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Research Article

The Filling Discs Dealing with Multiple Values of an Algebroid Function in the Unit Disc

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In this paper, by using the potential theory we prove the existence of filling discs dealing with multiple values of an algebroid function of finite order defined in the unit disc.

1. Introduction and Main Result

The value distribution theory of meromorphic functions due to Hayman (see [1] for standard references) was extended to the corresponding theory of algebroid functions by Selbreg [2], Ullrich [3], and Valiron [4] around 1930. The filling discs of an algebroid function are an important part of the value distribution theory. For an algebroid function defined on z-plane, the existence of its filling discs was proved by Sun [5] in 1995. In 1997, for the algebroid functions of infinite order and zero order, Gao [6] obtained the the corresponding results. The existence of the sequence of filling discs of algebroid functions dealing with multiple values, of finite or infinite order, was first proved by Gao [7, 8]. The existence of filling discs in the strong Borel direction of algebroid function with finite order was proved by Huo and Kong in [9]. Compared with the case of $\mathbb{C}$, it is interesting to investigate the algebroid functions defined in the unit disc, and there are some essential differences between these two cases. Recently, the first author [10] has investigated this problem and confirmed the existence of filling discs for this case. In this note, we will continue the work of Xuan [10] by considering the case dealing with multiple values and get more precise results.

Let $w = w(z)$ $(z \in \Delta)$ be the ν-valued algebroid function defined by irreducible equation

$$A_{\nu}(z)w^{\nu} + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0, \tag{1.1}$$

where $A_\nu(z),\ldots,A_0(z)$ are entire functions without any common zeros. The single-valued domain of definition of $w(z)$ is a ν-valued covering of the z-plane, a Riemann surface, denoted by $\widetilde{R}_z$. A point in $\widetilde{R}_z$, whose projection in the z-plane is z, is denoted by $\widetilde{z}$. The part of $\widetilde{R}_z$, which covers the disc $\{z : |z| < r\}$, is denoted by $|\widetilde{z}| < r$.

Denote

$$S(r,w) = \frac{1}{\pi}\iint_{|\widetilde{z}|\le r}\left[\frac{|w'(z)|}{1+|w(z)|^2}\right]^2 d\omega. \tag{1.2}$$

$S(r,w)$ is called the mean covering number of $|\widetilde{z}| \le r$ into w-sphere under the mapping $w = w(z)$. And $S(r,w)$ is conformal invariant. Let $n(r,a)$ be the number of zeros of $w(z)-a$, counted according to their multiplicities in $|\widetilde{z}| \le r$. $\overline{n}^{l)}(E, w=\alpha)$ denotes the number of zeros with multiplicity $\le l$ of $w(z) = \alpha$ in E, each zero being counted only once.

Let

$$\begin{aligned} N(r,a) &= \frac{1}{\nu}\int_0^r \frac{n(t,a)-n(0,a)}{t}dt + \frac{n(0,a)}{\nu}\log r, \\ m(r,a) &= \frac{1}{2\pi\nu}\int_{|\widetilde{z}|=r}\sum_{j=1}^{\nu}\log^+\left|\frac{1}{w_j(re^{i\theta})-a}\right| d\theta, \quad z = re^{i\theta}, \end{aligned} \tag{1.3}$$

where $|\widetilde{z}| = r$ is the boundary of $|\widetilde{z}| \le r$. The characteristic function of $w(z)$ is defined by

$$T(r,w) = \frac{1}{\nu}\int_0^r \frac{S(t,w)}{t}dt. \tag{1.4}$$

In view of [4], we have

$$T(r,w) = m(r,w) + N(r,\infty) + O(1). \tag{1.5}$$

The order of algebroid function $w(z)$ is defined by

$$\rho = \limsup_{r\to 1-}\frac{\log T(r,w)}{\log(1/(1-r))}. \tag{1.6}$$

In this paper we assume that $0 < \rho < +\infty$, V is the w-sphere, and C is a constant which can stand for different constant. Let $n(r,\widetilde{R}_z)$ be the number of the branch points of $\widetilde{R}_z$ in $|\widetilde{z}| \le r$, counted with the order of branch. Write

$$N\left(r,\widetilde{R}_z\right) = \frac{1}{\nu}\int_0^r \frac{n\left(t,\widetilde{R}_z\right)-n\left(0,\widetilde{R}_z\right)}{t}dt + \frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log r. \tag{1.7}$$

