Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 437912, 13 pages
doi:10.1155/2012/437912

Research Article

Existence of One-Signed Solutions of Discrete Second-Order Periodic Boundary Value Problems

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Received 2 September 2011; Accepted 24 October 2011

Academic Editor: Agacik Zafer

We prove the existence of one-signed periodic solutions of second-order nonlinear difference equation on a finite discrete segment with periodic boundary conditions by combining some properties of Green's function with the fixed-point theorem in cones.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Z}$ be the integers set, $T, a, b \in \mathbb{Z}$ with $T > 2$, $a > b$, and $[a, b]_{\mathbb{Z}} = \{a, a+1, \ldots, b\}$.

In recent years, the existence and multiplicity of positive solutions of periodic boundary value problems for difference equations have been studied extensively, see [1–5] and the references therein. In 2003, Atici and Cabada [2] studied the existence of solutions of second-order difference equation boundary value problem

$$\begin{gathered} \Delta^2 y(n-1) + a(n)y(n) + f\big(n, y(n)\big) = 0, \quad n \in [1, T]_{\mathbb{Z}}, \\ y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \end{gathered} \tag{1.1}$$

where a, f satisfy

(H1) $a : [1, T]_{\mathbb{Z}} \to (-\infty, 0]$ and $a(\cdot) \not\equiv 0$;

(H2) $f : [1, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to $y \in \mathbb{R}$.

The authors obtained the existence results of solutions of (1.1) under conditions (H1), (H2), and the used tool is upper and lower solutions techniques.

Naturally, whether there exists the Green function $G(t, s)$ of the homogeneous linear boundary value problem corresponding to (1.1) if $a(n) \geq 0$? Moreover, if the answer is positive, whether $G(t, s)$ keeps its sign? To the knowledge of the authors, there are very few works on the case $a(n) \geq 0$.

Recently, in 2003, Torres [6] investigated the existence of one-signed periodic solutions for second-order differential equation boundary value problem

$$\begin{gathered} x''(t) = f(t, x(t)), \quad t \in [1, T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{gathered} \tag{1.2}$$

by applying the fixed-point theorem in cones, and constructed Green's function of

$$\begin{gathered} x''(t) + a(t)x(t) = 0, \quad t \in [1, T], \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{gathered} \tag{1.3}$$

where $a \in L^p(0, T)$ satisfies either

(H3) $a \le 0$, $a(\cdot) \not\equiv 0$ on $[0, T]$;

(H4) $a \ge 0$, $a(\cdot) \not\equiv 0$ on $[0, T]$ and $\|a\|_p \le K(2p^*)$ for some $1 \le p \le +\infty$.

Motivated by Torres [6], in Section 2, the paper gives the new expression of Green's function of the linear boundary value problem

$$\Delta^2 y(t-1) + a(t)y(t) = 0, \quad t \in [1, T]_{\mathbb{Z}}, \tag{1.4}$$

$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \tag{1.5}$$

where $a \in \Lambda^+ \cup \Lambda^-$ and

$$\begin{gathered} \Lambda^- = \{a \mid a : [1, T]_{\mathbb{Z}} \longrightarrow (-\infty, 0],\ a(\cdot) \not\equiv 0\}, \\ \Lambda^+ = \left\{a \mid a : [1, T]_{\mathbb{Z}} \longrightarrow [0, \infty),\ a(\cdot) \not\equiv 0,\ \max_{t \in [1, T]_{\mathbb{Z}}} |a(t)| < 4 \sin^2 \frac{\pi}{2T}\right\}, \end{gathered} \tag{1.6}$$

and obtains the sign properties of Green's function of (1.4), (1.5).

In Section 3, we obtain the existence of one-signed periodic solutions of the discrete second-order nonlinear periodic boundary value problem

$$\begin{gathered} \Delta^2 y(t-1) = f(t, y(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \end{gathered} \tag{1.7}$$

where $f : [1, T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is continuous. For related results on the associated differential equations, see Torres [6].

2. Preliminaries

Let

$$E = \{y \mid y : [0, T+1]_{\mathbb{Z}} \longrightarrow \mathbb{R},\ y(0) = y(T),\ y(1) = y(T+1)\} \tag{2.1}$$

be a Banach space endowed with the norm $\|y\| = \max_{t \in [0, T+1]_{\mathbb{Z}}} |y(t)|$.

