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Research Article

Iterative Schemes for Fixed Point Computation of Nonexpansive Mappings

Rudong Chen

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China

Correspondence should be addressed to Rudong Chen, chenrd@tjpu.edu.cn

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Fixed point (especially, the minimum norm fixed point) computation is an interesting topic due to its practical applications in natural science. The purpose of the paper is devoted to finding the common fixed points of an infinite family of nonexpansive mappings. We introduce an iterative algorithm and prove that suggested scheme converges strongly to the common fixed points of an infinite family of nonexpansive mappings under some mild conditions. As a special case, we can find the minimum norm common fixed point of an infinite family of nonexpansive mappings.

1. Introduction

In many problems, it is needed to find a solution with minimum norm. In an abstract way, we may formulate such problems as finding a point $x^\dagger$ with the property

$$x^\dagger \in C, \quad \left\|x^\dagger\right\|^2 = \min_{x\in C}\|x\|^2, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H. In other words, $x^\dagger$ is the (nearest point or metric) projection of the origin onto C,

$$x^\dagger = P_C(0), \tag{1.2}$$

where P_C is the metric (or nearest point) projection from H onto C.

A typical example is the least-squares solution to the constrained linear inverse problem [1]

$$Ax = b, \quad x \in C, \tag{1.3}$$

where A is a bounded linear operator from H to another real Hilbert space H_1 and b is a given point in H_1. The least-squares solution to (1.3) is the least-norm minimizer of the minimization problem

$$\min_{x\in C}\|Ax-b\|^2. \tag{1.4}$$

Recently, some authors consider the minimum norm solution problem by using the iterative algorithm. For some related works, please refer to [2–4]. Yao and Xu [3] introduced the following algorithm:

$$x_{n+1}=P_C\big(\alpha_n f(x_n)+(1-\alpha_n)Tx_n\big),\quad n\geq 0. \tag{1.5}$$

They proved that the sequence $\{x_n\}$ converges in norm to the unique solution $\tilde{x}$ of VI $\langle (I-f)\tilde{x}, x-\tilde{x}\rangle\geq 0$, $x\in \mathrm{Fix}(T)$. Particularly, the sequence $\{x_n\}$ defined by

$$x_{n+1}=P_C((1-\alpha_n)Tx_n),\quad n\geq 0, \tag{1.6}$$

converges to the minimum norm fixed point of T. We note that the authors added an additional assumption, that is, $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n=1$. Iterative algorithm for finding the fixed points of nonexpansive mappings has been considered by many authors, see [5–20].

The purpose of this paper is to extend Yao and Xu's result to an infinite family of nonexpansive mappings $\{T_n\}_{n=0}^{\infty}$. We suggest a new algorithm. Particularly, we drop the above additional assumption and prove the suggested algorithm converges strongly to the common fixed points of $\{T_n\}_{n=0}^{\infty}$. As a special case, we can find the minimum norm fixed point of $\{T_n\}_{n=0}^{\infty}$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, and let C be a nonempty closed convex subset of H. We call $f: C\to H$ a κ-contraction if there exists a constant $\kappa\in[0,1)$ such that $\|f(x)-f(y)\|\leq\kappa\|x-y\|$ for all $x,y\in C$. A bounded linear operator B is said to be strongly positive on H if there exists a constant $\alpha>0$ such that

$$\langle Bx,x\rangle\geq\alpha\|x\|^2,\quad \forall x\in H. \tag{2.1}$$

Recall that the (nearest point or metric) projection from H onto C, denoted by P_C, is defined in such a way that, for each $x\in H$, P_Cx is the unique point in C with the property

$$\|x-P_Cx\|=\min\{\|x-y\|: y\in C\}. \tag{2.2}$$

It is known that P_C satisfies

$$\langle x-y, P_Cx-P_Cy\rangle\geq\|P_Cx-P_Cy\|^2,\quad \forall x,y\in H. \tag{2.3}$$

Moreover, P_C is characterized by the following properties:

$$\begin{gathered}\langle x - P_C x, y - P_C x\rangle \leq 0,\\ \|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2,\end{gathered} \tag{2.4}$$

for all $x \in H$ and $y \in C$.

We also need other sorts of nonlinear operators which are introduced below. Let $T : H \to H$ be a nonlinear operator.

