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Research Article

Multiple Positive Periodic Solutions for a Gilpin-Ayala Competition Predator-Prey System with Harvesting Terms

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By applying Mawhin's continuation theorem of coincidence degree theory, we study the existence of multiple positive periodic solutions for a Gilpin-Ayala competition predator-prey system with harvesting terms and obtain some sufficient conditions for the existence of multiple positive periodic solutions for the system under consideration. The result of this paper is completely new. An example is employed to illustrate our result.

1. Introduction

In 1973, Ayala et al. [1] conducted experiments on fruit fly dynamics to test the validity of ten models of competition. One of the models accounting best for the experimental results is given by

$$
\begin{aligned}
y_1' &= r_1 y_1 \left(1 - \left(\frac{y_1}{K_1} \right)^{\theta_1} - a_{12} \frac{y_2}{K_2} \right), \\
y_2' &= r_2 y_2 \left(1 - \left(\frac{y_2}{K_2} \right)^{\theta_2} - a_{21} \frac{y_1}{K_1} \right).
\end{aligned}
\tag{1.1}
$$

In order to fit data in their experiments and to yield significantly more accurate results, Gilpin and Ayala [2] claimed that a slightly more complicated model was needed and proposed the following competition model:

$$x_i'(t) = r_i x_i \left(1 - \left(\frac{x_i}{K_i} \right)^{\theta_i} - \sum_{j=1, j\neq i}^{n} a_{ij} \frac{x_j}{K_j} \right), \quad i = 1, 2, \dots, n, \tag{1.2}$$

where x_i is the population density of the ith species, r_i is the intrinsic exponential growth rate of the ith species, K_i is the environmental carrying capacity of species i in the absence of competition, θ_i provides a nonlinear measure of interspecific interference, and α_{ij} provides a measure of interspecific interference.

During the past decade, many generalizations and modifications of systems (1.1) and (1.2) have been proposed and studied; see, for example, [3–10].

Virtually all biological systems exist in environments which vary with time, frequently in a periodic way. Ecosystem effects and environmental variability are very important factors, and mathematical models cannot ignore, for example, year-to-year changes in weather, habitat destruction and exploitation, the expanding food surplus, and other factors that affect the population growth.

Since biological and environmental parameters are naturally subjected to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account the seasonality of the periodically changing environment. Also, the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry, and wildlife management; the study of population dynamics with harvesting is an important subject in mathematical bioeconomics, which is related to the optimal management of renewable resources and some other issues including control issues to regulate populations (see [11–16]).

Motivated by above, in this paper, we will investigate the following two species Gilpin-Ayala competition predator-prey system with harvesting terms:

$$\begin{aligned} y_1'(t) &= r_1(t) y_1(t) \left\{ 1 - \left(\frac{y_1(t)}{k_1(t)} \right)^{\theta_1} - a_{12}(t) \frac{y_2(t)}{k_2(t)} \right\} - h_1(t), \\ y_2'(t) &= r_2(t) y_2(t) \left\{ 1 - \left(\frac{y_2(t)}{k_2(t)} \right)^{\theta_2} - a_{21}(t) \frac{y_1(t)}{k_1(t)} \right\} - h_2(t), \end{aligned} \tag{1.3}$$

where $r_i(t) > 0$, $k_i(t) > 0$, $h_i(t) > 0$, $i = 1, 2$, $a_{12}(t)$ and $a_{21}(t) \in C([0, +\infty), (0, +\infty))$ are ω-periodic functions, θ_i, $i = 1, 2$ are positive constants, and y_1 and y_2 represent the number of individuals in the prey and predator population.

A very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic, which plays a similar role as a globally stable equilibrium does in an autonomous model; also, on the existence of positive periodic solutions to system (1.2), few results are found in literatures. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1.2). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena.

Therefore, it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is, by using Mawhin's continuation theorem of coincidence degree theory [17], to establish the existence of four positive periodic solutions for system (1.3). Our method used in this paper can be used to study the multiple existence of positive periodic solutions for n-species Gilpin-Ayala competition predator-prey system with harvesting terms.

The organization of this paper is as follows. In Section 2, we make some preparations. In Section 3, by using Mawhin's continuation theorem of coincidence degree theory, we establish sufficient conditions for the existence of multiple positive periodic solutions to system (1.3). An illustrative example is given in Section 3.