We say that the linear boundary value problem (1.4), (1.5) is nonresonant when its unique solution is the trivial one. If (1.4), (1.5) is nonresonant, and let $h : [1,T]_{\mathbb{Z}} \to \mathbb{R}$, by the virtue of the Fredholm's alternative theorem, we can get that the discrete second-order periodic boundary value problem

$$\Delta^2 y(t-1) + a(t)y(t) = h(t), \quad t \in [1,T]_{\mathbb{Z}} \tag{2.2}$$

$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T) \tag{2.3}$$

has a unique solution y,

$$y(t) = \sum_{s=1}^{T} G(t,s)h(s), \quad t \in [0,T+1]_{\mathbb{Z}}, \tag{2.4}$$

where $G(t,s)$ is Green's function related to (1.4), (1.5).

Definition 2.1 (see [7]). We say that a solution y of (1.4) has a generalized zero at t_0 provided that $y(t_0) = 0$ if $t_0 = 0$ and if $t_0 > 0$ either $y(t_0) = 0$ or $y(t_0-1)y(t_0) < 0$.

Theorem 2.2. *Assume that the distance between two consecutive generalized zeros of a nontrivial solution of* (1.4) *is greater than T. Then Green's function $G(t,s)$ has constant sign.*

Proof. Obviously, G is well defined on $[0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$. We only need to prove that G has no generalized zero in any point. Suppose on the contrary that there exists $(t_0, s_0) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$ such that (t_0, s_0) is a generalized zero of $G(t,s)$. It is well known that for a given $s_0 \in [1,T]_{\mathbb{Z}}$, $G(t,s_0)$ as a function of t is a solution of (1.4) in the intervals $[0, s_0-1]_{\mathbb{Z}}$ and $[s_0+1, T+1]_{\mathbb{Z}}$ such that

$$G(0,s_0) = G(T,s_0), \qquad G(1,s_0) = G(T+1,s_0). \tag{2.5}$$

Case 1 ($G(t_0,s_0) = 0$, $(t_0,s_0) \in [0,T+1]_{\mathbb{Z}} \times [1,T]_{\mathbb{Z}}$). If $t_0 \in [s_0+1, T+1]_{\mathbb{Z}}$, we can construct

$$y(t) = \begin{cases} G(t,s_0), & t \in [s_0, T+1]_{\mathbb{Z}}, \\ G(t-T,s_0), & t \in [T+1, s_0+T]_{\mathbb{Z}}. \end{cases} \tag{2.6}$$

Consequently, y is a solution of (1.4) in the whole interval $[s_0, s_0+T]_{\mathbb{Z}}$. Since $y(t_0) = 0$, we have $\Delta^2 y(t_0-1) = -a(t)y(t_0) = 0$, that is, $y(t_0-1)y(t_0+1) < 0$. Moreover, $y(s_0) = y(s_0+T)$, so there at least exists another generalized zero $t_1 \in [s_0+1, s_0+T]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, which is a contradiction.

Analogously, if $t_0 \in [0, s_0-1]_{\mathbb{Z}}$, we get a contradiction by the same reasoning with

$$y(t) = \begin{cases} G(t+T,s_0), & t \in [s_0-T, 0]_{\mathbb{Z}}, \\ G(t,s_0), & t \in [0, s_0]_{\mathbb{Z}}. \end{cases} \tag{2.7}$$

If $t_0 = s_0$, we can apply y as defined (2.7). Since $y(t_0) = y(s_0) = 0$ and $y(s_0-T) = y(s_0)$, which contradicts with the hypothesis.

Case 2 $(G(t_0-1,s_0)G(t_0,s_0)<0,\ (t_0,s_0)\in[1,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}})$. If $t_0\in[s_0+1,T+1]_{\mathbb{Z}}$, we can construct y defined as (2.6). It is not difficult to verify that y is a solution of (1.4) in the whole interval $[s_0,s_0+T]_{\mathbb{Z}}$. Also, we have that $y(t_0-1)y(t_0)<0$, that is, t_0 is a generalized zero of y. Moreover, $y(s_0)=y(s_0+T)$, so there at least exists another generalized zero $t_1\in[s_0+1,s_0+T]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, which is a contradiction.

Similarly, if $t_0\in[1,s_0-1]_{\mathbb{Z}}$, we can get a contradiction by the same reasoning as y defined (2.7).

