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Research Article

Periodic Solutions of a Class of Fourth-Order Superlinear Differential Equations

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This paper deals with the periodic solutions of a class of fourth-order superlinear differential equations. By using the classical variational techniques and symmetric mountain pass lemma, the periodic solutions of a single equation in literature are extended to that of equations, and also, the cubic growth of nonlinear term is extended to a general form of superlinear growth.

1. Introduction

The existence of periodic solutions of fourth-order differential equations has been studied by more and more researchers [1–6]. The application methods contain mainly Clark theorem [2–4], Cone theory [6], and so on.

For a single equation, Tersian and Chaparova [2] study the existence of infinitely many unbounded solutions, using symmetric mountain pass lemma:

$$\begin{aligned} &u^{iv} - pu'' + a(x)u - b(x)u^3 = 0, \quad x \in \mathbf{R}, \\ &u(0) = u(L) = 0, \qquad u''(0) = u''(L) = 0. \end{aligned} \tag{1.1}$$

It is a natural problem to wonder whether symmetric mountain pass lemma method may be applied not only to single equations but also to systems of differential equations.

In this paper we study the existence of periodic solutions of the fourth-order equations, by making use of the classical variational techniques and symmetric mountain pass lemma

$$\begin{gathered} u^{(4)} - cu'' + a(x)u - \frac{\partial F(x,u,v)}{\partial u} = 0, \quad 0 < x < L, \\ v^{(4)} - dv'' + b(x)v - \frac{\partial F(x,u,v)}{\partial v} = 0, \quad 0 < x < L, \\ u(0) = u''(0) = u(L) = u''(L) = 0, \\ v(0) = v''(0) = v(L) = v''(L) = 0. \end{gathered} \tag{1.2}$$

Through studying System (1.2), (1.1) of the corresponding conclusions are extended.

The paper is organized as follows. In Section 2, we consider the result of System (1.2) under certain conditions. In Section 3, we prove the main result of this paper and give an example.

2. Main Result

In this paper, we state our main result. First we give the following list of assumptions on the parameters in System (1.2):

(**A**) $a(x) > 0$, $b(x) > 0$, $c > -\pi^2/L^2$, $d > -\pi^2/L^2$.

($\mathbf{F_1}$) F is an even functional about (u,v). That is, $F(x,-u,-v) = F(x,u,v)$ for every $(u,v) \in \mathbf{R}^2$.

($\mathbf{F_2}$) There exists $\beta > 2$, as $u^2 + v^2 \neq 0$, we have

$$u \cdot \frac{\partial F(x,u,v)}{\partial u} + v \cdot \frac{\partial F(x,u,v)}{\partial v} \geq \beta F(x,u,v) > 0 \quad \text{for every } x \in \mathbf{R}. \tag{2.1}$$

($\mathbf{F_3}$) $F(x,u,v) = o(u^2+v^2)$ with respect to x consistently, as $u^2 + v^2 \to 0$.

Denote $a_1 = \min_{x\in[0,L]} a(x)$, $a_2 = \max_{x\in[0,L]} a(x)$, $b_1 = \min_{x\in[0,L]} b(x)$, $b_2 = \max_{x\in[0,L]} b(x)$.

From condition (**A**), we obtain $a_i > 0$, $b_i > 0$, when $i = 1,2$.

Remark 2.1. Let $z = (u,v) \in \mathbf{R}^2$, then condition ($\mathbf{F_2}$) is transformed to

$$(\nabla F(x,z), z) > \beta F(x,z) > 0 \quad \text{for every } z \neq 0, \tag{2.2}$$

where $(\cdot,\cdot)$ represents the usual inner product in $\mathbf{R}^2$.

Remark 2.2. From ($\mathbf{F_3}$), we obtain $\lim_{|z|\to 0} F(x,z)/|z|^2 = 0$, where $|\cdot|$ represents normal norm in $\mathbf{R}^2$. Besides, from the continuity of F, we obtain $F(x,0,0) = 0$.

Our main result is as follows.

Theorem 2.3. *Suppose $a(x)$, $b(x)$, and F satisfy* $(\mathbf{A})$, $(\mathbf{F_1})$–$(\mathbf{F_3})$. *Then System* (1.2) *has infinitely many distinct pairs of solutions $z_n = (u_n, v_n)$, which are critical points of the functional $I : X \to \mathbf{R}$, and $I(z_n) \to \infty$ as $n \to \infty$.*

In this paper, the existence of periodic solutions of a single equation in System (1.1) are extended to the case of equations, and also the cubic growth of nonlinear term is extended to a general form of superlinear growth.

3. Variational Structure and the Proof of Result

In this section, we prove the main result stated in Section 2.