2. Preliminaries

For the readers' convenience, we first summarize a few concepts from [17].

Let $\mathbb{X}$ and $\mathbb{Z}$ Banach spaces. Let $L : \operatorname{Dom} L \subset \mathbb{X} \longrightarrow \mathbb{Z}$ be a linear mapping and $N : \mathbb{X} \longrightarrow \mathbb{Z}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\operatorname{Im} L$ is a closed subspace of $\mathbb{Z}$ and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty. \tag{2.1}$$

If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : \mathbb{X} \longrightarrow \mathbb{Z}$ and $Q : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that

$$L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)\mathbb{X} \longrightarrow \operatorname{Im} L \tag{2.2}$$

is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of $\mathbb{X}$, the mapping N is called L-compact on $\mathbb{X}$, if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \longrightarrow \mathbb{X}$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L$.

In the proof of our existence result, we need the following continuation theorem cited from [17].

Lemma 2.1 (see [17]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\mathbb{X}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$;*

(b) *for each $x \in \partial\Omega$, $QNx \neq 0$;*

(c) *the Browner* $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

By making the substitutions $y_1(t) = e^{x_1(t)}$ and $y_2(t) = e^{x_2(t)}$, then system (1.2) is reformulated as

$$x_1'(t) = r_1(t)\left\{1 - \left(\frac{e^{x_1(t)}}{k_1(t)}\right)^{\theta_1} - a_{12}(t)\frac{e^{x_2(t)}}{k_2(t)}\right\} - h_1(t)e^{-x_1(t)},$$
$$x_2'(t) = r_2(t)\left\{1 - \left(\frac{e^{x_2(t)}}{k_2(t)}\right)^{\theta_2} - a_{21}(t)\frac{e^{x_1(t)}}{k_1(t)}\right\} - h_2(t)e^{-x_2(t)}. \tag{2.3}$$

It is obvious that periodic solutions of (2.3) are positive periodic solutions of (1.3).

For the sake of convenience, we introduce notations as follows:

$$f^L = \max_{t\in[0,\omega]} f(t), \qquad f^l = \min_{t\in[0,\omega]} f(t), \tag{2.4}$$

where f is a continuous ω-periodic function.

We also introduce four assumptions and eight positive numbers as follows.

Assumption (H$_1$) $(1 - a_{12}(k_2^L/k_2))^l > 2\sqrt{(h_1/r_1)^L/(k_1^{\theta_1})^l[(h_1/r_1)^l]^{1-\theta_1}}$.

Assumption (H$_2$) $(1 - a_{21}(k_1^L/k_1))^l > 2\sqrt{(h_2/r_2)^L/(k_2^{\theta_2})^l[(h_2/r_2)^l]^{1-\theta_2}}$.

Assumption (H$_3$) $(1 - a_{12}(k_2^L/k_2))^l > 2\sqrt{(h_1/r_1)^L((k_1^L)^{\theta_1-1}/(k_1^l)^{\theta_1})}$.

Assumption (H$_4$) $(1 - a_{21}(k_1^L/k_1))^l > 2\sqrt{(h_2/r_2)^L((k_2^L)^{\theta_1-2}/(k_2^l)^{\theta_2})}$.

$$u_\pm = \frac{(1 - a_{12}(k_2^L/k_2))^l \pm \sqrt{\left[(1 - a_{12}(k_2^L/k_2))^l\right]^2 - 4\left((h_1/r_1)^L/\left(k_1^{\theta_1}\right)^l\left[(h_1/r_1)^l\right]^{1-\theta_1}\right)}}{2/\left\{\left(k_1^{\theta_1}\right)^l[(h_1/r_1)^l]^{1-\theta_1}\right\}},$$

$$l_\pm = \frac{(1 - a_{21}(k_1^L/k_1))^l \pm \sqrt{\left[(1 - a_{21}(k_1^L/k_1))^l\right]^2 - 4\left((h_2/r_2)^L/\left(k_2^{\theta_2}\right)^l\left[(h_2/r_2)^l\right]^{1-\theta_2}\right)}}{2/\left\{(k_2^{\theta_2})^l[(h_2/r_2)^l]^{1-\theta_2}\right\}},$$

