Stability of the \( n \)-Dimensional Mixed-Type Additive and Quadratic Functional Equation in Non-Archimedean Normed Spaces

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1. Introduction

A classical question in the theory of functional equations is “when is it true that a function, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?” Such a problem, called a stability problem of the functional equation, was formulated by Ulam in 1940 (see [1]). In the following year, Hyers [2] gave a partial solution of Ulam’s problem for the case of approximate additive functions. Subsequently, his result was generalized by Aoki [3] for additive functions and by Rassias [4] for linear functions. Indeed, they considered the stability problem for unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [5–23]).

A non-Archimedean field is a field \( \mathbb{K} \) equipped with a function (valuation) \( |\cdot| : \mathbb{K} \to [0,\infty) \) such that
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\( (F_1) \ |r| = 0 \) if and only if \( r = 0 \);

\( (F_2) \ |rs| = |r||s| \);

\( (F_3) \ |r + s| \leq \max\{|r|, |s|\} \) for all \( r, s \in \mathbb{K} \).

Clearly, it holds that \( |1| = |-1| = 1 \) and \( |n| \leq 1 \) for all \( n \in \mathbb{N} \).

Let \( X \) be a vector space over a scalar field \( \mathbb{K} \) with a non-Archimedean and nontrivial valuation \( |\cdot| \). A function \( \|\cdot\| : X \to \mathbb{R} \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

\( (N_1) \ \|x\| = 0 \) if and only if \( x = 0 \);

\( (N_2) \ \|rx\| = |r|\|x\| \) for all \( r \in \mathbb{K} \) and \( x \in X \);

\( (N_3) \ \|x + y\| \leq \max\{\|x\|, \|y\|\} \) for all \( x, y \in X \).

Then \( (X, \|\cdot\|) \) is called a non-Archimedean space. Due to the fact that

\[
\|x_n - x_m\| \leq \max_{m \leq i < n} \|x_{i+1} - x_i\| \quad (n > m), \tag{1.1}
\]

a sequence \( \{x_n\} \) is Cauchy if and only if \( \{x_{n+1} - x_n\} \) converges to zero in a non-Archimedean space. A complete non-Archimedean space is a non-Archimedean space in which every Cauchy sequence is convergent.

Recently, Moslehian and Rassias [24] proved the Hyers-Ulam stability of the Cauchy functional equation

\[
f(x + y) = f(x) + f(y), \tag{1.2}
\]

and the quadratic functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.3}
\]

in non-Archimedean normed spaces.

We now consider the \( n \)-dimensional mixed-type quadratic and additive functional equation

\[
2f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) = (n + 1) \sum_{i=1}^{n} f(x_i) + (n - 1) \sum_{i=1}^{n} f(-x_i), \tag{1.4}
\]

whose solution is called a quadratic-additive function.

In 2009, Towanlong and Nakmahachalasint [25] obtained a stability result for the functional equation (1.4), in which they constructed a quadratic-additive function \( F \) by composing an additive function \( A \) and a quadratic function \( Q \), where \( A \) and \( Q \) approximate the odd part and the even part of the given function \( f \), respectively.

In this paper, we investigate a general stability problem for the \( n \)-dimensional mixed-type quadratic and additive functional equation (1.4) in non-Archimedean normed spaces.
2. Solutions of (1.4)

In this section, we prove the generalized Hyers-Ulam stability of the $n$-dimensional mixed-type quadratic and additive functional equation (1.4). Assume that $H$ is an additive group and $X$ is a complete non-Archimedean space.

For a given function $f : H \to X$, we use the abbreviations

$$f_o(x) := \frac{f(x) - f(-x)}{2},$$

$$Af(x, y) := f(x + y) - f(x) - f(y),$$

$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),$$

$$D_n f(x_1, x_2, \ldots, x_n) := 2f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i < j \leq n} f(x_i - x_j)$$

$$- (n + 1) \sum_{i=1}^{n} f(x_i) - (n - 1) \sum_{i=1}^{n} f(-x_i)$$

for all $x, y, x_1, x_2, \ldots, x_n \in H$ and for an arbitrarily fixed $n \in \mathbb{N}$.