(a) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.

(b) T is firmly nonexpansive if $2T - I$ is nonexpansive. Equivalently, $T = (I + S)/2$, where $S : H \to H$ is nonexpansive. Alternatively, T is firmly nonexpansive if and only if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad x, y \in H. \tag{2.5}$$

(c) T is averaged if $T = (1-\tau)I + \tau S$, where $\tau \in (0,1)$ and $S : H \to H$ is nonexpansive. In this case, we also say that T is τ-averaged. A firmly nonexpansive mapping is $1/2$-averaged.

It is well known that both P_C and $I - P_C$ are firmly nonexpansive. We will need to use the following notation:

(i) $\operatorname{Fix}(T)$ stands for the set of fixed points of T;

(ii) $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x;

(iii) $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

Let $T_1, T_2, \ldots$ be infinite mappings of C into itself, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i \in \mathbf{N}$. For any $n \in \mathbf{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \xi_n T_n U_{n,n+1} + (1-\xi_n)I,\\
&U_{n,n-1} = \xi_{n-1} T_{n-1} U_{n,n} + (1-\xi_{n-1})I,\\
&\vdots\\
&U_{n,k} = \xi_k T_k U_{n,k+1} + (1-\xi_k)I,\\
&U_{n,k-1} = \xi_{k-1} T_{k-1} U_{n,k} + (1-\xi_{k-1})I,\\
&\vdots\\
&U_{n,2} = \xi_2 T_2 U_{n,3} + (1-\xi_2)I,\\
&W_n = U_{n,1} = \xi_1 T_1 U_{n,2} + (1-\xi_1)I.
\end{aligned} \tag{2.6}$$

Such W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_2, T_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_2, \xi_1$. For the iterative algorithm for a finite family of nonexpansive mappings, we refer the reader to [21].

We have the following crucial lemmas concerning W_n which can be found in [22].

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in \mathbf{N}$. Then, for every $x \in C$ and $k \in \mathbf{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \le b < 1$ for any $i \in N$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

The following remark [23] is important to prove our main results.

Remark 2.3. Using Lemma 2.1, one can define a mapping W of C into itself as $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then one has

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0. \tag{2.7}$$

Throughout this paper, we will assume that $0 < \xi_i \le b < 1$ for every $i \in \mathbf{N}$.

Lemma 2.4 (see [24]). *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then T is demiclosed on K, that is, if $x_n \rightharpoonup x \in K$ weakly and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.5 (see [25]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.6 (see [26]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\gamma_n)a_n + \delta_n$ where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

In this section, we introduce our algorithm and prove its strong convergence.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C to C such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$. Let $f : C \to H$ be a κ-contraction and $B : H \to H$ be a self-adjoint, strongly* *positive bounded linear operator with coefficient* $\alpha > 0$*. Let* σ *be a constant such that* $0 < \sigma\kappa < \alpha$*. For an arbitrary initial point* x_0 *belonging to* C*, one defines a sequence* $\{x_n\}_{n\geq 0}$ *iteratively*

$$x_{n+1} = P_C\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B)W_n x_n\right], \quad \forall n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}$ *is a real sequence in* $[0,1]$*. Assume the sequence* $\{\alpha_n\}$ *satisfies the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges in norm to the unique solution* x^* *which solves the following variational inequality:*

$$x^* \in F \text{ such that } \left\langle \sigma f(x^*) - Bx^*, \tilde{x} - x^* \right\rangle \leq 0, \quad \forall \tilde{x} \in F. \tag{3.2}$$

Proof. Let $\tilde{x} \in F$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\| &= \left\| P_C\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B)W_n x_n\right] - \tilde{x} \right\| \\
&\leq \left\| \alpha_n \sigma f(x_n) + (I - \alpha_n B)W_n x_n - \tilde{x} \right\| \\
&\leq \alpha_n \sigma \left\| f(x_n) - f(\tilde{x}) \right\| + \|I - \alpha_n B\| \|W_n x_n - \tilde{x}\| + \alpha_n \left\| \sigma f(\tilde{x}) - B\tilde{x} \right\| \\
&\leq \alpha_n \sigma\kappa \|x_n - \tilde{x}\| + (1 - \alpha_n \alpha)\ \|x_n - \tilde{x}\| + \alpha_n \left\| \sigma f(\tilde{x}) - B\tilde{x} \right\| \\
&= [1 - (\alpha - \sigma\kappa)\alpha_n] \|x_n - \tilde{x}\| + \frac{(\alpha - \sigma\kappa)\alpha_n \left\| f(\tilde{x}) - B\tilde{x} \right\|}{(\alpha - \sigma\kappa)}.
\end{aligned} \tag{3.3}$$