If $t_0=s_0$, we can construct y defined by (2.7). Since $y(t_0-1)y(t_0)=y(s_0-1)y(s_0)<0$ and $y(s_0-T)=y(s_0)$, it is clear that there exists another generalized zero $t_1\in[s_0-T,s_0]_{\mathbb{Z}}$ of y. Note that the distance between t_0 and t_1 is smaller than T, this contradicts with the hypothesis. □

To apply the above result, we are going to study the two following cases.

Corollary 2.3. *If $a\in\Lambda^-$, then $G(t,s)<0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.*

Proof. If $a\in\Lambda^-$, by [7, Corollary 6.7], it is easy to verify that (1.4) is disconjugate on $[0,T+1]_{\mathbb{Z}}$, and any nontrivial solution of (1.4) has at most one generalized zero on $[0,T+1]_{\mathbb{Z}}$. Hence, by Theorem 2.2, Green's function $G(t,s)$ has constant sign. We claim that the sign is negative. In fact, $y(t)=\sum_{s=1}^{T}G(t,s)$ is the unique T-periodic solution of the equation

$$\Delta^2 y(t-1)+a(t)y(t)=1, \tag{2.8}$$

and summing both sides of (2.8) from $t=1$ to $t=T$, we can get

$$\sum_{t=1}^{T}a(t)y(t)=T>0. \tag{2.9}$$

Since $a(t)<0$, $y(t)<0$ for some $t\in[1,T]_{\mathbb{Z}}$, and as a consequence $G(t,s)<0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$. □

Remark 2.4. If $a(\cdot)\equiv a_0$ (a_0 is a negative constant), then by computing we can obtain

$$G(t,s)=\begin{cases}-\dfrac{\lambda_1^{t-s}+\lambda_1^{T-t+s}}{(\lambda_1-\lambda_1^{-1})(\lambda_1^T-1)}, & 1\le s\le t\le T+1,\\[2ex] -\dfrac{\lambda_1^{s-t}+\lambda_1^{T-s+t}}{(\lambda_1-\lambda_1^{-1})(\lambda_1^T-1)}, & 0\le t\le s\le T,\end{cases} \tag{2.10}$$

where $\lambda_1=(2-a_0+\sqrt{a_0^2-4a_0})/2>1$. Obviously, $G(t,s)<0,\ (t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.

If $a\ge 0$, then the solutions of (1.4) are oscillating, that is, there are infinite zeros, and to get the required distance between generalized zeros, a should satisfy Λ^+.

Corollary 2.5. *If $a\in\Lambda^+$, then $G(t,s)>0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.*

Proof. We claim that the distance between two consecutive generalized zeros of a nontrivial solution y of (1.4) is strictly greater than T. In fact, it is not hard to verify that

$\Delta^2 y(t-1)+\|a\|y(t)=0$ is disconjugate on $[0,T+1]_{\mathbb{Z}}$ under assumption $\|a\|<4\sin^2(\pi/2T)$. Since $a(t)\le\|a\|$, $t\in[1,T]_{\mathbb{Z}}$, by Sturm comparison theorem [7, Theorem 6.19], (1.4) is disconjugate on $[0,T+1]_{\mathbb{Z}}$, that is, any nontrivial solution of (1.4) has at most one generalized zero on $[0,T+1]$.

Hence, by Theorem 2.2, $G(t,s)$ has constant sign on $[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$, and the positive sign of G is determined as the proof process of Corollary 2.3. □

Remark 2.6. If $a(\cdot)\equiv\overline{a}$ ($\overline{a}$ is apositive constant), and $0<\overline{a}<4\sin^2(\pi/2T)$, then by computing we can obtain

$$G(t,s)=\begin{cases}\dfrac{\sin[\theta(t-s)]+\sin[\theta(T-t+s)]}{2\sin\theta(1-\cos(\theta T))}, & 1\le s\le t\le T+1,\\[2ex] \dfrac{\sin[\theta(s-t)]+\sin[\theta(T-s+t)]}{2\sin\theta(1-\cos(\theta T))}, & 0\le t\le s\le T,\end{cases} \tag{2.11}$$

where $\theta=\arccos((2-\overline{a})/2)$ and $0<\theta<\pi/T$. Clearly, $G(t,s)>0$, $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.