**Theorem 2.1.** Assume that $n \geq 2$ is an integer. Let $H$ and $X$ be an additive group and a complete non-Archimedean space, respectively. A function $f : H \to X$ is a solution of (1.4) if and only if $f_o$ is quadratic, $f_o$ is additive, and $f_o(0) = 0$.

**Proof.** If a function $f : H \to X$ is a solution of (1.4), then we have $f_o(0) = 0$.

$$Qf_o(x, y) = f_o(x + y) + f_o(x - y) - 2f_o(x) - 2f_o(y)$$

$$= \frac{1}{2}D_n f_o(x, y, 0, \ldots, 0) + \frac{1}{2} (n - 2)(n + 3) f_o(0)$$

$$= 0,$$

$$Af_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = \frac{1}{2}D_n f_o(x, y, 0, \ldots, 0) = 0$$

for all $x, y \in H$, that is, $f_o$ is quadratic and $f_o$ is additive.

Conversely, assume that $f_o$ is quadratic, $f_o$ is additive, and $f_o(0) = 0$. We apply an induction on $j$ to prove $D_n f_o(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in H$. For $j = 2$, we have

$$D_n f_o(x_1, x_2, 0, \ldots, 0)$$

$$= 2f_o(x_1 + x_2) + 2f_o(x_1 - x_2) - 4f_o(x_1) - 4f_o(x_2) - (n - 2)(n + 3) f_o(0)$$

$$= 0.$$
If \( n > 2 \) and \( D_n f_e(x_1, x_2, \ldots, x_j, 0, \ldots, 0) = 0 \) for some integer \( j (2 \leq j < n) \) and for all \( x_1, x_2, \ldots, x_j \in H \), then a routine calculation yields

\[
D_n f_e (x_1, x_2, \ldots, x_j, 0, \ldots, 0) = Q f_e (x_1 + \cdots + x_j, x_{j+1} - x_j) + \frac{1}{2} D_n f_e (x_1, \ldots, x_{j-1}, 2x_j, 0, \ldots, 0) + \frac{1}{2} D_n f_e (x_1, \ldots, x_{j-1}, 2x_{j+1}, 0, \ldots, 0) - \sum_{k=1}^{j-1} (Q f_e (x_k, x_j) + Q f_e (x_k, x_{j+1}))
\]

(2.4)

\[
- \frac{j}{2} Q f_e (x_{j+1}, x_{j+1}) - \frac{j}{2} Q f_e (x_j, x_j)
\]

= 0

for all \( x_1, x_2, \ldots, x_{j+1} \in H \). Hence, we conclude that

\[
D_n f_e (x_1, x_2, \ldots, x_n) = 0
\]

(2.5)

for all \( x_1, x_2, \ldots, x_n \in H \).

Since \( f_o \) is additive, a long calculation yields

\[
D_n f_o (x_1, x_2, \ldots, x_n)
\]

\[
= \sum_{1 \leq i, j \leq n, i \neq j} A f_o (x_i - x_j) + 2 \sum_{i=1}^{n-1} A f_o \left( \sum_{j=1}^{i} x_j, x_{i+1} \right)
\]

(2.6)

\]

= 0.

Hence, it follows from (2.5) and (2.6) that

\[
D_n f(x_1, x_2, \ldots, x_n) = D_n f_e (x_1, x_2, \ldots, x_n) + D_n f_o (x_1, x_2, \ldots, x_n) = 0
\]

(2.7)

for all \( x_1, x_2, \ldots, x_n \in H \); that is, \( f \) is a solution of (1.4).

\[
3. \textbf{Generalized Hyers-Ulam Stability of (1.4)}
\]

In the following theorem, we will investigate the stability problem of the functional equation (1.4).

\textbf{Theorem 3.1.} Assume that \( n \geq 2 \) is an integer. Let \( H \) and \( X \) be an additive group and a complete non-Archimedean space, respectively. Assume that \( \varphi : H^n \to [0, \infty) \) is a function such that

\[
\lim_{m \to \infty} \varphi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0
\]