It follows by induction that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\| &\leq \max\left\{ \|x_n - \tilde{x}\|, \frac{\left\| f(\tilde{x}) - B\tilde{x} \right\|}{(\alpha - \sigma\kappa)} \right\} \\
&\leq \max\left\{ \|x_0 - \tilde{x}\|, \frac{\left\| f(\tilde{x}) - B\tilde{x} \right\|}{(\alpha - \sigma\kappa)} \right\}.
\end{aligned} \tag{3.4}$$

This indicates that $\{x_n\}$ is bounded. It is easy to deduce that $\{f(x_n)\}$, $\{W_n x_n\}$, and $\{BW_n x_n\}$ are also bounded.

Set $S = 2P_C - I$. It is known that S is nonexpansive. Note that $W_n = \xi_1 T_1 U_{n,2} x_n + (1 - \xi_1) x_n$. Then, we can rewrite (3.1) as

$$
\begin{aligned}
x_{n+1} &= \frac{I+S}{2}\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B) W_n x_n\right] \\
&= \frac{1-\alpha_n}{2} W_n x_n + \frac{\alpha_n}{2}\left(\sigma f(x_n) - B W_n x_n + W_n x_n\right) \\
&\quad + \frac{1}{2} S\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B) W_n x_n\right] \\
&= \frac{1-\alpha_n}{2}\left[(1-\xi) I + \xi T_1 U_{n,2}\right] x_n + \frac{\alpha_n}{2}\left(\sigma f(x_n) - B W_n x_n + W_n x_n\right) \\
&\quad + \frac{1}{2} S\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B) W_n x_n\right] \\
&= \frac{(1-\xi)(1-\alpha_n)}{2} x_n + \frac{\xi(1-\alpha_n)}{2} T_1 U_{n,2} x_n + \frac{\alpha_n}{2}\left(\sigma f(x_n) - B W_n x_n + W_n x_n\right) \\
&\quad + \frac{1}{2} S\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B) W_n x_n\right].
\end{aligned} \tag{3.5}
$$

Note that

$$
\begin{gathered}
0 < \lim_{n \to \infty} \frac{(1-\xi)(1-\alpha_n)}{2} = \frac{1-\xi}{2} < 1, \\
\frac{\xi(1-\alpha_n)}{2} + \frac{1}{2} = \frac{1+\xi}{2} - \frac{\xi}{2}\alpha_n.
\end{gathered} \tag{3.6}
$$

From (3.5), we have

$$
\begin{aligned}
x_{n+1} &= \left[1 - \left(\frac{1+\xi}{2} + \frac{1-\xi}{2}\alpha_n\right)\right] x_n + \left(\frac{1+\xi}{2} + \frac{1-\xi}{2}\alpha_n\right) \\
&\qquad \times \frac{(\xi(1-\alpha_n)/2) T_1 U_{n,2} x_n + (\alpha_n/2)\left(\sigma f(x_n) - B W_n x_n + W_n x_n\right)}{((1+\xi)/2) + ((1-\xi)/2)\alpha_n} \\
&\qquad + \frac{(1/2) S\left[\alpha_n \sigma f(x_n) + (I - \alpha_n B) W_n x_n\right]}{((1+\xi)/2) + ((1-\xi)/2)\alpha_n} \\
&= \left[1 - \left(\frac{1+\xi}{2} + \frac{1-\xi}{2}\alpha_n\right)\right] x_n + \left(\frac{1+\xi}{2} + \frac{1-\xi}{2}\alpha_n\right) y_n,
\end{aligned} \tag{3.7}
$$

where

$$y_n = \frac{(\xi(1-\alpha_n)/2)T_1U_{n,2}x_n + (\alpha_n/2)\left(\sigma f(x_n) - BW_nx_n + W_nx_n\right)}{((1+\xi)/2) + ((1-\xi)/2)\alpha_n} + \frac{(1/2)S\left[\alpha_n\sigma f(x_n) + (I-\alpha_nB)W_nx_n\right]}{((1+\xi)/2) + ((1-\xi)/2)\alpha_n}$$
$$= \frac{\xi(1-\alpha_n)T_1U_{n,2}x_n + \alpha_n\left(\sigma f(x_n) - BW_nx_n + W_nx_n\right) + S\left[\alpha_n\sigma f(x_n) + (I-\alpha_nB)W_nx_n\right]}{1+\xi+(1-\xi)\alpha_n}. \quad (3.8)$$

