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Research Article

Coefficient Conditions for Harmonic Close-to-Convex Functions

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New sufficient conditions, concerned with the coefficients of harmonic functions $f(z) = h(z)+\overline{g(z)}$ in the open unit disk $\mathbb{U}$ normalized by $f(0) = h(0) = h'(0) - 1 = 0$, for $f(z)$ to be harmonic close-to-convex functions are discussed. Furthermore, several illustrative examples and the image domains of harmonic close-to-convex functions satisfying the obtained conditions are enumerated.

1. Introduction

For a continuous complex-valued function $f(z) = u(x, y) + iv(x, y)$ $(z = x + iy)$, we say that $f(z)$ is harmonic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ if both $u(x, y)$ and $v(x, y)$ are real harmonic in $\mathbb{U}$, that is, $u(x, y)$ and $v(x, y)$ satisfy the Laplace equations

$$\Delta u = u_{xx} + u_{yy} = 0, \qquad \Delta v = v_{xx} + v_{yy} = 0. \tag{1.1}$$

A complex-valued harmonic function $f(z)$ in $\mathbb{U}$ is given by $f(z) = h(z)+\overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$. We call $h(z)$ and $g(z)$ the analytic part and the coanalytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense preserving in $\mathbb{U}$ is $|h'(z)| > |g'(z)|$ in $\mathbb{U}$ (see [1] or [2]). Let $\mathscr{H}$ denote the class of harmonic functions $f(z)$ in $\mathbb{U}$ with $f(0) = h(0) = 0$ and $h'(0) = 1$. Thus, every normalized harmonic function $f(z)$ can be written by

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in \mathscr{H}, \tag{1.2}$$

where $a_1 = 1$ and $b_0 = 0$, for convenience.

We next denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f(z) \in \mathcal{H}$ that are univalent and sense preserving in $\mathbb{U}$. Due to the sense-preserving property of $f(z)$, we see that $|b_1| = |g'(0)| < |h'(0)| = 1$. If $g(z) \equiv 0$, then $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ consisting of normalized analytic univalent functions. Furthermore, for every function $f(z) \in \mathcal{S}_{\mathcal{H}}$, the function

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1 - |b_1|^2} = z + \sum_{n=2}^{\infty} \frac{a_n - \overline{b_1} b_n}{1 - |b_1|^2} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n - b_1 a_n}{1 - |b_1|^2} z^n} \tag{1.3}$$

is also a member of $\mathcal{S}_{\mathcal{H}}$. Therefore, we consider the subclass $\mathcal{S}_{\mathcal{H}}^0$ of $\mathcal{S}_{\mathcal{H}}$ defined as

$$\mathcal{S}_{\mathcal{H}}^0 = \left\{ f(z) \in \mathcal{S}_{\mathcal{H}} : b_1 = g'(0) = 0 \right\}. \tag{1.4}$$

Conversely, if $F(z) \in \mathcal{S}_{\mathcal{H}}^0$, then $f(z) = F(z) + \overline{b_1 F(z)} \in \mathcal{S}_{\mathcal{H}}$ for any b_1 $(|b_1| < 1)$.

We say that a domain $\mathbb{D}$ is a close-to-convex domain if the complement of $\mathbb{D}$ can be written as a union of nonintersecting half-lines (except that the origin of one half-line may lie on one of the other half-lines). Let $\mathcal{C}$, $\mathcal{C}_{\mathcal{H}}$, and $\mathcal{C}_{\mathcal{H}}^0$ be the respective subclasses of $\mathcal{S}$, $\mathcal{S}_{\mathcal{H}}$, and $\mathcal{S}_{\mathcal{H}}^0$ consisting of all functions $f(z)$, which map $\mathbb{U}$ onto a certain close-to-convex domain.

Bshouty and Lyzzaik [3] have stated the following result.

Theorem 1.1. *If $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ satisfies*

$$g'(z) = z h'(z), \qquad \operatorname{Re}\left(1 + \frac{z h''(z)}{h'(z)}\right) > -\frac{1}{2} \tag{1.5}$$

for all $z \in \mathbb{U}$, then $f(z) \in \mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}^0$.

A simple and interesting example is below.