(3.1)
for all \( x_1, x_2, \ldots, x_n \in H \). Moreover, assume that the limit

\[
\tilde{\varphi}(x) := \lim_{m \to \infty} \max_{0 \leq i < m} \left\{ \frac{\varphi(n^i x, \ldots, n^i x)}{|4||n|^{2i+2}}, \frac{\varphi(-n^i x, \ldots, -n^i x)}{|4||n|^{2i+2}} \right\}
\]

exists for each \( x \in H \). If a function \( f : H \to X \) satisfies the inequality

\[
\|D_n f(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n)
\]

for any \( x_1, x_2, \ldots, x_n \in H \), then there exists a unique quadratic-additive function \( T : H \to X \) such that

\[
\|f(x) - T(x)\| \leq \tilde{\varphi}(x)
\]

for each \( x \in H \). In particular, \( T \) is given by

\[
T(x) = \lim_{m \to \infty} \left( \frac{f(n^m x) + f(-n^m x)}{2n^{2m}} + \frac{f(n^m x) - f(-n^m x)}{2n^m} \right)
\]

for all \( x \in H \).

Proof. If we replace \( x_i \) in (3.1) with 0 for each \( i \in \{1, 2, \ldots, n\} \), then we have

\[
\lim_{m \to \infty} \frac{\varphi(0, 0, \ldots, 0)}{|n|^{2m}} = 0.
\]

Since \( |n| \leq 1 \), it holds that \( \varphi(0, 0, \ldots, 0) = 0 \) and

\[
\| (n - 1)(n + 2)f(0) \| = \| D_n f(0, 0, \ldots, 0) \| \leq \varphi(0, 0, \ldots, 0) = 0.
\]

Hence, we conclude that \( f(0) = 0 \).

Let \( J_m f : H \to Y \) be a function defined by

\[
J_m f(x) = \frac{f(n^m x) + f(-n^m x)}{2n^{2m}} + \frac{f(n^m x) - f(-n^m x)}{2n^m}
\]
for all \( x \in H \) and \( m \in \{0, 1, 2, \ldots\} \). A tedious calculation, together with \((F_2), (N_3), \) and \((3.3)\), yields

\[
\| J_i f(x) - J_{i+1} f(x) \| = \left\| \frac{D_n f(n^ix, \ldots, n^ix)}{4n^{2i+2}} - \frac{D_n f(-n^ix, \ldots, -n^ix)}{4n^{2i+2}} \right\|
\]

\[
- \frac{D_n f(n^ix, \ldots, n^ix)}{4n^{i+1}} + \frac{D_n f(-n^ix, \ldots, -n^ix)}{4n^{i+1}} \right\|
\]

\[
\leq \max \left\{ \frac{\| D_n f(n^ix, \ldots, n^ix) \| \| D_n f(-n^ix, \ldots, -n^ix) \|}{|4||n|^{2i+2}}, \frac{\| D_n f(n^ix, \ldots, n^ix) \| \| D_n f(-n^ix, \ldots, -n^ix) \|}{|4||n|^{i+1}} \right\}
\]

\[
\leq \max \left\{ \frac{\varphi(n^ix, \ldots, n^ix)}{4||n|^{2i+2}}, \frac{\varphi(-n^ix, \ldots, -n^ix)}{4||n|^{i+1}} \right\}
\]

for all \( x \in H \) and \( i \in \{0, 1, 2, \ldots\} \). It follows from \((3.1)\) and \((3.9)\) that the sequence \( \{ J_m f(x) \} \) is Cauchy. Since \( X \) is complete, we conclude that \( \{ J_m f(x) \} \) is convergent.

Let us define

\[
T(x) := \lim_{m \to \infty} J_m f(x) \quad (3.10)
\]

for any \( x \in H \). It follows from \((N_3)\) and \((3.9)\) that

\[
\| f(x) - J_m f(x) \| = \left\| \sum_{i=0}^{m-1} (J_i f(x) - J_{i+1} f(x)) \right\|
\]

\[
\leq \max_{0 \leq i < m} \| J_i f(x) - J_{i+1} f(x) \| \quad (3.11)
\]

\[
\leq \max_{0 \leq i < m} \left\{ \frac{\varphi(n^ix, \ldots, n^ix)}{4||n|^{2i+2}}, \frac{\varphi(-n^ix, \ldots, -n^ix)}{4||n|^{i+1}} \right\}
\]

for all \( m \in \{0, 1, 2, \ldots\} \) and \( x \in H \). In view of \((3.2)\), if we let \( m \to \infty \) in \((3.11)\), then we obtain the inequality \((3.4)\).