If $a(\cdot)\equiv\overline{a}$ and $\overline{a}=4\sin^2(\pi/2T)$, then $\theta=\pi/T$, and by computing we get

$$G(t,s)=\begin{cases}\dfrac{1}{2\sin(\pi/T)}\sin\left[\dfrac{\pi}{T}(t-s)\right], & 1\le s\le t\le T+1,\\[2ex] \dfrac{1}{2\sin(\pi/T)}\sin\left[\dfrac{\pi}{T}(s-t)\right], & 0\le t\le s\le T.\end{cases} \tag{2.12}$$

Obviously, Green's function $G(t,s)=0$ for $t=s$ and $G(t,s)>0$ for $t\neq s$.

If $a(\cdot)\equiv\overline{a}$ and $\overline{a}=4\sin^2(\pi/T)$, then $\theta=2\pi/T$, and it is not difficult to verify that

$$\varphi(t)=\sin\left(\frac{2\pi}{T}t\right),\qquad \varphi(t)=\cos\left(\frac{2\pi}{T}t\right),\quad t\in[0,T+1]_{\mathbb{Z}} \tag{2.13}$$

are nontrivial solutions of (1.4), (1.5). That is, the problem (1.4), (1.5) has no Green's function.

If $a(\cdot)\equiv\overline{a}$ and $4\sin^2(\pi/2T)<\overline{a}<4\sin^2(\pi/T)$, then Green's function may change its sign. For example, let $T=6$, $\overline{a}=4\sin^2(\pi/8)=2-\sqrt{2}$, it is easy to verify that $2-\sqrt{3}=4\sin^2(\pi/12)<\overline{a}<4\sin^2(\pi/6)=1$ and $\theta=\pi/4$, thus

$$G(t,s)=\begin{cases}\sin\left[\dfrac{\pi}{4}(t-s-1)\right], & 1\le s\le t\le T+1,\\[2ex] \sin\left[\dfrac{\pi}{4}(s-t-1)\right], & 0\le t\le s\le T.\end{cases} \tag{2.14}$$

Clearly, $G(t,s)=-\sin(\pi/4)<0$ for $t=s$, $G(t,s)=0$ for $|t-s|=1$, and $G(t,s)=\sin(\pi/4)>0$ for $|t-s|=2$.

Consequently, $a\in\Lambda^+$ is the optimal condition of $G(t,s)>0$, $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.

Next, we provide a way to get the expression of $G(t,s)$. Let u be the unique solution of the initial value problem

$$\Delta^2 u(t-1)+a(t)u(t)=0,\quad t\in[1,T]_{\mathbb{Z}},\quad u(0)=0,\quad \Delta u(0)=1, \tag{2.15}$$

and v be the unique solution of the initial value problem

$$\Delta^2 v(t-1) + a(t)v(t) = 0, \quad t \in [1,T]_{\mathbb{Z}}, \quad v(T) = 0, \quad \Delta v(T) = -1. \tag{2.16}$$

Lemma 2.7. *Let $a \in \Lambda^- \cup \Lambda^+$. Then Green's function $G(t,s)$ of* (1.4), (1.5) *is explicitly given by*

$$G(t,s) = \frac{[u(s)+v(s)][u(t)+v(t)]}{v(0)[2+v(1)-u(T+1)]} - \frac{1}{v(0)}\begin{cases} u(t)v(s), & 0 \le t \le s \le T, \\ u(s)v(t), & 1 \le s \le t \le T+1. \end{cases} \tag{2.17}$$

Proof. Suppose that Green's function of (1.4), (1.5) is of the form

$$G(t,s) = \left[\alpha(s)u(t) + \beta(s)v(t)\right] - \frac{1}{v(0)}\begin{cases} u(t)v(s), & 0 \le t \le s \le T, \\ u(s)v(t), & 1 \le s \le t \le T+1, \end{cases} \tag{2.18}$$

where $\alpha(s)$, $\beta(s)$ can be determined by imposing the boundary conditions.