Set $z_n = \sigma f(x_n) - BW_nx_n + W_nx_n$ and $\tilde{z}_n = \alpha_n\sigma f(x_n) + (I-\alpha_nB)W_nx_n$ for all n. Then

$$y_n = \frac{\xi(1-\alpha_n)T_1U_{n,2}x_n + \alpha_nz_n + S\tilde{z}_n}{1+\xi+(1-\xi)\alpha_n}, \quad \forall n \geq 0. \quad (3.9)$$

It follows that

$$\begin{aligned}
y_{n+1} - y_n &= \frac{\xi(1-\alpha_{n+1})T_1U_{n+1,2}x_{n+1} + \alpha_{n+1}z_{n+1} + S\tilde{z}_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}} \\
&\quad - \frac{\xi(1-\alpha_n)T_1U_{n,2}x_n + \alpha_nz_n + S\tilde{z}_n}{1+\xi+(1-\xi)\alpha_n} \\
&= \frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}}\left(T_1U_{n+1,2}x_{n+1} - T_1U_{n,2}x_n\right) \\
&\quad + \left(\frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}} - \frac{\xi(1-\alpha_n)}{1+\xi+(1-\xi)\alpha_n}\right)T_1U_{n,2}x_n \\
&\quad + \frac{\alpha_{n+1}z_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}} - \frac{\alpha_nz_n}{1+\xi+(1-\xi)\alpha_n} \\
&\quad + \frac{S\tilde{z}_{n+1} - S\tilde{z}_n}{1+\xi+(1-\xi)\alpha_{n+1}} + \left(\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}} - \frac{1}{1+\xi+(1-\xi)\alpha_n}\right)S\tilde{z}_n.
\end{aligned} \quad (3.10)$$

Thus,

$$\begin{aligned}
\left\|y_{n+1} - y_n\right\| &\leq \frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}}\left\|T_1U_{n+1,2}x_{n+1} - T_1U_{n,2}x_n\right\| \\
&\quad + \left|\frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}} - \frac{\xi(1-\alpha_n)}{1+\xi+(1-\xi)\alpha_n}\right|\left\|T_1U_{n,2}x_n\right\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}}\|z_{n+1}\|+\frac{\alpha_n}{1+\xi+(1-\xi)\alpha_n}\|z_n\| \\
&+\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}}\|S\widetilde{z}_{n+1}-S\widetilde{z}_n\| \\
&+\left|\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}}-\frac{1}{1+\xi+(1-\xi)\alpha_n}\right|\|S\widetilde{z}_n\|.
\end{aligned}
\tag{3.11}
$$