Valiron is the first one to introduce the concept of a proximate order $\rho(1/(1-r))$ for a meromorphic function $w(z)$ with finite positive order and $U(1/(1-r)) = (1/(1-r))^{\rho(1/(1-r))}$

is called type function of $w(z)$ or $T(r,w)$ such that $\rho(1/(1-r))$ is nondecreasing, piecewise continuous, and differentiable, and

$$\lim_{r\to 1-}\rho\left(\frac{1}{1-r}\right)=\rho,$$
$$\lim_{r\to 1-}\frac{U(k/(1-r))}{U(1/(1-r))}=k^{\rho}\quad (k \text{ is any given positive constant}),$$
$$\lim_{r\to 1-}\frac{T(r,w)}{U(1/(1-r))}=1,$$
$$\lim_{r\to 1-}\frac{(1/(1-r))^{\rho-\varepsilon}}{U(1/(1-r))}=0,\quad 0<\varepsilon<\rho. \tag{1.8}$$

For an algebroid function $w(z)$ of finite positive order, we can apply the same method to get its type function $U(1/(1-r))$.

Our main result is the following.

Theorem 1.1. *Suppose that $w(z)$ is the ν-valued algebroid function of finite order ρ in $|z|<1$ defined by* (1.1) *and $l(\geq 2\nu+1)$ is an integer, then there exists a sequence of discs*

$$\Gamma_n:\{|z-z_n|<r_n\sigma_n\},\quad n=1,2,\ldots, \tag{1.9}$$

where

$$z_n=r_ne^{i\theta_n},\quad \lim_{n\to\infty}r_n=1,\quad \sigma_n>0,\quad \lim_{n\to\infty}\sigma_n=0. \tag{1.10}$$

Such that for each α

$$\overline{n}^{l)}(\Gamma_n\cap\Delta,w=\alpha)\geq\frac{1}{(1-r_n)^{\rho+1-\varepsilon_n}}, \tag{1.11}$$

except for those complex numbers contained in the union of 2ν spherical discs each with radius $(1-r_n)^{\rho/11}$, where $\lim_{n\to+\infty}\varepsilon_n=0$, $\Delta=\{z:|z|<1\}$.

The discs with the above property are called filling discs dealing with multiple values.

Remark 1.2. In [10], the result says that $\overline{n}(\Gamma_n\cap\Delta,w=\alpha)\geq 1/(1-r_n)^{\rho-\varepsilon_n}$. Theorem 1.1 is really the improvement of [10].

Remark 1.3. The existence of filling discs in Borel radius of meromorphic functions was proved by Kong [11]. In view of our theorem, we can get the similar results of [11] (when $\nu=1$). But we must point out that the structure and definition of filling discs between this paper and [11] are different. There are also some papers relevant to the singular points of algebroid functions in the unit disc (see [12–14]).

2. Two Lemmas

Lemma 2.1 (see [15] or [16]). *Suppose that* $w(z)$ *is the* ν*-valued algebroid function in* $\{z : |z| < R\}$ *defined by* (1.1), $l(\geq 2\nu + 1)$ *is an integer and* $a_1, a_2, a_3, \ldots, a_q$ $(q \geq 3)$ *are distinct points given arbitrarily in* w*-sphere, and the spherical distance of any two points is no smaller than* $\delta \in (0, 1/2)$, *then for any* $r \in (0, R)$, *one has*

$$\left(q - 2 - \frac{2}{l}\right) S(r, w) \leq \sum_{j=1}^{q} \overline{n}^{l)}(R, a_j) + \frac{l+1}{l} n\left(R, \widetilde{R}_z\right) + \frac{CR}{(R - r)\delta^{10}}. \tag{2.1}$$

Combining the potential theory with Lemma 2.1, one proves Lemma 2.2, which is crucial to the theorem.