From the basis theory of Green's function, we know that

$$G(0,s) = G(T,s), \qquad G(1,s) = G(T+1,s), \quad \forall s \in [1,T]_{\mathbb{Z}}, \; v(0) = u(T). \tag{2.19}$$

Hence, $\beta(s)v(0) = G(0,s) = G(T,s) = \alpha(s)u(T)$, $s \in [1,T]_{\mathbb{Z}}$, combining with $v(0) = u(T)$, we can get

$$\alpha(s) = \beta(s), \quad s \in [1,T]_{\mathbb{Z}}. \tag{2.20}$$

Moreover, since $G(1,s) = G(T+1,s)$, it follows that

$$\alpha(s) = \frac{u(s)+v(s)}{v(0)[2+v(1)-u(T+1)]}. \tag{2.21}$$

□

Note that $\alpha(\cdot)$ has the same sign with $a(\cdot)$. In fact, by the comparison theorem [7, Theorem 6.6], it is easy to prove that $u, v \ge 0$ on $[0,T]_{\mathbb{Z}}$. If $a(t) \ge 0$, then

$$\Delta^2 u(t-1) = -a(t)u(t) \le 0, \qquad \Delta u(t) \le \Delta u(t-1), \quad t \in [1,T]_{\mathbb{Z}}. \tag{2.22}$$

Thus $\Delta u(T) < \Delta u(0) = 1$. Similarly, we can get that $\Delta v(0) > \Delta v(T) = -1$. Since $v(0) = u(T)$, we have

$$2 + v(1) - u(T+1) = 2 + \Delta v(0) - \Delta u(T) > 0. \tag{2.23}$$

That is $\alpha(t) = (u(t)+v(t))/(v(0)[2+v(1)-u(T+1)]) > 0$.

If $a(\cdot) \le 0$, by the similar method, we can prove $\alpha(\cdot) < 0$.

Lemma 2.8. *Let $a \in \Lambda^- \cup \Lambda^+$. Then the periodic boundary value problem* (2.2), (2.3) *has the unique solution*

$$y(t) = \sum_{s=1}^{T} G(t,s)h(s), \quad t \in [0, T+1]_{\mathbb{Z}}, \tag{2.24}$$

where $G(t,s)$ *is defined by* (2.17).

Proof. We check that y satisfies (2.2). In fact,

$$\begin{aligned}
y(t) &= \sum_{s=1}^{T}(u(t)+v(t))\alpha(s)h(s) - \frac{1}{v(0)}\sum_{s=1}^{t-1}u(t)v(s)h(s) - \frac{1}{v(0)}\sum_{s=t}^{T}u(s)v(t)h(s) \\
&= (u(t)+v(t))\sum_{s=1}^{T}\alpha(s)h(s) - \frac{u(t)}{v(0)}\sum_{s=1}^{t-1}v(s)h(s) - \frac{v(t)}{v(0)}\sum_{s=t}^{T}u(s)h(s), \\
y(t+1) &= (u(t+1)+v(t+1))\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{u(t+1)}{v(0)}\sum_{s=1}^{t}v(s)h(s) - \frac{v(t+1)}{v(0)}\sum_{s=t+1}^{T}u(s)h(s), \\
y(t-1) &= (u(t-1)+v(t-1))\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{u(t-1)}{v(0)}\sum_{s=1}^{t-2}v(s)h(s) - \frac{v(t-1)}{v(0)}\sum_{s=t-1}^{T}u(s)h(s), \\
\Delta^2 y(t-1) + a(t)y(t) &= y(t+1) - (2-a(t))y(t) + y(t-1) \\
&= \left[\Delta^2 u(t-1) + a(t)u(t) + \Delta^2 v(t-1) + a(t)v(t)\right]\sum_{s=1}^{T}\alpha(s)h(s) \\
&\quad - \frac{\Delta^2 u(t-1) + a(t)u(t)}{v(0)}\sum_{s=1}^{t-2}v(s)h(s) \\
&\quad - \frac{\Delta^2 v(t-1) + a(t)v(t)}{v(0)}\sum_{s=t+1}^{T}u(s)h(s) \\
&\quad - \frac{u(t-1)h(t-1)}{v(0)}\left[\Delta^2 v(t-1) + a(t)v(t)\right] \\
&\quad - \frac{u(t)h(t)}{v(0)}\left[\Delta^2 v(t-1) + a(t)v(t)\right] \\
&\quad + \frac{u(t)v(t-1)h(t) - u(t-1)v(t)h(t)}{v(0)}
\end{aligned}$$

$$= \frac{h(t)}{v(0)} \begin{vmatrix} u(t) & v(t) \\ u(t-1) & v(t-1) \end{vmatrix} = \frac{h(t)}{v(0)} \begin{vmatrix} u(1) & v(1) \\ u(0) & v(0) \end{vmatrix} = h(t). \tag{2.25}$$

