ldots, p_n$ be real constants with $p_1+p_2+\cdots+p_n \in (-\infty, 1) \cup (2, \infty)$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and the inequality*

$$\left\| Df(x_1, x_2, \ldots, x_n) \right\| \le \theta \|x_1\|^{p_1} \|x_2\|^{p_2} \cdots \|x_n\|^{p_n} \tag{4.27}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, then there exists a unique function $F : X \to Y$ such that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and

$$\left\| f(x) - F(x) \right\| \le \begin{cases} \dfrac{(n+1)\theta\|x\|^p}{2n(n-n^p)} & \text{if } p < 1, \\ \dfrac{\theta\|x\|^p}{n^p - n^2} & \text{if } p > 2 \end{cases} \tag{4.28}$$

for all $x \in X \setminus \{0\}$, where $p := p_1 + p_2 + \cdots + p_n$.

Proof. If we put

$$\varphi(x_1, x_2, \ldots, x_n) := \theta \|x_1\|^{p_1} \|x_2\|^{p_2} \cdots \|x_n\|^{p_n} \tag{4.29}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$ and L is defined in (4.23), then our assertion follows from Theorems 3.1 and 3.3. □

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