$$\overline{u}_\pm = \frac{(1 - a_{12}(k_2^L/k_2))^l \pm \sqrt{\left[(1 - a_{12}(k_2^L/k_2))^l\right]^2 - 4(h_1/r_1)^L\left((k_1^L)^{\theta_1-1}/\left(k_1^l\right)^{\theta_1}\right)}}{2\left(k_1^L\right)^{\theta_1-1}/\left(k_1^l\right)^{\theta_1}},$$

$$\bar{l}_\pm = \frac{(1 - a_{21}(k_1^L/k_1))^l \pm \sqrt{\left[(1 - a_{21}(k_1^L/k_1))^l\right]^2 - 4(h_2/r_2)^L\left((k_2^L)^{\theta_2-1}/\left(k_2^l\right)^{\theta_2}\right)}}{2\left(k_2^L\right)^{\theta_2-1}/\left(k_2^l\right)^{\theta_2}}. \tag{2.5}$$

3. Main Result

Our main result of this paper is as follows.

Theorem 3.1. *Assume that one of the following conditions holds.*

(i) *If* $0 < \theta_i < 1$, $i = 1, 2$, *then* (H_1)-(H_2).

(ii) *If* $\theta_i \geq 1$, $i = 1, 2$, *then* (H_3)-(H_4).

(iii) *If* $\theta_1 \geq 1$ *and* $0 < \theta_2 < 1$, *then* (H_2)-(H_3).

(iv) *If* $\theta_2 \geq 1$ *and* $0 < \theta_1 < 1$, *then* (H_1)-(H_4).

Then system (1.3) *has at least four positive periodic solutions.*

Proof. Let

$$X = Z = \left\{ z = (x_1, x_2)^T \in C(R, R^n) : z(t + \omega) = z(t) \right\}, \tag{3.1}$$

and define

$$\|z\| = \max_{t \in [0,\omega]} |x_1(t)| + \max_{t \in [0,\omega]} |x_2(t)|, \quad z \in X \text{ or } Z. \tag{3.2}$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. Let

$$N : \mathbb{X} \longrightarrow \mathbb{Z}, \quad Nz(t) = \begin{pmatrix} F_1(z(t)) \\ F_2(z(t)) \end{pmatrix}, \tag{3.3}$$

where

$$\begin{gathered} F_1(z(t)) = r_1(t) \left\{ 1 - \left(\frac{e^{x_1(t)}}{k_1(t)} \right)^{\theta_1} - a_{12}(t) \frac{e^{x_2(t)}}{k_2(t)} \right\} - h_1(t) e^{-x_1(t)}, \\ F_2(z(t)) = r_2(t) \left\{ 1 - \left(\frac{e^{x_2(t)}}{k_2(t)} \right)^{\theta_2} - a_{21}(t) \frac{e^{x_1(t)}}{k_1(t)} \right\} - h_2(t) e^{-x_2(t)}, \end{gathered} \tag{3.4}$$

and $Lu = \dot{z} = \mathrm{d}z(t)/\mathrm{d}t$. We put $Pz = (1/\omega) \int_0^\omega z(t)\mathrm{d}t,\ z \in X;\ Qz = (1/\omega) \int_0^\omega z(t)\mathrm{d}t,\ z \in Z$. Thus it follows that $\operatorname{Ker} L = R^2,\ \operatorname{Im} L = \{z \in Z : \int_0^\omega z(t)\mathrm{d}t = 0\}$ is closed in Z, $\dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{3.5}$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by

$$K_P(z) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^s z(s)\mathrm{d}s. \tag{3.6}$$