Lemma 2.2. *Suppose that* $w(z)$ *is the* ν*-valued algebroid function of finite order* ρ *satisfying* $0 < \rho < +\infty$ *in* $|z| < 1$ *defined by* (1.1) *and* $l(\geq 2\nu + 1)$ *is an integer. For any* $\varepsilon \in (0, \rho)$, $0 < R < 1$, *there exists* $a_0 \in (1/2, 1)$, *such that for any* $a \in (a_0, 1)$, *put*

$$r_n = 1 - a^n, \qquad m = \left[\frac{2\pi}{1 - a}\right], \qquad \theta_q = \frac{2\pi(q+1)}{m},$$
$$\Omega_{pq} = \left\{1 - a^p \leq |z| < 1 - a^{p+2}\right\} \cap \left\{\left|\arg z - \theta_q\right| \leq \frac{2\pi}{m}\right\} \quad (p = 1, 2, 3, \ldots;\ q = 0, 1, 2, \ldots, m - 1), \tag{2.2}$$

where $[x]$ *stands for the inter part of* x.

Then, among p, q, *there exists at least one pair* p_0, q_0, *such that* $1 - a^{p_0} > R$, *and in* $\Omega_{p_0 q_0}$,

$$\overline{n}^{l)}(\Omega_{p_0 q_0}, w = \alpha) \geq \frac{1}{a^{\rho + 1 - \varepsilon}}, \tag{2.3}$$

except for those complex numbers contained in the union of 2ν *spherical discs each with radius* $\delta = a^{p_0 \rho / 11}$.

Proof. Suppose the conclusion is false. Then there exists a sequence $\{a_i\}_{i=1}^{\infty}$ $(0 < a_i < 1)$, where $\lim_{i \to \infty} a_i = 1$. For any $a \in \{a_i\}$, any $p > P = \log(1 - R) / \log a$ and $q \in \{0, 1, 2, \ldots, m - 1\}$, there exist $2\nu + 1$ complex numbers which satisfy that the spherical distance of any two of those points is no smaller than $\delta = a^{p\rho/40}$. Denote

$$\left\{\alpha_j = \alpha_j(p, q)\right\}_{j=1}^{2\nu+1}. \tag{2.4}$$

For any p, q mentioned above, we have

$$\overline{n}^{l)}(\Omega_{pq}, w = \alpha_j) < \frac{1}{a^{p(\rho + 1 - \varepsilon)}}. \tag{2.5}$$

For any $r > R$, let $T = [\log(1 - r) / \log a]$, then we have $1 - a^T \leq r < 1 - a^{T+1}$.

For any given positive integers N and M, set

$$b = a^{1/M} \in (0,1), \quad \gamma_{pt} = 1 - b^{Mp+t}, \quad t = 0,1,2,\dots,M-1,$$
$$L_{pt} = \left\{\gamma_{pt} \le |z| < \gamma_{p,t+1}\right\},$$
$$\theta_{qj} = \frac{2\pi q}{m} + \frac{2\pi j}{Nm},$$
$$\triangle_{qj} = \left\{z : |z| < 1 - a^T, \theta_{qj} \le \arg z < \theta_{q,j+1}\right\}. \tag{2.6}$$

Then

$$\left\{1 - a \le |z| < 1 - a^T\right\} = \bigcup_{t=0}^{M-1}\bigcup_{p=1}^{T-1} L_{pt},$$
$$\left\{|z| < 1 - a^T\right\} = \bigcup_{j=0}^{N-1}\bigcup_{q=0}^{m-1} \triangle_{qj}. \tag{2.7}$$

Thus there exists t_0, j_0 which are related to T. We can assume $t_0 = 0, \ j_0 = 0$, such that

$$\sum_{p=1}^{T-1} n\left(L_{p0}, \widetilde{R}_z\right) \le \frac{1}{M} n\left(1 - a^T, \widetilde{R}_z\right),$$
$$\sum_{q=0}^{m-1} n\left(\triangle_{q0}, \widetilde{R}_z\right) \le \frac{1}{N} n\left(1 - a^T, \widetilde{R}_z\right). \tag{2.8}$$

Set

$$\Omega_{pq}^0 = \left\{1 - \frac{b^{Mp} + b^{Mp+1}}{2} \le |z| < 1 - \frac{b^{Mp+M} + b^{Mp+M+1}}{2}\right\} \cap \left\{\frac{\theta_{q0} + \theta_{q1}}{2} \le \arg z < \frac{\theta_{q+1,0} + \theta_{q+1,1}}{2}\right\},$$
$$\overline{\Omega}_{pq} = \left\{1 - b^{Mp} \le |z| < 1 - b^{Mp+M+1}\right\} \cap \left\{\theta_{q0} \le \arg z < \theta_{q+1,1}\right\}. \tag{2.9}$$