Example 1.2. The function

$$f(z) = \frac{1 - (1 - z)^2}{2(1 - z)^2} + \overline{\frac{z^2}{2(1 - z)^2}} = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \sum_{n=2}^{\infty} \frac{n-1}{2} \overline{z}^n \tag{1.6}$$

satisfies the conditions of Theorem 1.1, and therefore $f(z)$ belongs to the class $\mathcal{C}_{\mathcal{H}}^0$. We now show that $f(\mathbb{U})$ is actually a close-to-convex domain. It follows that

$$\begin{aligned} f(z) &= \left(\frac{z}{2(1 - z)^2} + \frac{z}{2(1 - z)}\right) + \overline{\left(\frac{z}{2(1 - z)^2} - \frac{z}{2(1 - z)}\right)} \\ &= \operatorname{Re}\left(\frac{z}{(1 - z)^2}\right) + i \operatorname{Im}\left(\frac{z}{1 - z}\right). \end{aligned} \tag{1.7}$$

Setting

$$f\left(re^{i\theta}\right) = \frac{-2r^2 + r\left(1+r^2\right)\cos\theta}{\left(1+r^2-2r\cos\theta\right)^2} + \frac{r\sin\theta}{1+r^2-2r\cos\theta}i = u + iv \tag{1.8}$$

for any $z = re^{i\theta} \in \mathbb{U}$ $(0 \leqq r < 1,\ 0 \leqq \theta < 2\pi)$, we see that

$$-4\left(u+v^2\right) = \frac{4r(r-\cos\theta)(1-r\cos\theta)}{\left(1+r^2-2r\cos\theta\right)^2} = \frac{4r(r-t)(1-rt)}{\left(1+r^2-2rt\right)^2} \equiv \phi(t) \quad (-1 \leqq t = \cos\theta \leqq 1). \tag{1.9}$$

Since

$$\phi'(t) = \frac{-4r\left(1-r^2\right)^2}{\left(1+r^2-2rt\right)^3} \leqq 0, \tag{1.10}$$

we obtain that

$$\phi(t) \leqq \phi(-1) = \frac{4r}{(1+r)^2} \equiv \psi(r). \tag{1.11}$$

Also, noting that

$$\psi'(r) = \frac{4(1-r)}{(1+r)^3} > 0, \tag{1.12}$$

we know that

$$\psi(r) < \psi(1) = 1, \tag{1.13}$$

which implies that

$$u > -v^2 - \frac{1}{4}. \tag{1.14}$$

Thus, $f(z)$ maps $\mathbb{U}$ onto the following close-to-convex domain as shown in Figure 1.

Remark 1.3. Let $\mathcal{M}$ be the class of all functions satisfying the conditions of Theorem 1.1. Then, it was earlier conjectured by Mocanu [4, 5] that $\mathcal{M} \subset \mathcal{S}^0_{\mathcal{H}}$. Furthermore, we can immediately see that the function $f(z)$ in Example 1.2 is a member of the class $\mathcal{M}$ and it shows that $f(z) \in \mathcal{M}$ is not necessarily starlike with respect to the origin in $\mathbb{U}$ ($f(z)$ is starlike with respect to the origin in $\mathbb{U}$ if and only if $tw \in f(\mathbb{U})$ for all $w \in f(\mathbb{U})$ and t $(0 \leqq t \leqq 1)$).

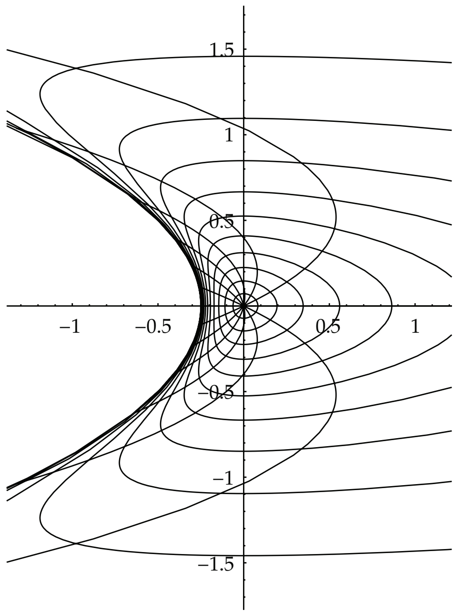

Figure 1: The image of $f(z) = (1-(1-z)^2)/2(1-z)^2 + \overline{z^2/2(1-z)^2}$.