On the other hand, it is easy to verify that $y(0) = y(T),\ y(1) = y(T+1)$. □

Denote that

$$m = \min_{t,s \in [1,T]_{\mathbb{Z}}} G(t,s), \qquad M = \max_{t,s \in [1,T]_{\mathbb{Z}}} G(t,s). \tag{2.26}$$

As a direct application, we can compute the maximum and the minimum of the Green's function when $a(\cdot) \equiv a_0$, it follows that

$$0 > m \geq -\frac{2\lambda_1^{T/2}}{\left(\lambda_1 - \lambda_1^{-1}\right)\left(\lambda_1^T - 1\right)}, \qquad M = -\frac{\lambda_1^T + 1}{\left(\lambda_1 - \lambda_1^{-1}\right)\left(\lambda_1^T - 1\right)}, \tag{2.27}$$

where λ_1 is defined in Remark 2.4. Similarly, when $0 < a(\cdot) \equiv \overline{a} < 4\sin^2(\pi/2T)$, we can get

$$m = \frac{1}{2\sin\theta}\cot\left(\frac{\theta T}{2}\right) > 0, \qquad M \leq \frac{1}{2\sin\theta\sin(\theta T/2)}, \tag{2.28}$$

where θ is defined in Remark 2.6.

3. Main Results

In this section, we consider the existence of one-signed solutions of (1.7). The following well-known fixed-point theorem in cones is crucial to our arguments.

Theorem 3.1 (see [8]). *Let E be a Banach space and $K \subset E$ be a cone. Suppose Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Assume that $A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|,\ u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Au\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leq \|u\|,\ u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Assume that there exist $a \in \Lambda^+$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \geq 0, \quad \forall y \in \left[\frac{m}{M}r, \frac{M}{m}R\right],\ t \in [1,T]_{\mathbb{Z}}. \tag{3.1}$$

If one of the following conditions holds:

(i)

$$
\begin{aligned}
&f(t,y)+a(t)y \ge \frac{M}{Tm^2}y, \quad \forall y \in \left[\frac{m}{M}r, r\right],\ t \in [1,T]_{\mathbb{Z}},\\
&f(t,y)+a(t)y \le \frac{1}{TM}y, \quad \forall y \in \left[R, \frac{M}{m}R\right],\ t \in [1,T]_{\mathbb{Z}},
\end{aligned} \tag{3.2}
$$

(ii)

$$
\begin{aligned}
&f(t,y)+a(t)y \le \frac{1}{TM}y, \quad \forall y \in \left[\frac{m}{M}r, r\right],\ t \in [1,T]_{\mathbb{Z}},\\
&f(t,y)+a(t)y \ge \frac{M}{Tm^2}y, \quad \forall y \in \left[R, \frac{M}{m}R\right],\ t \in [1,T]_{\mathbb{Z}},
\end{aligned} \tag{3.3}
$$

then problem (1.7) *has a positive solution.*

Proof. From Corollary 2.5, we get that $M > m > 0$. It is easy to see that the equation $\Delta^2 y(t-1) = f(t, y(t))$ is equivalent to

$$\Delta^2 y(t-1) + a(t)y(t) = f(t, y(t)) + a(t)y(t). \tag{3.4}$$

Define the open sets

$$\Omega_1 = \{y \in E : \|y\| < r\}, \qquad \Omega_2 = \left\{y \in E : \|y\| < \frac{M}{m}R\right\}, \tag{3.5}$$

and the cone P in E,

$$P = \left\{y \in E : \min_{t\in[0,T+1]_{\mathbb{Z}}} y(t) > \frac{m}{M}\|y\|\right\}. \tag{3.6}$$

Clearly, if $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then $(m/M)r \le y(t) \le (M/m)R$, for all $t \in [0, T+1]_{\mathbb{Z}}$.