Then we have

$$\Omega_{pq}^0 \subset \overline{\Omega}_{pq} \subset \Omega_{pq}. \tag{2.10}$$

Since $\{\overline{\Omega}_{pq}\}_{p,q}$ covers $\bigcup_{p=1}^{T-1} L_{p0}$ and $\bigcup_{q=0}^{m-1} \triangle_{q0}$ twice at most. We obtain

$$\sum_{p=1}^{T-1}\sum_{q=0}^{m-1} n\left(\overline{\Omega}_{pq}, \widetilde{R}_z\right) \le \left(1 + \frac{1}{M} + \frac{1}{N}\right) n\left(1 - a^T, \widetilde{R}_z\right). \tag{2.11}$$

Obviously, each $\overline{\Omega}_{pq}$ can be mapped conformally to the unit disc $|\zeta| < 1$ such that the center of $\overline{\Omega}_{pq}$ is mapped to $\zeta = 0$, and the image of Ω_{pq}^0 is contained in the disc $|\zeta| < \eta(<1)$. Since all $\overline{\Omega}_{pq}$, Ω_{pq}^0 are similar, C is independent of p,q. Since S is conformally invariant, in view of Lemma 2.1, we obtain

$$\begin{aligned}\left(2\nu-1-\frac{2}{l}\right)S\left(1-a^T,w\right) &\le \left(2\nu-1-\frac{2}{l}\right)\sum_{p=P+1}^{T-1}\sum_{q=0}^{m-1}S\left(\Omega_{pq}^0,w\right)+\left(2\nu-1-\frac{2}{l}\right)S\left(1-a^2,w\right)\\ &\le \sum_{p=P+1}^{T-1}\sum_{q=0}^{m-1}\left[\sum_{j=1}^{2\nu+1}\overline{n}^{l)}\left(\overline{\Omega}_{pq},w=\alpha_j\right)+\frac{l+1}{l}n\left(\overline{\Omega}_{pq},\widetilde{R}_z\right)+\frac{C}{\delta^{10}(1-\eta)}\right]\\ &\quad+\left(2\nu-1-\frac{2}{l}\right)S\left(1-a^2,w\right)\\ &\le 3\nu Tma^{-T(\rho+1-\varepsilon)}+\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)n\left(1-a^T,\widetilde{R}_z\right)\\ &\quad+CT\left(a^{-T\rho/11}\right)^{10}+\left(2\nu-1-\frac{2}{l}\right)S\left(1-a^2,w\right).\end{aligned}\tag{2.12}$$

For sufficiently large integer $T(=[\log(1-r)/\log a])$, $r\in[1-a^T,1-a^{T+1})$. Thus we get

$$\begin{aligned}\left(2\nu-1-\frac{2}{l}\right)S\left(1-\frac{1-r}{a},w\right) &\le \left(2\nu-1-\frac{2}{l}\right)S\left(1-a^T,w\right)\\ &\le \frac{1}{(1-r)^{\rho+1-(\varepsilon/2)}}+\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)n\left(r,\widetilde{R}_z\right)\\ &\quad+C\left(\frac{1}{1-r}\right)^{10\rho/11}+C,\end{aligned}\tag{2.13}$$

where C is a constant.

For any integer $T(=[\log(1-r)/\log a])$, we have $(1-a^T)/(a-a^T)\in(1/a,(1+a)/a)$. We can choose one $a(>((l+1)/l)((2\nu-2)/(2\nu-1-(2/l)))\in(0,1)$ as $l\ge 2\nu+1)$ such that $(l/(l+1))((2\nu-1-(2/l))/(2\nu-2))\in(1/a,(1+a)/a)$. For a certain sufficiently fixed large inter T, we have

$$k=\left(1-\frac{1}{1+a+a^2+\cdots+a^{T-1}}\right)^{-1}=\frac{1-a^T}{a-a^T}<\frac{l}{l+1}\cdot\frac{2\nu-1-(2/l)}{2\nu-2}.\tag{2.14}$$

This yields the following:

$$\begin{aligned}\frac{1-((1-t)/a)}{t} &= \frac{1}{a}-\left(\frac{1}{a}-1\right)\frac{1}{t}\\ &\ge \frac{1}{a}-\left(\frac{1}{a}-1\right)\frac{1}{1-a^T}=\frac{1}{a}\left(1-\frac{1}{1+a+\cdots+a^{T-1}}\right)=\frac{1}{ak},\end{aligned}\tag{2.15}$$

where $t\in[1-a^T,1-a^{T+1})$.