3.1. Variational Structure

Denote

$$X(L) = \left(H^2(0,L) \cap H_0^1(0,L)\right)^2. \tag{3.1}$$

Then $X(L)$ is a Hilbert space. The norm is

$$\|z\|^2 = \|u\|_c^2 + \|v\|_d^2, \tag{3.2}$$

where

$$\begin{aligned} \|u\|_c &= \left\{\int_0^L \left[\left|u''(x)\right|^2 + c\left|u'(x)\right|^2 + a(x)|u(x)|^2\right]dx\right\}^{1/2}, \\ \|v\|_d &= \left\{\int_0^L \left[\left|v''(x)\right|^2 + d\left|v'(x)\right|^2 + b(x)|v(x)|^2\right]dx\right\}^{1/2}, \end{aligned} \tag{3.3}$$

$z = (u, v) \in X(L)$. The corresponding inner product are

$$\begin{aligned} \langle z_1, z_2\rangle| &= \int_0^L \left[\left(z_1'', z_2''\right) + c\left(u_1', u_2'\right) + d\left(v_1', v_2'\right) + a(x)(u_1, u_2) + b(x)(v_1, v_2)\right]dx, \\ \langle u_1, u_2\rangle|_c &= \int_0^L \left[\left(u_1'', u_2''\right) + c\left(u_1', u_2'\right) + a(x)(u_1, u_2)\right]dx, \\ \langle v_1, v_2\rangle|_d &= \int_0^L \left[\left(v_1'', v_2''\right) + d\left(v_1', v_2'\right) + b(x)(v_1, v_2)\right]dx. \end{aligned} \tag{3.4}$$

For every $z = (u, v) \in X(L)$, using Poincaré inequality [7], we obtain

$$\int_0^L u^2 dx \le \frac{L_2}{\pi^2} \int_0^L u'^2 dx, \qquad \int_0^L u'^2 dx \le \frac{L_2}{\pi^2} \int_0^L u''^2 dx. \tag{3.5}$$

Thus, we can define another norm $\|\cdot\|_1$ in $X(L)$. That is, for every $z \in X(L)$,

$$\|z\|_1 = \left\{ \int_0^L \left|z''(x)\right|^2 dx \right\}^{1/2}. \tag{3.6}$$

The inner product in $X(L)$ as follows:

$$\langle z_1, z_2 \rangle|_1 = \int_0^L \left(z_1''(x), z_2''(x)\right) dx, \quad z_1, z_2 \in X(L). \tag{3.7}$$

The two different norms (3.2) and (3.6) are equivalent in $X(L)$.

In this section we consider System (1.2). The Fréchet derivative of I is given by the following:

$$I(u, v) = \frac{1}{2} \int_0^L \left[u''^2 + cu'^2 + a(x)u^2 + v''^2 + dv'^2 + b(x)v^2\right] dx - \int_0^L F(x, u, v) dx, \tag{3.8}$$

where $z = (u, v) \in X(L)$.

Remark 3.1. In general, the growth of F is limited by the differentiability of functional I, but we apply truncation techniques in [8]. First, introduce auxiliary functional and the auxiliary functional is Fréchet differentiable. Second, we use critical point theory to prove the existence of critical point of auxiliary functional, then prove the existence of the original equation. However, in order to avoid technical complexity, we assume directly functional I is Fréchet differentiable.

In fact, for every $z = (u, v) \in X(L)$, $\overline{z} = (\overline{u}, \overline{v}) \in X(L)$, we obtain

$$\left\langle I'(z), \overline{z} \right\rangle = \left\langle I_u'(u, v), \overline{u} \right\rangle + \left\langle I_v'(u, v), \overline{v} \right\rangle, \tag{3.9}$$

where

$$\begin{aligned} \left\langle I_u'(u, v), \overline{u} \right\rangle &= \int_0^L \left[u''\overline{u}'' + cu'\overline{u}' + a(x)u\overline{u} - \frac{\partial F(x, u, v)}{\partial u}\overline{u}\right] dx, \\ \left\langle I_v'(u, v), \overline{v} \right\rangle &= \int_0^L \left[v''\overline{v}'' + cv'\overline{v}' + a(x)v\overline{v} - \frac{\partial F(x, u, v)}{\partial v}\overline{v}\right] dx. \end{aligned} \tag{3.10}$$

and $I_u'(u, v), I_v'(u, v) \in [H^2(0, L) \cap H_0^1(0, L)]^*$, $I'(z) \in X(L)^*$.

It is similar to the discussion of [8], the solutions of System (1.2) corresponds to the critical point of the functional I, so we need to discuss the critical point of functional I. In order to prove Theorem 2.3, we introduce below definition and lemma.