From Lemma 2.8, we define the operator $A : E \to E$ by

$$(Ay)(t) = \sum_{s=1}^{T} G(t,s)\left[f(s,y(s)) + a(s)y(s)\right], \quad t \in [0,T+1]_{\mathbb{Z}}. \tag{3.7}$$

From (3.1), if $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$
\begin{aligned}
Ay(t) &\ge \frac{m}{M} M \sum_{s=1}^{T}\left[f(s,y(s)) + a(s)y(s)\right]\\
&> \frac{m}{M}\max_{t\in[0,T+1]_{\mathbb{Z}}} \sum_{s=1}^{T} G(t,s)\left[f(s,y(s)) + a(s)y(s)\right] = \frac{m}{M}\|Ay\|.
\end{aligned} \tag{3.8}
$$

Thus $A(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset P$. Moreover, E is a finite space, it is easy to prove that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator. Clearly, y is the solution of problem (1.7) if and only if y is the fixed point of the operator A.

We only prove (i). (ii) can be obtained by the similar method. If $y \in \partial\Omega_1 \cap P$, then $\|y\| = r$ and $(m/M)r \le y(t) \le r$ for all $t \in [0, T+1]_{\mathbb{Z}}$. Therefore, from (i),

$$Ay(t) \ge m\sum_{s=1}^{T}\left[f(s, y(s)) + a(s)y(s)\right] \ge \frac{M}{Tm}\sum_{s=1}^{T} y(s) \ge r = \|y\|. \tag{3.9}$$

If $y \in \partial\Omega_2 \cap P$, then $\|y\| = (M/m)R$ and $R \le y(t) \le (M/m)R$ for all $t \in [0, T+1]_{\mathbb{Z}}$. As a consequence,

$$Ay(t) \le M\sum_{s=1}^{T}\left[f(s, y(s)) + a(s)y(s)\right] \le \frac{1}{T}\sum_{s=1}^{T} y(s) \le \|y\|. \tag{3.10}$$

From Theorem 3.1, A has a fixed point $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and satisfies

$$\frac{m}{M}r \le y(t) \le \frac{M}{m}R. \tag{3.11}$$

Therefore, y is a positive solution of (1.7). □

Similar to the proof of Theorem 3.2, we can prove the following.

Corollary 3.3. *Assume that there exist $a \in \Lambda^+$ and $0 < r < R$ such that*

$$f(t, y) + a(t)y \le 0, \quad \forall y \in \left[-\frac{M}{m}R, -\frac{m}{M}r\right], \ t \in [1, T]_{\mathbb{Z}}. \tag{3.12}$$

If one of the following conditions holds

(i)

$$\begin{aligned} f(t, y) + a(t)y &\le \frac{M}{Tm^2}y, \quad \forall y \in \left[-r, -\frac{m}{M}r\right], \ t \in [1, T]_{\mathbb{Z}}, \\ f(t, y) + a(t)y &\ge \frac{1}{TM}y, \quad \forall y \in \left[-\frac{M}{m}R, -R\right], \ t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{3.13}$$

(ii)

$$\begin{aligned} f(t, y) + a(t)y &\ge \frac{1}{TM}y, \quad \forall y \in \left[-r, -\frac{m}{M}r\right], \ t \in [1, T]_{\mathbb{Z}}, \\ f(t, y) + a(t)y &\le \frac{M}{Tm^2}y, \quad \forall y \in \left[-\frac{M}{m}R, -R\right], \ t \in [1, T]_{\mathbb{Z}}, \end{aligned} \tag{3.14}$$

then (1.7) *has a negative solution.*

Applying the sign properties of $G(t,s)$ when $a \in \Lambda^-$ and the similar argument to prove Theorem 3.2 with obvious changes, we can prove the following.

Theorem 3.4. *Assume that there exist $a \in \Lambda^-$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \le 0, \quad \forall y \in \left[\frac{M}{m}r, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}. \tag{3.15}$$

If one of the following conditions holds

(i)

$$\begin{aligned} f(t,y) + a(t)y &\le \frac{m}{TM^2}y, \quad \forall y \in \left[\frac{M}{m}r, r\right],\ t \in [1,T]_{\mathbb{Z}}, \\ f(t,y) + a(t)y &\ge \frac{1}{Tm}y, \quad \forall y \in \left[R, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}, \end{aligned} \tag{3.16}$$