Remark 1.4. For the function $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ given by

$$g'(z) = z^{n-1}h'(z) \quad (n = 2, 3, 4, \ldots), \tag{1.15}$$

letting $w(t) = f(e^{it}) = h(e^{it}) + \overline{g(e^{it})}$ $(-\pi \leqq t < \pi)$, we know that

$$\operatorname{Im}\left(\frac{w''(t)}{w'(t)}\right) \leqq 0 \quad (-\pi \leqq t < \pi), \tag{1.16}$$

which means that $f(z)$ maps the unit circle $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$ onto a union of several concave curves (see [6, Theorem 2.1]).

Jahangiri and Silverman [7] have given the following coefficient inequality for $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$.

Theorem 1.5. *If $f(z) \in \mathcal{H}$ satisfies*

$$\sum_{n=2}^{\infty} n \mid a_n \mid + \sum_{n=1}^{\infty} n \mid b_n \mid \leqq 1, \tag{1.17}$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 1.6. The function

$$f(z) = z + \frac{1}{5}\overline{z}^5 \tag{1.18}$$

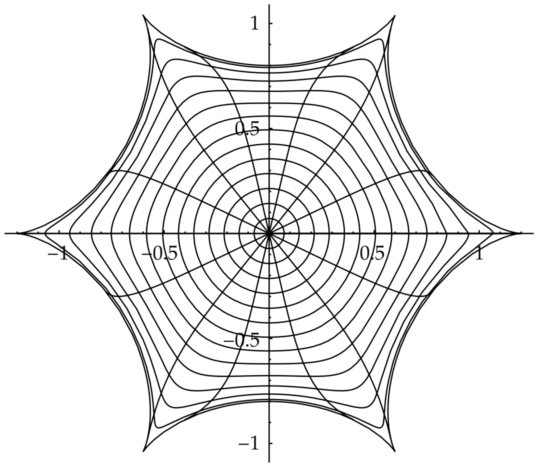

Figure 2: The image of $f(z) = z + (1/5)\overline{z}^5$.

belongs to the class $\mathcal{C}_{\mathcal{H}}^0 \subset \mathcal{C}_{\mathcal{H}}$ and satisfies the condition of Theorem 1.5. Indeed, $f(z)$ maps $\mathbb{U}$ onto the following hypocycloid of six cusps (cf. [8] or [6]) as shown in Figure 2.

The object of this paper is to find some sufficient conditions for functions $f(z) \in \mathcal{H}$ to be in the class $\mathcal{C}_{\mathcal{H}}$. In order to establish our results, we have to recall here the following lemmas due to Clunie and Sheil-Small [1].

Lemma 1.7. *If $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $|h'(0)| > |g'(0)|$ and $h(z) + \varepsilon g(z)$ is close-to-convex for each ε ($|\varepsilon| = 1$), then $f(z) = h(z) + \overline{g(z)}$ is harmonic close-to-convex.*

Lemma 1.8. *If $f(z) = h(z) + \overline{g(z)}$ is locally univalent in $\mathbb{U}$ and $h(z) + \varepsilon g(z)$ is convex for some ε ($|\varepsilon| \leqq 1$), then $f(z)$ is univalent close-to-convex.*

We also need the following result due to Hayami et al. [9].

Lemma 1.9. *If a function $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is analytic in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j} j(j+1)\binom{\alpha}{k-j}A_j\right\}\binom{\beta}{n-k}\right| + \left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(-1)^{k-j} j(j-1)\binom{\alpha}{k-j}A_j\right\}\binom{\beta}{n-k}\right|\right] \leqq 2 \tag{1.19}$$

for some real numbers α and β, then $F(z)$ is convex in $\mathbb{U}$.

2. Main Results

Our first result is contained in the following theorem.

Theorem 2.1. *If $f(z) \in \mathscr{H}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left|na_n - e^{i\varphi}(n-1)a_{n-1}\right| + \sum_{n=1}^{\infty}\left|nb_n - e^{i\varphi}(n-1)b_{n-1}\right| \leqq 1 \tag{2.1}$$

for some real number φ $(0 \leqq \varphi < 2\pi)$, then $f(z) \in \mathcal{C}_{\mathscr{H}}$.