Then

$$QNz = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega F_1(z(s))\mathrm{d}s \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega F_2(z(s))\mathrm{d}s \end{pmatrix},$$

$$K_P(I-Q)N(z) = \begin{pmatrix} \displaystyle\int_0^t F_1(z(s))\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t F_1(z(s))\mathrm{d}s\,\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_1(z(s))\mathrm{d}s \\ \displaystyle\int_0^t F_2(z(s))\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t F_2(z(s))\mathrm{d}s\,\mathrm{d}t + \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega F_2(z(s))\mathrm{d}s \end{pmatrix}. \tag{3.7}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we are in the position of searching for an appropriate open, bounded subset Ω for the application of Lemma 2.1. Corresponding to the operator equation

$$Lx = \lambda N x, \quad \lambda \in (0,1), \tag{3.8}$$

we obtain

$$\begin{aligned} x_1'(t) &= \lambda r_1(t)\left\{1 - \left(\frac{e^{x_1(t)}}{k_1(t)}\right)^{\theta_1} - a_{12}(t)\frac{e^{x_2(t)}}{k_2(t)}\right\} - \lambda h_1(t)e^{-x_1(t)}, \\ x_2'(t) &= \lambda r_2(t)\left\{1 - \left(\frac{e^{x_2(t)}}{k_2(t)}\right)^{\theta_2} - a_{21}(t)\frac{e^{x_1(t)}}{k_1(t)}\right\} - \lambda h_2(t)e^{-x_2(t)}. \end{aligned} \tag{3.9}$$

Assume that $x \in \mathbb{X}$ is a solution of system (3.9) for some $\lambda \in (0,1)$. Let $\xi_i, \eta_i \in [0,\omega]$ such that

$$x_i(\xi_i) = \max_{t\in[0,\omega]} x_i(t), \quad x_i(\eta_i) = \min_{t\in[0,\omega]} x_i(t), \quad i = 1,2. \tag{3.10}$$

It is clear that $x_i'(\xi_i) = 0$ and $x_i'(\eta_i) = 0$, $i = 1, 2$. From this and system (3.9), we have

$$r_1(\xi_1)\left\{1 - \left(\frac{e^{x_1(\xi_1)}}{k_1(\xi_1)}\right)^{\theta_1} - a_{12}(\xi_1)\frac{e^{x_2(\xi_1)}}{k_2(\xi_1)}\right\} - h_1(\xi_1)e^{-x_1(\xi_1)} = 0,$$
$$r_2(\xi_2)\left\{1 - \left(\frac{e^{x_2(\xi_2)}}{k_2(\xi_2)}\right)^{\theta_2} - a_{21}(\xi_2)\frac{e^{x_1(\xi_2)}}{k_1(\xi_2)}\right\} - h_2(\xi_2)e^{-x_2(\xi_2)} = 0, \tag{3.11}$$

$$r_1(\eta_1)\left\{1 - \left(\frac{e^{x_1(\eta_1)}}{k_1(\eta_1)}\right)^{\theta_1} - a_{12}(\eta_1)\frac{e^{x_2(\eta_1)}}{k_2(\eta_1)}\right\} - h_1(\eta_1)e^{-x_1(\eta_1)} = 0,$$
$$r_2(\eta_2)\left\{1 - \left(\frac{e^{x_2(\eta_2)}}{k_2(\eta_2)}\right)^{\theta_2} - a_{21}(\eta_2)\frac{e^{x_1(\eta_2)}}{k_1(\eta_2)}\right\} - h_2(\eta_2)e^{-x_2(\eta_2)} = 0. \tag{3.12}$$

The first equation of (3.11) implies

$$1 - \left(\frac{e^{x_1(\xi_1)}}{k_1(\xi_1)}\right)^{\theta_1} = \frac{h_1\xi_1}{r_1(\xi_1)}e^{-x_1(\xi_1)} + a_{12}(\xi_1)\frac{e^{x_2(\xi_1)}}{k_2(\xi_1)} > 0; \tag{3.13}$$

that is,

$$x_1(\xi_1) < \ln k_1^L. \tag{3.14}$$

Similarly from the second equation of (3.11), we have

$$x_2(\xi_2) < \ln k_2^L. \tag{3.15}$$

And, the first equation of (3.12) implies

$$\frac{h_1(\eta_1)}{r_1(\eta_1)}e^{-x_1(\eta_1)} = 1 - \left(\frac{e^{x_1(\eta_1)}}{k_1(\eta_1)}\right)^{\theta_1} - a_{12}(\eta_1)\frac{e^{x_2(\eta_1)}}{k_2(\eta_1)} < 1; \tag{3.16}$$