From the nonexpansivity of S, we get

$$
\begin{aligned}
\|S\widetilde{z}_{n+1}-S\widetilde{z}_n\| &\le \|\widetilde{z}_{n+1}-\widetilde{z}_n\| \\
&= \left\|\alpha_{n+1}\sigma f(x_{n+1})+(I-\alpha_{n+1}B)W_{n+1}x_{n+1}-\left(\alpha_n\sigma f(x_n)+(I-\alpha_nB)W_nx_n\right)\right\| \\
&\le \alpha_{n+1}\left\|\sigma f(x_{n+1})-BW_{n+1}x_{n+1}\right\|+\alpha_n\left\|\sigma f(x_n)-BW_nx_n\right\|+\|W_{n+1}x_{n+1}-W_nx_n\| \\
&\le \alpha_{n+1}\left\|\sigma f(x_{n+1})-BW_{n+1}x_{n+1}\right\|+\alpha_n\left\|\sigma f(x_n)-BW_nx_n\right\| \\
&\quad+\|W_{n+1}x_{n+1}-W_{n+1}x_n\|+\|W_{n+1}x_n-W_nx_n\| \\
&\le \alpha_{n+1}\left\|\sigma f(x_{n+1})-BW_{n+1}x_{n+1}\right\|+\alpha_n\left\|\sigma f(x_n)-BW_nx_n\right\|+\|x_{n+1}-x_n\| \\
&\quad+\|W_{n+1}x_n-W_nx_n\|.
\end{aligned}
\tag{3.12}
$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$
\begin{aligned}
\|T_1U_{n+1,2}x_n-T_1U_{n,2}x_n\| &\le \|U_{n+1,2}x_n-U_{n,2}x_n\| \\
&= \|\xi_2T_2U_{n+1,3}x_n-\xi_2T_2U_{n,3}x_n\| \\
&\le \xi_2\|U_{n+1,3}x_n-U_{n,3}x_n\| \\
&\le \cdots \\
&\le \xi_2\cdots\xi_n\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\| \\
&\le M\prod_{i=2}^{n}\xi_i,
\end{aligned}
\tag{3.13}
$$

where $M>0$ is a constant such that $\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|\le M$ for all $n\ge 0$. So,

$$
\begin{aligned}
\|T_1U_{n+1,2}x_{n+1}-T_1U_{n,2}x_n\| &\le \|T_1U_{n+1,2}x_{n+1}-T_1U_{n+1,2}x_n\|+\|T_1U_{n+1,2}x_n-T_1U_{n,2}x_n\| \\
&\le \|x_{n+1}-x_n\|+M\prod_{i=2}^{n}\xi_i.
\end{aligned}
\tag{3.14}
$$

Hence,

$$\begin{aligned}
\|y_{n+1}-y_n\| \le{}& \frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}}\|x_{n+1}-x_n\| + M\prod_{i=2}^{n}\xi_i \\
&+\left|\frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}}-\frac{\xi(1-\alpha_n)}{1+\xi+(1-\xi)\alpha_n}\right|\|T_1U_{n,2}x_n\| \\
&+\frac{\alpha_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}}\|z_{n+1}\|+\frac{\alpha_n}{1+\xi+(1-\xi)\alpha_n}\|z_n\| \\
&+\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}} \\
&\times\left(\alpha_{n+1}\|\sigma f(x_{n+1})-BW_{n+1}x_{n+1}\|+\alpha_n\|\sigma f(x_n)-BW_nx_n\|+\|x_{n+1}-x_n\|\right) \\
&+\|W_{n+1}x_n-W_nx_n\|+\left|\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}}-\frac{1}{1+\xi+(1-\xi)\alpha_n}\right|\|S\tilde{z}_n\| \\
={}& \frac{1+\xi-\xi\alpha_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}}\|x_{n+1}-x_n\| \\
&+\left|\frac{\xi(1-\alpha_{n+1})}{1+\xi+(1-\xi)\alpha_{n+1}}-\frac{\xi(1-\alpha_n)}{1+\xi+(1-\xi)\alpha_n}\right|\|T_1U_{n,2}x_n\| \\
&+\frac{\alpha_{n+1}}{1+\xi+(1-\xi)\alpha_{n+1}}\|z_{n+1}\|+\frac{\alpha_n}{1+\xi+(1-\xi)\alpha_n}\|z_n\| \\
&+\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}} \\
&\times\left(\alpha_{n+1}\|\sigma f(x_{n+1})-BW_{n+1}x_{n+1}\|+\alpha_n\|\sigma f(x_n)-BW_nx_n\|\right) \\
&+\|W_{n+1}x_n-W_nx_n\|+\left|\frac{1}{1+\xi+(1-\xi)\alpha_{n+1}}-\frac{1}{1+\xi+(1-\xi)\alpha_n}\right|\|S\tilde{z}_n\|.
\end{aligned} \tag{3.15}$$

Since $\alpha_n \to 0$, we have $\xi(1-\alpha_{n+1})/(1+\xi+(1-\xi)\alpha_{n+1})-\xi(1-\alpha_n)/(1+\xi+(1-\xi)\alpha_n) \to 0$, $\|W_{n+1}x_n-W_nx_n\| \to 0$, and $1/(1+\xi+(1-\xi)\alpha_{n+1})-1/(1+\xi+(1-\xi)\alpha_n) \to 0$. Therefore,