Hence

$$\int_{1-a^T}^{r}\frac{S(1-((1-t)/a),w)}{t}dt=\int_{1-a^T}^{r}\frac{S(1-((1-t)/a),w)}{1-((1-t)/a)}\frac{1-((1-t)/a)}{t}dt \geq \frac{1}{ak}\int_{1-a^T}^{r}\frac{S(1-((1-t)/a),w)}{1-((1-t)/a)}dt = \frac{1}{k}\int_{1-a^{T-1}}^{1-((1-r)/a)}\frac{S(x,w)}{x}dx. \tag{2.16}$$

Next, we deduce the following:

$$\int_{1-a^T}^{r}\frac{1}{(1-t)^{\rho+1-(\varepsilon/2)}t}dt \leq \frac{1}{1-a^T}\int_{1-a^T}^{r}\frac{1}{(1-t)^{\rho+1-(\varepsilon/2)}}dt = -\frac{1}{1-a^T}\int_{1-a^T}^{r}\frac{1}{(1-t)^{\rho+1-(\varepsilon/2)}}d(1-t) \leq \frac{1}{1-a^T}\cdot\frac{1}{\rho-(\varepsilon/2)}\cdot\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}, \tag{2.17}$$

when $r \geq t \geq 1-a^T$.

In view of $1-t \leq t$ for $t \in [1-a^T, 1-a^{T+1})$, we have

$$\int_{1-a^T}^{r}\frac{1}{(1-t)^{10\rho/11}t}dt \leq \int_{1-a^T}^{r}\frac{1}{(1-t)^{(10\rho/11)+1}}dt \leq \frac{1}{10\rho/11}\cdot\frac{1}{(1-r)^{10\rho/11}}. \tag{2.18}$$

Dividing both sides of (2.13) by νt and integrating it from $1-a^T$ to r, we have

$$\begin{aligned}
&\left(2\nu-1-\frac{2}{l}\right)\frac{1}{\nu}\int_{1-a^T}^{r}\frac{S(1-((1-t)/a),w)}{t}dt\\
&\quad\leq \frac{1}{\nu}\int_{1-a^T}^{r}\frac{1}{(1-t)^{\rho+1-(\varepsilon/2)}t}dt+\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)\\
&\qquad\times\left[\frac{1}{\nu}\int_{1-a^T}^{r}\frac{n\left(t,\widetilde{R}_z\right)-n\left(0,\widetilde{R}_z\right)}{t}dt+\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log r-\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log\left(1-a^T\right)\right]\\
&\qquad+\frac{C}{\nu}\int_{1-a^T}^{r}\frac{1}{(1-t)^{10\rho/11}t}dt\leq\frac{1}{\nu}\cdot\frac{1}{1-a^T}\cdot\frac{1}{\rho-(\varepsilon/2)}\cdot\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}\\
&\qquad+\frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)\left[\frac{1}{\nu}\int_{0}^{r}\frac{n\left(t,\widetilde{R}_z\right)-n\left(0,\widetilde{R}_z\right)}{t}dt+\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log r\right]\\
&\qquad-\frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log\left(1-a^T\right)+\frac{C}{\nu}\frac{1}{10\rho/11}\frac{1}{(1-r)^{10\rho/11}}
\end{aligned}$$

$$= \frac{1}{\nu}\cdot\frac{1}{\rho-(\varepsilon/2)}\cdot\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}+\frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)N\left(r,\widetilde{R}_z\right)$$
$$-\frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log\left(1-a^T\right)+\frac{C}{\nu}\frac{1}{10\rho/11}\frac{1}{(1-r)^{10\rho/11}}. \tag{2.19}$$

Note that T is fixed, we see that $T(1-a^{T-1},w)$ is a finite constant.