that is,

$$x_1(\eta_1) > \ln\left(\frac{h_1}{r_1}\right)^l. \tag{3.17}$$

Similarly from the second equation of (3.12), we have

$$x_2(\eta_2) > \ln\left(\frac{h_2}{r_2}\right)^l. \tag{3.18}$$

Case i (for $0 < \theta_i < 1,\ i = 1, 2$). In view of (3.15), we have

$$e^{x_2(\xi_1)} < e^{x_2(\xi_2)} < k_2^L. \tag{3.19}$$

Then, the first equation of (3.11) can be reduced to

$$\left(\frac{e^{x_1(\xi_1)}}{k_1(\xi_1)}\right)^{\theta_1} + a_{12}(\xi_1)\frac{k_2^L}{k_2(\xi_1)} + \frac{h_1(\xi_1)}{r_1(\xi_1)}e^{-x_1(\xi_1)} - 1 > 0. \tag{3.20}$$

Multiplying inequality (3.20) by $e^{x_1(\xi_1)}$ gives

$$\frac{e^{x_1(\xi_1)(\theta_1-1)}}{[k_1(\xi_1)]^{\theta_1}}e^{2x_1(\xi_1)} - \left[1 - a_{12}(\xi_1)\frac{k_2^L}{k_2(\xi_1)}\right]e^{x_1(\xi_1)} + \frac{h_1(\xi_1)}{r_1(\xi_1)} > 0. \tag{3.21}$$

Because

$$e^{x_1(\xi_1)(\theta_1-1)} = \frac{1}{e^{x_1(\xi_1)(1-\theta_1)}} < \frac{1}{\left[(h_1/r_1)^l\right]^{1-\theta_1}}, \tag{3.22}$$

then we have

$$\frac{1}{\left(k_1^{\theta_1}\right)^l\left[(h_1/r_1)^l\right]^{1-\theta_1}}e^{2x_1(\xi_1)} - \left(1 - a_{12}\frac{k_2^L}{k_2}\right)^l e^{x_1(\xi_1)} + \left(\frac{h_1}{r_1}\right)^L > 0, \tag{3.23}$$

which implies

$$x_1(\xi_1) > \ln u_+ \quad \text{or} \quad x_1(\xi_1) < \ln u_-. \tag{3.24}$$

Similarly from the first equation of (3.12), we have

$$x_1(\eta_1) > \ln u_+ \quad \text{or} \quad x_1(\eta_1) < \ln u_-. \tag{3.25}$$

From the second equations of (3.11) and (3.12), by a parallel argument to (3.24) and (3.25), we obtain

$$\begin{aligned} x_2(\xi_2) > \ln l_+ \quad &\text{or} \quad x_2(\xi_2) < \ln l_-, \\ x_2(\eta_2) > \ln l_+ \quad &\text{or} \quad x_2(\eta_2) < \ln l_-. \end{aligned} \tag{3.26}$$

From (3.14), (3.17), (3.24), and (3.25), we obtain that, for all $t \in \mathbb{R}$,

$$\ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_- \quad \text{or} \quad \ln u_+ < x_1(t) < \ln k_1^L. \tag{3.27}$$

Similarly, from (3.15), (3.18), and (3.26), we obtain that, for all $t \in \mathbb{R}$,

$$\ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_- \quad \text{or} \quad \ln l_+ < x_2(t) < \ln k_2^L. \tag{3.28}$$

Obviously, $\ln u_\pm, \ln l_\pm, \ln k_1^L, \ln k_2^L, \ln(h_1/r_1)^l$, and $\ln(h_2/r_2)^l$ are independent of λ. Now let

$$\begin{aligned}
\Omega_1 &= \left\{x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_-, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_-\right\},\\
\Omega_2 &= \left\{x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_-, \ln l_+ < x_2(t) < \ln k_2^L\right\},\\
\Omega_3 &= \left\{x = (x_1, x_2)^T \in \mathbb{X} : \ln u_+ < x_1(t) < \ln k_1^L, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_-\right\},\\
\Omega_4 &= \left\{x = (x_1, x_2)^T \in \mathbb{X} : \ln u_+ < x_1(t) < \ln k_1^L, \ln l_+ < x_2(t) < \ln k_2^L\right\}.
\end{aligned} \tag{3.29}$$

Then $\Omega_i,\ i = 1,2,3,4$, are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ i,j = 1,2,3,4$. Thus $\Omega_i,\ i = 1,2,3,4$, satisfy the requirement (a) in Lemma 2.1.