Proof. Let $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ be analytic in $\mathbb{U}$. If $F(z)$ satisfies

$$\sum_{n=2}^{\infty}\left|nA_n - e^{i\varphi}(n-1)A_{n-1}\right| \leqq 1, \tag{2.2}$$

then it follows that

$$\begin{aligned}\left|\left(1-e^{i\varphi}z\right)F'(z) - 1\right| &= \left|\sum_{n=2}^{\infty}\left(nA_n - e^{i\varphi}(n-1)A_{n-1}\right)z^{n-1}\right| \\ &\leqq \sum_{n=2}^{\infty}\left|nA_n - e^{i\varphi}(n-1)A_{n-1}\right| \cdot |\,z\,|^{n-1} \\ &< \sum_{n=2}^{\infty}\left|nA_n - e^{i\varphi}(n-1)A_{n-1}\right| \leqq 1 \quad (z \in \mathbb{U}).\end{aligned} \tag{2.3}$$

This gives us that

$$\operatorname{Re}\left(\left(1-e^{i\varphi}z\right)F'(z)\right) > 0 \quad (z \in \mathbb{U}), \tag{2.4}$$

that is, $F(z) \in \mathcal{C}$. Then, it is sufficient to prove that

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b_1} = z + \sum_{n=2}^{\infty}\frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1}z^n \in \mathcal{C} \tag{2.5}$$

for each ε $(|\varepsilon| = 1)$ by Lemma 1.7. From the assumption of the theorem, we obtain that

$$\begin{aligned}&\sum_{n=2}^{\infty}\left|n\frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} - e^{i\varphi}(n-1)\frac{a_{n-1} + \varepsilon b_{n-1}}{1 + \varepsilon b_1}\right| \\ &\quad \leqq \frac{1}{1-|\,b_1\,|}\sum_{n=2}^{\infty}\left[\left|na_n - e^{i\varphi}(n-1)a_{n-1}\right| + \left|nb_n - e^{i\varphi}(n-1)b_{n-1}\right|\right] \leqq \frac{1-|\,b_1\,|}{1-|\,b_1\,|} = 1.\end{aligned} \tag{2.6}$$

This completes the proof of the theorem. □

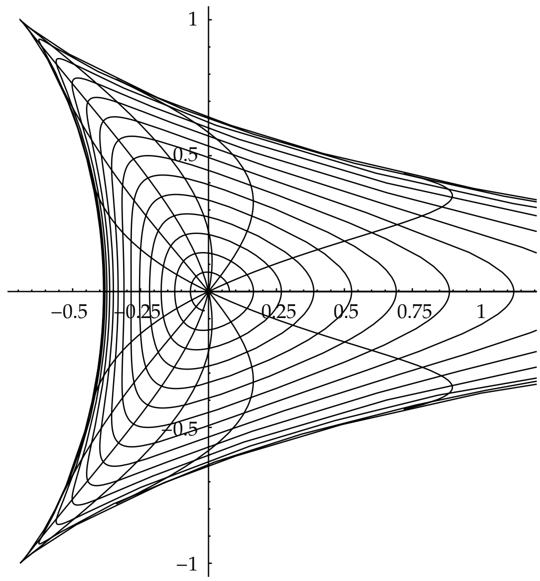

Figure 3: The image of $f(z) = -\overline{z} - 2\log|1-z|$.

Example 2.2. The function

$$f(z) = -\log(1-z) + \overline{(-mz - \log(1-z))} = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n + (1-m)\overline{z} + \sum_{n=2}^{\infty} \frac{1}{n} \overline{z}^n \quad (0 < m \leqq 1) \tag{2.7}$$

satisfies the condition of Theorem 2.1 with $\varphi = 0$ and belongs to the class $\mathcal{C}_{\mathcal{H}}$. In particular, putting $m = 1$, we obtain Figure 3.

By making use of Lemma 1.8 with $\varepsilon = 0$ and applying Lemma 1.9, we readily obtain the next theorem.

Theorem 2.3. *If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies*

$$\sum_{n=2}^{\infty} \left[\left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j+1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| + \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j-1) \binom{\alpha}{k-j} a_j \right\} \binom{\beta}{n-k} \right| \right] \leqq 2 \tag{2.8}$$

for some real numbers α and β, then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Putting $\alpha = \beta = 0$ in the above theorem, we arrive at the following result due to Jahangiri and Silverman [7].

Theorem 2.4. *If* $f(z) \in \mathcal{H}$ *is locally univalent in* $\mathbb{U}$ *with*

$$\sum_{n=2}^{\infty} n^2 |a_n| \leqq 1, \tag{2.9}$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Furthermore, taking $\alpha = 1$ and $\beta = 0$ in the theorem, we have the following corollary.