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\le 0. \tag{3.16}$$

By Lemma 2.5, we get

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.17}$$

Hence, from (3.7), we deduce

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\left(\frac{1+\xi}{2}+\frac{1-\xi}{2}\alpha_n\right)\|y_n-x_n\|=0. \tag{3.18}$$

Observe that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|P_C\left[\alpha_n\sigma f(x_n)+(I-\alpha_n B)W_n\right]x_n - Wx_n\right\| \\ &\le \alpha_k\left\|\sigma f(x_n)-BW_nx_n\right\| + \|W_nx_n - Wx_n\| \longrightarrow 0.\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we deduce

$$\lim_{k\to\infty}\|Wx_n - x_n\| = 0. \tag{3.20}$$

Next we prove

$$\limsup_{k\to\infty}\left\langle \sigma f(x^*) - Bx^*, x^k - x^*\right\rangle \le 0, \tag{3.21}$$

where x^* is the unique solution of VI (3.2).

Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \sigma f(x^*) - Bx^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle \sigma f(x^*) - Bx^*, x_{n_i} - x^*\rangle. \tag{3.22}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ which converges weakly to a point $\tilde{x}$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges weakly to $\tilde{x}$. Therefore, from (3.20) and Lemma 2.4, we have $x_{n_i} \rightharpoonup \tilde{x} \in \mathrm{Fix}(W) = F$. Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle \sigma f(x^*) - Bx^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle \sigma f(x^*) - Bx^*, x_{n_i} - x^*\rangle \\ &= \langle \sigma f(x^*) - Bx^*, \tilde{x} - x^*\rangle \le 0.\end{aligned} \tag{3.23}$$

Finally, we show that $x_n \to x^*$. We observe that

$$\|x_{n+1} - x^*\|^2 = \langle x_{n+1} - \tilde{z}_n, x_{n+1} - x^*\rangle + \langle \tilde{z}_n - x^*, x_{n+1} - x^*\rangle. \tag{3.24}$$

Since $\langle x_{n+1} - \tilde{z}_n, x_{n+1} - x^* \rangle \le 0$, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \langle \tilde{z}_n - x^*, x_{n+1} - x^* \rangle \\
&= \left\langle \alpha_n \sigma \left(f(x_n) - f(x^*) \right) + (I - \alpha_n B)(W_n x_n - x^*), x_{n+1} - x^* \right\rangle \\
&\quad + \alpha_n \left\langle \sigma f(x^*) - Bx^*, x_{n+1} - x^* \right\rangle \\
&\le \left(\alpha_n \sigma \left\| f(x_n) - f(x^*) \right\| + \|I - \alpha_n B\| \|W_n x_n - x^*\| \right) \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \left\langle \sigma f(x^*) - Bx^*, x_{n+1} - x^* \right\rangle \\
&\le (1 - \alpha_n(\alpha - \sigma\kappa)) \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \left\langle \sigma f(x^*) - Bx^*, x_{n+1} - x^* \right\rangle \\
&\le \frac{[1 - \alpha_n(\alpha - \sigma\kappa)]^2}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|x_{n+1} - x^*\|^2 \\
&\quad + \alpha_n \left\langle \sigma f(x^*) - Bx^*, x_{n+1} - x^* \right\rangle.
\end{aligned} \tag{3.25}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le [1 - \alpha_n(\alpha - \sigma\kappa)] \|x_n - x^*\|^2 \\
+ 2\alpha_n \left\langle \sigma f(x^*) - Bx^*, x_{n+1} - x^* \right\rangle.
\end{aligned} \tag{3.26}$$

Hence, all conditions of Lemma 2.6 are satisfied. Therefore, we immediately deduce that $x_k \to x^*$. This completes the proof. □

From (3.1) and Theorem 3.1, we can deduce easily the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings from C to C such that the common fixed point set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For an arbitrary initial point x_0, one defines a sequence $\{x_n\}_{n \ge 0}$ iteratively*

$$x_{n+1} = P_C[(1 - \alpha_n) W_n x_n], \quad n \ge 0, \tag{3.27}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Assume the sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n \to \infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by (3.27) *converges to the minimum norm common fixed point x^* of $\{T_n\}$.*

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