Case ii (for $\theta_i \geq 1, i = 1,2$). From (3.20), it follows that

$$\frac{e^{x_1(\xi_1)(\theta_1-1)}}{[k_1(\xi_1)]^{\theta_1}} e^{2x_1(\xi_1)} - \left[1 - a_{12}(\xi_1)\frac{k_2^L}{k_2(\xi_1)}\right] e^{x_1(\xi_1)} + \frac{h_1(\xi_1)}{r_1(\xi_1)} > 0. \tag{3.30}$$

Because

$$e^{x_1(\xi_1)(\theta_1-1)} < \left(k_1^L\right)^{\theta_1-1}, \tag{3.31}$$

then we have

$$\frac{(k_1^L)^{\theta_1-1}}{\left(k_1^l\right)^{\theta_1}} e^{2x_1(\xi_1)} - \left(1 - a_{12}\frac{k_2^L}{k_2}\right)^l e^{x_1(\xi_1)} + \left(\frac{h_1}{r_1}\right)^L > 0, \tag{3.32}$$

which implies

$$x_1(\xi_1) > \ln \overline{u}_+ \quad \text{or} \quad x_1(\xi_1) < \ln \overline{u}_-. \tag{3.33}$$

Similarly from the first equation of (3.12), we have

$$x_1(\eta_1) > \ln \overline{u}_+ \quad \text{or} \quad x_1(\eta_1) < \ln \overline{u}_-. \tag{3.34}$$

From the second equations of (3.11) and (3.12), by a parallel argument to (3.33) and (3.34), we obtain

$$x_2(\xi_2) > \ln \bar{l}_+ \quad \text{or} \quad x_2(\xi_2) < \ln \bar{l}_-,$$
$$x_2(\eta_2) > \ln \bar{l}_+ \quad \text{or} \quad x_2(\eta_2) < \ln \bar{l}_-. \tag{3.35}$$

From (3.14), (3.17), (3.33), and (3.34), we obtain that, for all $t \in \mathbb{R}$,

$$\ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln \overline{u}_- \quad \text{or} \quad \ln \overline{u}_+ < x_1(t) < \ln k_1^L. \tag{3.36}$$

Similarly, from (3.15), (3.18), and (3.35) we obtain that, for all $t \in \mathbb{R}$,

$$\ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln \bar{l}_- \quad \text{or} \quad \ln \bar{l}_+ < x_2(t) < \ln k_2^L. \tag{3.37}$$

Obviously, $\ln \overline{u}_\pm, \ln \bar{l}_\pm, \ln k_1^L, \ln k_2^L, \ln(h_1/r_1)^l$, and $\ln(h_2/r_2)^l$ are independent of λ. Now let

$$\Omega_1 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln \overline{u}_-, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln \bar{l}_- \right\},$$
$$\Omega_2 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln \overline{u}_-, \ln \bar{l}_+ < x_2(t) < \ln k_2^L \right\},$$
$$\Omega_3 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln \overline{u}_+ < x_1(t) < \ln k_1^L, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln \bar{l}_- \right\},$$
$$\Omega_4 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln \overline{u}_+ < x_1(t) < \ln k_1^L, \ln \bar{l}_+ < x_2(t) < \ln k_2^L \right\}. \tag{3.38}$$

Then $\Omega_i,\ i = 1, 2, 3, 4$, are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ i, j = 1, 2, 3, 4$. Thus $\Omega_i,\ i = 1, 2, 3, 4$, satisfy the requirement (a) in Lemma 2.1.

Case iii (for $\theta_1 \geq 1$ and $0 < \theta_2 < 1$). From Case i and Case ii, we can easily obtain that

$$\Omega_1 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln \overline{u}_-, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_- \right\},$$
$$\Omega_2 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln \overline{u}_-, \ln l_+ < x_2(t) < \ln k_2^L \right\},$$
$$\Omega_3 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln \overline{u}_+ < x_1(t) < \ln k_1^L, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_- \right\},$$
$$\Omega_4 = \left\{ x = (x_1, x_2)^T \in \mathbb{X} : \ln \overline{u}_+ < x_1(t) < \ln k_1^L, \ln l_+ < x_2(t) < \ln k_2^L \right\}. \tag{3.39}$$

Then $\Omega_i,\ i=1,2,3,4$, are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ i,j=1,2,3,4$. Thus $\Omega_i,\ i=1,2,3,4$, satisfy the requirement (a) in Lemma 2.1.