Hence,

$$\begin{aligned}
&\left(2\nu-1-\frac{2}{l}\right)\frac{1}{k}\frac{1}{\nu}\int_{1-a^{T-1}}^{1-((1-r)/a)}\frac{S(t,w)}{t}dt\\
&\quad\le\left(2\nu-1-\frac{2}{l}\right)\frac{1}{\nu}\int_{1-a^T}^{r}\frac{S(1-((1-t)/a))}{t}dt\\
&\quad\le\frac{1}{1-a^T}\cdot\frac{1}{\nu}\cdot\frac{1}{\rho-(\varepsilon/2)}\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}+\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)N\left(r,\widetilde{R}_z\right)\\
&\qquad-\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log\left(1-a^T\right)+\frac{C}{\nu}\frac{1}{10\rho/11}\frac{1}{(1-r)^{10\rho/11}}.
\end{aligned}\tag{2.20}$$

Then

$$\begin{aligned}
&\left(2\nu-1-\frac{2}{l}\right)\frac{1}{k\nu}\int_{0}^{1-((1-r)/a)}\frac{S(t,w)}{t}dt\le\frac{1}{1-a^T}\cdot\frac{1}{\nu}\frac{1}{\rho-(\varepsilon/2)}\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}\\
&\quad+\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)N\left(r,\widetilde{R}_z\right)-\frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)\frac{n\left(0,\widetilde{R}_z\right)}{\nu}\log\left(1-a^T\right)\\
&\quad+\frac{C}{\nu}\frac{1}{10\rho/11}\frac{1}{(1-r)^{10\rho/11}}+\frac{1}{k}T\left(1-a^{T-1},w\right).
\end{aligned}\tag{2.21}$$

In view of [3], we know that

$$N\left(r,\widetilde{R}_z\right)\le 2(\nu-1)T(r,w)+O(1). \tag{2.22}$$

We obtain

$$\begin{aligned}
\left(2\nu-1-\frac{2}{l}\right)\cdot\frac{1}{k}T\left(1-\frac{1-r}{a},w\right)&\le\frac{1}{1-a^T}\cdot\frac{1}{\nu\left(\rho-(\varepsilon/2)\right)}\frac{1}{(1-r)^{\rho-(\varepsilon/2)}}\\
&\quad+\frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)2(\nu-1)T(r,w)\\
&\quad+C\log\left(1-a^T\right)+\frac{C}{\nu}\frac{1}{10\rho/11}\frac{1}{(1-r)^{10\rho/11}}+C,
\end{aligned}\tag{2.23}$$

where C is a constant.

Dividing both sides of the above inequality by $U(1/(1-r)) = (1/(1-r))^{\rho(1/(1-r))}$, we have

$$\left(2\nu - 1 - \frac{2}{l}\right)\cdot\frac{1}{k}\frac{T(1-((1-r)/a),w)}{U(1/(1-r))} \le \frac{1}{1-a^T}\cdot\frac{1}{\nu(\rho-(\varepsilon/2))}\frac{(1-r)^{\rho(1/(1-r))}}{(1-r)^{\rho-(\varepsilon/2)}} + \frac{l+1}{l}\cdot\left(1+\frac{1}{M}+\frac{1}{N}\right)2(\nu-1)\frac{T(r,w)}{U(1/(1-r))} + C\frac{\log(1-a^T)}{(1/(1-r))^{\rho(1/(1-r))}} + \frac{C}{\nu}\frac{1}{10\rho/11}\frac{(1-r)^{\rho(1/(1-r))}}{(1-r)^{10\rho/11}} + \frac{C}{(1/(1-r))^{\rho(1/(1-r))}}. \tag{2.24}$$

We note that

$$\frac{T(1-((1-r)/a),w)}{U(1/(1-r))} = \frac{T(1-((1-r)/a),w)}{U(a/(1-r))}\frac{U(a/(1-r))}{U(1/(1-r))}. \tag{2.25}$$

In view of the properties of the $U(1/(1-r))$, we obtain

$$\begin{aligned}\limsup_{r\to 1-}\frac{T(1-((1-r)/a),w)}{U(1/(1-r))} &\ge \limsup_{r\to 1-}\frac{T(1-((1-r)/a),w)}{U(a/(1-r))}\liminf_{r\to 1-}\frac{U(a/(1-r))}{U(1/(1-r))}\\ &= \limsup_{r\to 1-}\frac{T(1-((1-r)/a),w)}{U(a/(1-r))}\lim_{r\to 1-}\frac{U(a/(1-r))}{U(1/(1-r))} = a^{\rho}.\end{aligned} \tag{2.26}$$