Definition 3.2 (see [9]). Let X be a real Banach space, $I \in C^1(X, R)$, I is a Fréchet continuously differentiable functional in $X(L)$. I is said to be satisfying *Palais-Smale* (PS) condition if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $\{I'(u_n)\} \to 0$ as $j \to \infty$, possesses a convergent subsequence.

Lemma 3.3 (see [8]). *Let X be an infinite dimensional Banach space and $(X_n)_n$ be a sequence of finite dimensional subspaces of X such that* $\dim X_n = n$,

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X, \qquad \overline{\bigcup_{n=1}^{\infty} X_n} = X. \tag{3.11}$$

Let $I \in C^1(X, R)$ be an even functional, $I(0) = 0$, and I satisfy (PS) condition. Suppose that

($\mathbf{A_1}$) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and*

($\mathbf{A_2}$) *for every n there is an $R_n > 0$ such that $I \leq 0$ on $X_n \setminus B_{R_n}$.*

Then I possesses infinitely many pairs of critical points with unbounded sequence of critical values.

3.2. *The Proof of Result*

Step 1 (Functional I satisfies (PS) condition). Let $\{z_n\} = \{(u_n, v_n)\}$ be a (PS) sequence in X, that is, $\{I(z_n)\}$ is bounded and $I'(z_n) \to 0$, as $n \to \infty$. Suppose that $\{z_n\}$ is unbounded in X, that is, $\|z_n\| \to \infty$ as $n \to \infty$. Since

$$I(z_n) + \frac{1}{\gamma}\|I'(z_n)\|\|z_n\| \geq I(z_n) - \frac{1}{\gamma}\langle I'(z_n), z_n\rangle = \frac{1}{\gamma}\|z_n\|^2, \tag{3.12}$$

it follows that

$$\frac{I(z_n)}{\|z_n\|^2} + \frac{\|I'(z_n)\|}{\gamma\|z_n\|} \geq \frac{1}{\gamma}, \tag{3.13}$$

where $\gamma \geq 4$. Letting $n \to \infty$ in (3.13), we have a contradiction with $\|z_n\| \to \infty$ as $n \to \infty$.

Therefore $\{z_n\}$ is a bounded sequence in $X(L)$. Passing if necessary to a subsequence we may assume that $\{z_n\}$ is weakly convergent to a function $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, and $z_n \to z$ in $C[(0, L)]$.

From the Lebesgue theorem, $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, and $z_n \to z$ in $C[(0,L)]$, letting $n \to \infty$ in (3.9)

$$\begin{aligned} \left\langle I'(z_n, z_n)\right\rangle &= \|z_n\|^2 - \int_0^L \frac{\partial F(x,u_n,v_n)}{\partial u} u_n dx - \frac{\partial F(x,u_n,v_n)}{\partial v} v_n dx, \\ \left\langle I'(z_n, z)\right\rangle &= \langle z_n, z\rangle - \int_0^L \frac{\partial F(x,u_n,v_n)}{\partial u} u\, dx - \frac{\partial F(x,u_n,v_n)}{\partial v} v\, dx, \end{aligned} \tag{3.14}$$

we obtain

$$\lim_{n\to\infty} \|z_n\|^2 = \int_0^L \frac{\partial F(x,u,v)}{\partial u} u\, dx + \frac{\partial F(x,u,v)}{\partial v} v\, dx = \|z\|^2. \tag{3.15}$$

From (3.15) and $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, we have $\|z_n - z\| \to 0$ as $n \to \infty$.

Remark 3.4. γ is the largest sum of the order of u and v.

Step 2 (Geometric conditions). Let $e_1 = (1,0)$, $e_2 = (0,1)$, then $\{e_1, e_2\}$ constitutes a pair of standard orthogonal base in $\mathbf{R}^2$. Let us define X_{2m} to be the subspace of $X(L)$

$$X_{2m} = \operatorname{span}\left\{\sin\frac{k\pi x}{L} e_i,\ i = 1,2,\ k = 1,2,\dots,m\right\}, \tag{3.16}$$

for every $m \in \mathbf{N}$. We have $\dim X_{2m} = 2m$, $X_1 \subset X_2 \cdots \subset X_{2m} \subset X$, $\overline{\bigcup_{n=1}^{\infty} X_n} = X$.