Case iv (for $\theta_2 \geq 1$ and $0<\theta_1<1$). From Case i and Case ii, we can easily obtain that

$$\begin{aligned}
\Omega_1 &= \left\{ x=(x_1,x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_-, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln \bar{l}_- \right\}, \\
\Omega_2 &= \left\{ x=(x_1,x_2)^T \in \mathbb{X} : \ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_-, \ln \bar{l}_+ < x_2(t) < \ln k_2^L \right\}, \\
\Omega_3 &= \left\{ x=(x_1,x_2)^T \in \mathbb{X} : \ln u_+ < x_1(t) < \ln k_1^L, \ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln \bar{l}_- \right\}, \\
\Omega_4 &= \left\{ x=(x_1,x_2)^T \in \mathbb{X} : \ln u_+ < x_1(t) < \ln k_1^L, \ln \bar{l}_+ < x_2(t) < \ln k_2^L \right\}.
\end{aligned} \tag{3.40}$$

Then $\Omega_i,\ i=1,2,3,4$, are bounded open subsets of $\mathbb{X}$, $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ i,j=1,2,3,4$. Thus $\Omega_i,\ i=1,2,3,4$, satisfy the requirement (a) in Lemma 2.1.

Now, we prove that (b) of Lemma 2.1 holds; that is, we prove that, when $x \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap \mathbb{R}^2$, $QNu \neq (0,0)^T$ $(i=1,2,3,4)$. If it is not true, then, when $x=(x_1,x_2)^T \in \partial\Omega_i \cap \mathbb{R}^2$, $i=1,2,3,4$, $x=(x_1,x_2)^T$ is constant vector satisfying

$$QNx = m\left((Nx_1, Nx_2)^T\right) = m\begin{pmatrix} r_1(t)\left\{1-\left(\dfrac{e^{x_1}}{k_1(t)}\right)^{\theta_1} - a_{12}(t)\dfrac{e^{x_2}}{k_2(t)}\right\} - h_1(t)e^{-x_1} \\ r_2(t)\left\{1-\left(\dfrac{e^{x_2}}{k_2(t)}\right)^{\theta_2} - a_{21}(t)\dfrac{e^{x_1}}{k_1}\right\} - h_2(t)e^{-x_2} \end{pmatrix} = (0,0)^T. \tag{3.41}$$

Thus there exists a point t_0 such that

$$\begin{pmatrix} r_1(t_0)\left\{1-\left(\dfrac{e^{x_1}}{k_1(t_0)}\right)^{\theta_1} - a_{12}(t_0)\dfrac{e^{x_2}}{k_2(t_0)}\right\} - h_1(t_0)e^{-x_1} \\ r_2(t_0)\left\{1-\left(\dfrac{e^{x_2}}{k_2(t_0)}\right)^{\theta_2} - a_{21}(t_0)\dfrac{e^{x_1}}{k_1(t_0)}\right\} - h_2(t_0)e^{-x_2} \end{pmatrix} = (0,0)^T. \tag{3.42}$$

For $0<\theta_i<1,\ i=1,2$, following the arguments of (3.11)–(3.26), we obtain

$$\begin{aligned}
&\ln\left(\frac{h_1}{r_1}\right)^l < x_1(t) < \ln u_- \quad \text{or} \quad \ln u_+ < x_1(t) < \ln k_1^L, \\
&\ln\left(\frac{h_2}{r_2}\right)^l < x_2(t) < \ln l_- \quad \text{or} \quad \ln l_+ < x_2(t) < \ln k_2^L.
\end{aligned} \tag{3.43}$$

Hence $x \in \Omega_1 \cap \mathbb{R}^2$ or $x \in \Omega_2 \cap \mathbb{R}^2$ or $x \in \Omega_3 \cap \mathbb{R}^2$ or $x \in \Omega_4 \cap \mathbb{R}^2$. This contradicts the fact that $x=(x_1,x_2)^T \in \partial\Omega_i \cap \mathbb{R}^2$ $(i=1,2,3,4)$. This proves that (b) in Lemma 2.1 holds. By a similar argument as above process, for $\theta_i \geq 1$ $(i = 1,2)$ or $\theta_1 \geq 1$ and $0 < \theta_2 < 1$ or $\theta_2 \geq 1$ and $0 < \theta_1 < 1$, we can easily prove that (b) in Lemma 2.1 holds.