(ii)

$$\begin{aligned} f(t,y) + a(t)y &\ge \frac{1}{Tm}y, \quad \forall y \in \left[\frac{M}{m}r, r\right],\ t \in [1,T]_{\mathbb{Z}}, \\ f(t,y) + a(t)y &\le \frac{m}{TM^2}y, \quad \forall y \in \left[R, \frac{m}{M}R\right],\ t \in [1,T]_{\mathbb{Z}}, \end{aligned} \tag{3.17}$$

then (1.7) *has a positive solution.*

Proof. Since $m < M < 0$, define the open sets

$$\Omega_1 = \{y \in E : \|y\| < r\}, \qquad \Omega_2 = \left\{y \in E : \|y\| < \frac{m}{M}R\right\}, \tag{3.18}$$

and define the cone P in E,

$$P = \left\{y \in E : \min_{t\in[0,T+1]_{\mathbb{Z}}} y(t) > \frac{M}{m}\|y\|\right\}. \tag{3.19}$$

If $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then

$$\frac{M}{m}r \le y(t) \le \frac{m}{M}R, \quad \forall t \in [0,T+1]_{\mathbb{Z}}. \tag{3.20}$$

Define the operator A as (3.7), and the proof is analogous to that of Theorem 3.2 and is omitted. □

Corollary 3.5. *Assume that there exist $a \in \Lambda^-$ and $0 < r < R$ such that*

$$f(t,y) + a(t)y \ge 0, \quad \forall y \in \left[-\frac{m}{M}R, -\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}}. \tag{3.21}$$

If one of the following conditions holds

(i)

$$f(t,y) + a(t)y \ge \frac{m}{TM^2}y, \quad \forall y \in \left[-r, -\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y) + a(t)y \le \frac{1}{Tm}y, \quad \forall y \in \left[-\frac{m}{M}R, -R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.22}$$

(ii)

$$f(t,y) + a(t)y \le \frac{1}{Tm}y, \quad \forall y \in \left[-r, -\frac{M}{m}r\right],\ t \in [1,T]_{\mathbb{Z}},$$
$$f(t,y) + a(t)y \ge \frac{m}{TM^2}y, \quad \forall y \in \left[-\frac{m}{M}R, -R\right],\ t \in [1,T]_{\mathbb{Z}}, \tag{3.23}$$

then (1.7) *has a negative solution.*

Example 3.6. Let us consider the periodic boundary value problem

$$\Delta^2 y(n-1) = f(n, y(n)), \quad n \in [1,T]_{\mathbb{Z}},$$
$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \tag{3.24}$$

where

$$f(n,y) = \begin{cases} 2n^2 + 3y, & y \in \left[\dfrac{\sqrt{2}}{2}r, r\right],\ n \in [1,T]_{\mathbb{Z}}, \\ (2n^2+3y)\dfrac{R-y}{R-r} - 0.25y\dfrac{y-r}{R-r}, & y \in [r,R],\ n \in [1,T]_{\mathbb{Z}}, \\ -0.25y, & y \in \left[R, \sqrt{2}R\right],\ n \in [1,T]_{\mathbb{Z}}. \end{cases} \tag{3.25}$$

Consider the auxiliary problem

$$\Delta^2 y(n-1) + a(n)y(n) = f(n, y(n)) + a(n)y(n), \quad n \in [1,T]_{\mathbb{Z}},$$
$$y(0) = y(T), \qquad \Delta y(0) = \Delta y(T), \tag{3.26}$$

take $r = 2\sqrt{2}$, $R = 8\sqrt{2}$, $T = 3, a(n) \equiv \overline{a} = 2 - \sqrt{3} < 4\sin^2(\pi/2T), \theta = \pi/6, \cos\theta = (2-\overline{a})/2 = \sqrt{3}/2, \sin\theta = 1/2$, $m = (1/2\sin\theta)\cot(\theta T/2) = 1$, $M = 1/(2\sin\theta\sin(\theta T/2)) = \sqrt{2}$. By computing, $f(n,y) + a(n)y \ge 0$, $y \in [(\sqrt{2}/2)r, \sqrt{2}R]$, $n \in [1,T]_{\mathbb{Z}}; f(n,y) + a(n)y \ge (\sqrt{2}/3)y$, $y \in [(\sqrt{2}/2)r, r], n \in [1,T]_{\mathbb{Z}}$; $f(n,y) + a(n)y \le (1/3\sqrt{2})y, y \in [R, \sqrt{2}R]$, $n \in [1,T]_{\mathbb{Z}}$. Consequently, from Theorem 3.2, the problem (3.24) has a positive solution.

Acknowledgment

This work was supported by the NSFC (no. 11061030) and the Fundamental Research Funds for the Gansu Universities.

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