Letting $r \to 1-$ in (2.24), we have

$$\left(2\nu - 1 - \frac{2}{l}\right)\frac{1}{k}a^{\rho} \le \frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)2(\nu-1), \tag{2.27}$$

that is,

$$2\nu - 1 - \frac{2}{l} \le \frac{l+1}{l}\left(1+\frac{1}{M}+\frac{1}{N}\right)2(\nu-1)ka^{-\rho}. \tag{2.28}$$

Letting $a \to 1-$, $M \to +\infty$, and $N \to +\infty$, we obtain

$$k \ge \frac{l}{l+1}\frac{2\nu-1-(2/l)}{2\nu-2}. \tag{2.29}$$

This is contradictory to $k < (l/(l+1))((2\nu-1-(2/l))/(2\nu-2))$, and the lemma is proved. □

3. Proof of the Theorem

Proof. Choose $\varepsilon_n = \rho/2n, R_n = 1-(1/2^n)$.

In view of Lemma 2.2, there exists $a_n \in (1-(1/n),1)$, $m_n = [2\pi/(1-a_n)], p_n, q_n, \theta_{q_n} = (2\pi(q_n)+1)/m_n$, and

$$\Omega_{p_n q_n} = \left\{1 - a_n^{p_n} \le |z| \le 1 - a_n^{p_n+2}\right\} \bigcap \left\{ \left|\arg z - \theta_{q_n}\right| \le \frac{2\pi}{m_n} \right\}, \quad (n = 1,2,\ldots). \tag{3.1}$$

Let

$$\theta_n = \theta_{q_n}, \qquad z_n = \left(1 - a_n^{p_n}\right) e^{i\theta_n}. \tag{3.2}$$

Then

$$1 > r_n = |z_n| = 1 - a_n^{p_n} > R_n = 1 - \frac{1}{2^n} \longrightarrow 1 - (n \longrightarrow +\infty), \quad \lim_{n\to+\infty} a_n^{p_n} = 0. \tag{3.3}$$

Set

$$\begin{aligned} B_n &= \left[\left(1 - a_n^{p_n+2}\right) - \left(1 - a_n^{p_n}\right)\right] + \left(1 - a_n^{p_n+2}\right)\frac{2\pi}{m_n} \\ &\le \left[\left(1 - a_n^{2p_n}\right) - \left(1 - a_n^{p_n}\right)\right] + \left(1 - a_n^{2p_n}\right)\frac{2\pi}{m_n} \\ &= \left(1 - a_n^{p_n}\right)a_n^{p_n} + \left(1 - a_n^{p_n}\right)\left(1 + a_n^{p_n}\right)\frac{2\pi}{m_n} \\ &\le \left(1 - a_n^{p_n}\right)\left[a_n^{p_n} + \left(1 + a_n^{p_n}\right)\frac{2\pi}{m_n}\right]. \end{aligned} \tag{3.4}$$

Take

$$\sigma_n = a_n^{p_n} + \left(1 + a_n^{p_n}\right)\frac{2\pi}{m_n}, \tag{3.5}$$

then

$$\sigma_n \longrightarrow 0 \quad (n \longrightarrow +\infty). \tag{3.6}$$

Put

$$\Gamma_n = \{|z - z_n| < r_n \sigma_n\}. \tag{3.7}$$

Then

$$\Omega_{p_n q_n} \subset \Gamma_n. \tag{3.8}$$

In view of Lemma 2.2, for each n,

$$\overline{n}^{l)}(\Gamma_n \cap \triangle, w = \alpha) \geq \frac{1}{a_n^{\rho+1-\varepsilon_n}}, \tag{3.9}$$

except for those complex numbers contained in the union of 2ν spherical discs each with radius $a_n^{p_n\rho/11} = (1 - r_n)^{\rho/11}$. Theorem 1.1 is proved. □

Remark 3.1. By using the same method, we can prove the existence of filling discs for *K-quasimeromorphic mappings* whose general case is carefully discussed in another paper.

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