A direct computation gives

$$\deg\{JQN, \Omega_i \cap \ker L, 0\} = -1 \quad \text{or} \quad 1 \neq 0, \quad i = 1,2,3,4. \tag{3.44}$$

Here J is taken as the identity mapping. So far we have proved that Ω_i $(i = 1,2,3,4)$ satisfies all the conditions in Lemma 2.1. Hence system (2.3) has at least four periodic solutions in $\overline{\Omega}_i$ $(i = 1,2,3,4)$; that is, system (1.3) has at least four positive periodic solutions. This completes the proof. □

An Example

In system (1.2), let

$$\begin{aligned} &r_1(t) = k_1(t) = \sin t + 2, \qquad r_2(t) = k_2(t) = \cos t + 2, \qquad h_1(t) = \frac{1}{10000(\sin t + 2)}, \\ &h_2(t) = \frac{1}{10000(\cos t + 2)}, \qquad a_{12}(t) = \frac{1}{100(\cos t + 2)}, \qquad a_{21}(t) = \frac{1}{100(\sin t + 2)}. \end{aligned} \tag{3.45}$$

(i) If $\theta_1 = \theta_2 = 1/2$, then system (1.2) has at least four positive 2π-periodic solutions.

(ii) If $\theta_1 = \theta_2 = 1$, then system (1.2) has at least four positive 2π-periodic solutions.

(iii) If $\theta_1 = 1$ and $\theta_2 = 1/2$ or $\theta_2 = 1$ and $\theta_1 = 1/2$, then system (1.2) has at least four positive 2π-periodic solutions.

Proof. By calculation, for $\theta_1 = \theta_2 = 1/2$,

$$\begin{aligned} \left(1 - a_{12}\frac{k_2^L}{k_2}\right)^l &= \frac{299}{300} > 2\sqrt{\frac{3}{100}} = 2\sqrt{\frac{(h_1/r_1)^L}{\left(k_1^{\theta_1}\right)^l \left[(h_1/r_1)^l\right]^{1-\theta_1}}}, \\ \left(1 - a_{21}\frac{k_1^L}{k_1}\right)^l &= \frac{299}{300} > 2\sqrt{\frac{3}{100}} = 2\sqrt{\frac{(h_2/r_2)^L}{\left(k_2^{\theta_2}\right)^l \left[(h_2/r_2)^l\right]^{1-\theta_2}}}. \end{aligned} \tag{3.46}$$

It is obvious that (H_1)-(H_2) hold. By Theorem 3.1 and system (1.2) have at least four positive almost periodic solutions. For $\theta_1 = \theta_2 = 1$,

$$\left(1 - a_{12}\frac{k_2^L}{k_2}\right)^l = \frac{299}{300} > \frac{2}{100} = 2\sqrt{\left(\frac{h_1}{r_1}\right)^L \frac{\left(k_1^L\right)^{\theta_1 - 1}}{\left(k_1^l\right)^{\theta_1}}}, \tag{3.47}$$

$$\left(1 - a_{21}\frac{k_1^L}{k_1}\right)^l = \frac{299}{300} > \frac{2}{100} = 2\sqrt{\left(\frac{h_2}{r_2}\right)^L \frac{\left(k_2^L\right)^{\theta_2 - 1}}{\left(k_2^l\right)^{\theta_2}}}. \tag{3.48}$$

It is obvious that (H_3)-(H_4) hold. By Theorem 3.1, system (1.2) has at least four positive 2π-periodic solutions. For $\theta_1 = 1$ and $\theta_2 = 1/2$ or $\theta_2 = 1$ and $\theta_1 = 1/2$, from (i) and (ii), the result follows from Theorem 3.1. This completes the proof. □

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