For a given constant $\rho > 0$, define a bounded closed set $K \subset X_{2m}$

$$K = \left\{ z = (u,v) \in X_{2m} \mid z = \sum_{k=1}^{m}\left[\alpha_k \sin\frac{k\pi x}{L} e_1 + \beta_k \sin\frac{k\pi x}{L} e_2\right], \sum_{k=1}^{m}\left(\alpha_k^2 + \beta_k^2\right) = \rho^2 \right\}. \tag{3.17}$$

Define mapping $H : X_{2m} \to \mathbf{R}^{2m}$. For any $z \in X_{2m}$, we obtain

$$H(z) = \frac{(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_m, \beta_m)}{\rho}. \tag{3.18}$$

It is clear that H is a linear odd mapping. For every $z \in X_{2m}$, we have

$$\begin{aligned} \|z\|_1^2 &= \int_0^L \left[\left|u''(x)\right|^2 + \left|v(x)''\right|^2\right] dx \\ &= \frac{\pi^4}{2L^3} \sum_{k=1}^{m} k^4 \left(\alpha_k^2 + \beta_k^2\right). \end{aligned} \tag{3.19}$$

So

$$\frac{\rho^2\pi^4}{2L^3}|H(z)|^2 \le |z|_1^2 \le \frac{\rho^2(m\pi)^4}{2L^3}|H(z)|^2. \tag{3.20}$$

From (3.20), we obtain H is an odd homeomorphism from X_{2m} to $\mathbf{R}^{2m}$. Then H is an odd homeomorphism from K to S^{2m-1}, since $H(K) = S^{2m-1}$.

On one hand, from functional (3.8) and using Sobolev's embedding theorem, we obtain

$$\begin{aligned} I(z) &\ge \frac{1}{2}\|z\|^2 - \varepsilon|z|^2 L \\ &\ge \frac{1}{2}\|z\|^2 - \varepsilon\frac{\pi^2}{L}\|z\|^2. \end{aligned} \tag{3.21}$$

Thus condition ($\mathbf{A_1}$) is fulfilled if $\varepsilon = L/4\pi^2$, $\rho = \|z\|/2$.

On the other hand, as $-F(x,u,v) < 0$, then there exists σ, such that $-F(x,u,v) < -(1/4)\sigma\|z\|_1^4$.

Denote $A(n) = (n\pi/L)^4 + p(n\pi/L)^2 + a$. From functional (3.8), we obtain

$$\begin{aligned} I(z) &\le \frac{1}{2}\int_0^L \left(z''^2 + pz'^2 + az^2\right)dx - \int_0^L F(x,z)dx \\ &\le \frac{L}{4}A(n)\|z\|_1^2 - \int_0^L F(x,z)dx \\ &\le \frac{L}{4}A(n)\|z\|_1^2 - \frac{L}{4}\sigma\|z\|_1^4, \end{aligned} \tag{3.22}$$

where $p = \max\{c,d\}$, $a = \max\{a_2, b_2\}$. Here choosing $R_n = \|z\|_1 \ge \sqrt{A(n)/\sigma}$, we obtain

$$I(z) \le 0. \tag{3.23}$$

So ($\mathbf{A_2}$) holds. The proof of Theorem 2.3 is completed.

Example 3.5. In System (1.2), consider the problem:

$$F(x,u,v) = p_0(x)u^n + p_1(x)u^{n-1}v + \cdots + p_i(x)u^{n-i}v^i + \cdots + p_{n-1}(x)uv^{n-1} + p_n(x)v^n, \tag{3.24}$$

where $p_i(x) \ge 0$, but there exists at least one $p_i(x) \ne 0$, n is an even and $n \ge 4$, $i = 0,1,2,\ldots,n$.

It is obvious that $F(x,-u,-v) = F(x,u,v)$ and $F(x,u,v) = o(u^2+v^2)$ as $u^2+v^2 \to 0$.

For the superlinear property, we calculate that

$$\begin{aligned} u \cdot \frac{\partial F(x,u,v)}{\partial u} &+ v \cdot \frac{\partial F(x,u,v)}{\partial v} \\ &= np_0(x)u^n + (n-1)p_1(x)u^{n-1}v + \cdots + (n-i)p_i(x)u^{n-i}v^i + \cdots + p_{n-1}(x)uv^{n-1} \\ &\quad + p_1(x)u^{n-1}v + \cdots + ip_i(x)u^{n-i}v^i + \cdots + (n-1)p_{n-1}(x)uv^{n-1} + np_n(x)v^n \\ &= nF(x,u,v) \\ &\geq 4F(x,u,v). \end{aligned} \tag{3.25}$$

Therefore, there exists $\beta = 4 > 2$, as $u^2 + v^2 \neq 0$, we have

$$u \cdot \frac{\partial F(x,u,v)}{\partial u} + v \cdot \frac{\partial F(x,u,v)}{\partial v} \geq 4F(x,u,v) > 0 \quad \text{for every } x \in \mathbf{R}. \tag{3.26}$$

So F satisfies the conditions $(\mathbf{F_1})$–$(\mathbf{F_3})$. We only choose $a(x) > 0$, $b(x) > 0$, $c > -\pi^2/L^2$, $d > -\pi^2/L^2$, then the condition $(\mathbf{A})$ is satisfied. Therefore, System (1.2) has infinitely many distinct pairs of solutions by using Theorem 2.3.

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