Corollary 2.5. *If* $f(z) \in \mathcal{H}$ *is locally univalent in* $\mathbb{U}$ *and satisfies*

$$\sum_{n=2}^{\infty} \{ n|(n+1)a_n - (n-1)a_{n-1}| + (n-1)|na_n - (n-2)a_{n-1}| \} \leqq 2, \tag{2.10}$$

then $f(z) \in \mathcal{C}_{\mathcal{H}}$.

Example 2.6. The function

$$f(z) = -\int_0^z \frac{\log(1-t)}{t} dt + \overline{(z + (1-z)\log(1-z))} = z + \sum_{n=2}^{\infty} \frac{1}{n^2} z^n + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \overline{z}^n \tag{2.11}$$

satisfies the conditions of Corollary 2.5 and belongs to the class $\mathcal{C}_{\mathcal{H}}$ as shown in Figure 4.

3. Appendix

A sequence $\{c_n\}_{n=0}^{\infty}$ of nonnegative real numbers is called a convex null sequence if $c_n \to 0$ as $n \to \infty$ and

$$c_n - c_{n+1} \geqq c_{n+1} - c_{n+2} \geqq 0 \tag{3.1}$$

for all n $(n = 0, 1, 2, \ldots)$.

The next lemma was obtained by Fejér [10].

Lemma 3.1. *Let* $\{c_n\}_{k=0}^{\infty}$ *be a convex null sequence. Then, the function*

$$p(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \tag{3.2}$$

is analytic and satisfies $\operatorname{Re}(p(z)) > 0$ *in* $\mathbb{U}$.

Applying the above lemma, we deduce the following theorem.

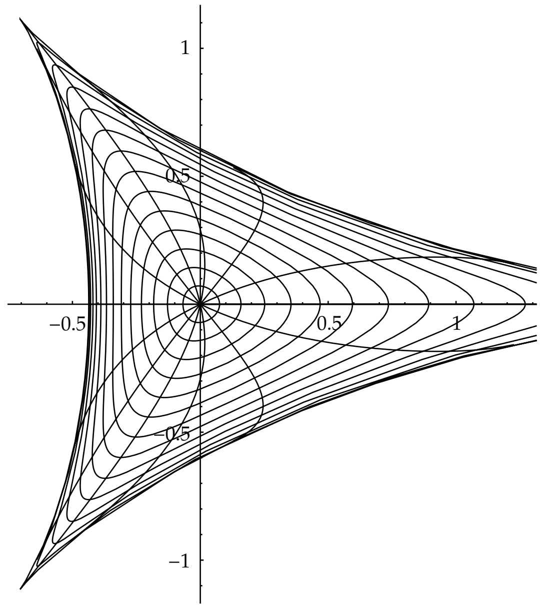

Figure 4: The image of $f(z) = -\int_0^z (\log(1-t)/t)\ dt + \overline{(z + (1-z)\log(1-z))}$.

Theorem 3.2. *For some* b $(|b| < 1)$ *and some convex null sequence* $\{c_n\}_{n=0}^{\infty}$ *with* $c_0 = 2$, *the function*

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n + \overline{b\left(z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \right)} \tag{3.3}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \tag{3.4}$$

for each ε $(|\varepsilon| = 1)$. Then, we know that

$$F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2). \tag{3.5}$$

By virtue of Lemmas 1.7 and 3.1, it follows that $\mathrm{Re}(F'(z)) > 0$ $(z \in \mathbb{U})$, that is, $F(z) \in \mathcal{C}$. Thus, we conclude that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{H}}$. □

In the same manner, we also have the following theorem.

Theorem 3.3. *For some b $(|b| < 1)$ and some convex null sequence $\{c_n\}_{n=0}^{\infty}$ with $c_0 = 2$, the function*

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} c_j\right) z^n + b\overline{\left(z + \sum_{n=2}^{\infty} \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} c_j\right) z^n\right)} \tag{3.6}$$

belongs to the class $\mathcal{C}_{\mathcal{H}}$.

Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{1}{n}\left(1 + \sum_{j=1}^{n-1} c_j\right) z^n \tag{3.7}$$

for each ε $(|\varepsilon| = 1)$. Then, we know that

$$(1 - z)F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \qquad (c_0 = 2). \tag{3.8}$$

Therefore, by the help of Lemmas 1.7 and 3.1, we obtain that $\operatorname{Re}((1 - z)F'(z)) > 0$ $(z \in \mathbb{U})$, that is, $F(z) \in \mathcal{C}$, which implies that $f(z) = h(z) + \overline{g(z)} \in \mathcal{C}_{\mathcal{H}}$. □

Remark 3.4. The sequence

$$\{c_n\}_{n=0}^{\infty} = \left\{2, 1, \frac{2}{3}, \ldots, \frac{2}{n+1}, \ldots\right\} \tag{3.9}$$

is a convex null sequence because

$$\begin{gathered}\lim_{n\to\infty} c_n = \lim_{n\to\infty}\left(\frac{2}{n+1}\right) = 0, \qquad c_n - c_{n+1} = \frac{2}{(n+1)(n+2)} \geqq 0, \\ (c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = \frac{4}{(n+1)(n+2)(n+3)} \geqq 0 \quad (n = 0, 1, 2, \ldots).\end{gathered} \tag{3.10}$$

Setting $b = 1/4$ in Theorem 3.2 with the above sequence $\{c_n\}_{n=0}^{\infty}$, we derive the following example.

Example 3.5. The function

$$f(z) = -z - 2\int_0^z \frac{\log(1 - t)}{t} dt - \overline{\frac{1}{4}\left(z + 2\int_0^z \frac{\log(1 - t)}{t} dt\right)} = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n + \frac{1}{4}\overline{\left(z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n\right)} \tag{3.11}$$

is in the class $\mathcal{C}_{\mathcal{H}}$ as shown in Figure 5.

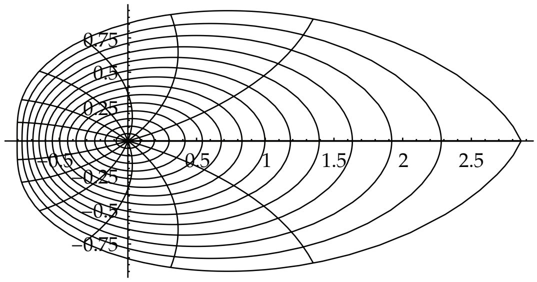

Figure 5: The image of $f(z)$ in Example 3.5.

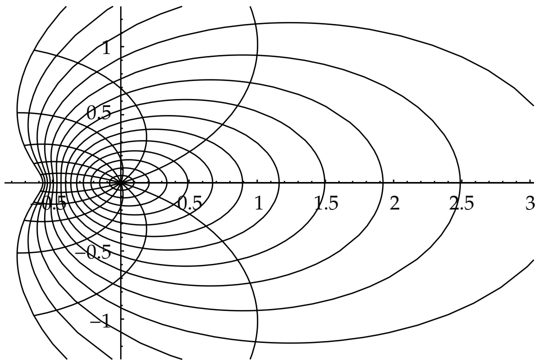

Figure 6: The image of $f(z)$ in Example 3.7.

Moreover, we know the following remark.

Remark 3.6. The sequence

$$\{c_n\}_{n=0}^{\infty} = \left\{2, 1, \frac{1}{2}, \ldots, 2^{1-n}, \ldots\right\} \tag{3.12}$$

is a convex null sequence because

$$\begin{gathered} \lim_{n\to\infty} c_n = \lim_{n\to\infty} 2^{1-n} = 0, \qquad c_n - c_{n+1} = 2^{-n} \geqq 0, \\ (c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = 2^{-(n+1)} \geqq 0 \quad (n = 0, 1, 2, \ldots). \end{gathered} \tag{3.13}$$

Hence, letting $b = 1/4$ in Theorem 3.3 with the sequence $\{c_n\}_{n=0}^{\infty} = \{2^{1-n}\}_{n=0}^{\infty}$, we have the following example.

Example 3.7. The function

$$\begin{aligned} f(z) &= -3\log(1-z) + 4\log\left(1-\frac{z}{2}\right) + \overline{\left(-\frac{3}{4}\log(1-z) + \log\left(1-\frac{z}{2}\right)\right)} \\ &= z + \sum_{n=2}^{\infty}\frac{1}{n}\left(1+\sum_{j=1}^{n-1}2^{1-j}\right)z^n + \overline{\frac{1}{4}\left(z + \sum_{n=2}^{\infty}\frac{1}{n}\left(1+\sum_{j=1}^{n-1}2^{1-j}\right)z^n\right)} \end{aligned} \tag{3.14}$$

is in the class $\mathcal{C}_{\mathcal{H}}$ as shown in Figure 6.

Dedication

This paper is dedicated to Professor Owa on the occasion of his retirement from Kinki University.

Acknowledgment

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