Replacing \( x_i \) in \((3.3)\) with \( n^m x_i \) for \( i \in \{1, 2, \ldots, n\} \) and considering \((F_2)\) and \((N_3)\), we get

\[
\| D_n f(x_1, x_2, \ldots, x_n) \| = \left\| \frac{D_n f(n^m x_1, \ldots, n^m x_n) - D_n f(-n^m x_1, \ldots, -n^m x_n)}{2n^m} \right\|
\]

\[
+ \frac{D_n f(n^m x_1, \ldots, n^m x_n) + D_n f(-n^m x_1, \ldots, -n^m x_n)}{2n^{2m}} \right\|
\]
\begin{equation}
\leq \max \left\{ \frac{\varphi(n^m x_1, \ldots, n^m x_n)}{2\|n\|^{2m}}, \frac{\varphi(-n^m x_1, \ldots, -n^m x_n)}{2\|n\|^{2m}}, \frac{\varphi(n^m x_1, \ldots, n^m x_n)}{2\|n\|^{2m}}, \frac{\varphi(-2^m x_1, \ldots, -2^m x_n)}{2\|n\|^{2m}} \right\}
\end{equation}

(3.12)

for all \( m \in \{0, 1, 2, \ldots\} \) and \( x_1, x_2, \ldots, x_n \in H \). If we let \( m \to \infty \) in the last inequality, then it follows from the condition (3.1) that \( D_nT(x_1, x_2, \ldots, x_n) = 0 \) for all \( x_1, x_2, \ldots, x_n \in H \); that is, \( T \) is a quadratic-additive function.

Assume that \( T' : H \to X \) is another quadratic-additive function satisfying (3.4). By the definition of \( D_n \), a routine calculation yields

\[
-D_nT'(n^i x, \ldots, n^i x) - \frac{D_nT'(-n^i x, \ldots, -n^i x)}{4n^{2i+2}} - \frac{D_nT'(n^i x, \ldots, n^i x)}{4n^{i+1}} + \frac{D_nT'(-n^i x, \ldots, -n^i x)}{4n^{i+1}} = -\frac{1}{2n^{2i+1}} \left( T'(n^{i+1} x) + T'(-n^{i+1} x) \right) + \frac{1}{2n^{i+2}} \left( T'(n^i x) + T'(-n^i x) \right)
\]

\[
-\frac{1}{2n^{i+1}} \left( T'(n^{i+1} x) - T'(-n^{i+1} x) \right) + \frac{1}{2n^{i+2}} \left( T'(n^i x) - T'(-n^i x) \right)
\]

(3.13)

for each \( i \in \{0, 1, 2, \ldots\} \) and \( x \in H \). Hence, it follows from (3.8) that

\[
\sum_{i=0}^{k-1} \left( -\frac{D_nT'(n^i x, \ldots, n^i x)}{4n^{2i+2}} - \frac{D_nT'(-n^i x, \ldots, -n^i x)}{4n^{2i+2}} \right. \left. - \frac{D_nT'(n^i x, \ldots, n^i x)}{4n^{i+1}} + \frac{D_nT'(-n^i x, \ldots, -n^i x)}{4n^{i+1}} \right) = T'(x) - J_k T'(x)
\]

(3.14)

for any \( k \in \mathbb{N} \) and \( x \in H \). Since \( T' \) is a solution of (1.4), it follows from the last equality that

\[
T'(x) = J_k T'(x)
\]

(3.15)

for any \( k \in \mathbb{N} \) and \( x \in H \). Obviously, this equality also holds for \( T \).

Consequently, by considering that \( |n| \leq 1 \), it follows from (N3), (3.1), (3.4), and (3.8) that

\[
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\|
\]

\[
\leq \lim_{k \to \infty} \max \{ \|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\| \}
\]
\[
\leq \lim_{k \to \infty} |2|^{-1n}|n|^{-2k} \max \left\{ \| T(n^k x) - f(n^k x) \|, \| T(-n^k x) - f(-n^k x) \|, \right. \\
\| f(n^k x) - T(n^k x) \|, \left. \| f(-n^k x) - T(-n^k x) \| \right\} \\
\leq \lim_{k \to \infty} \lim_{m \to \infty} \max_{k \leq m+k} \left\{ \frac{\varphi(n^i x, \ldots, n^i x)}{|8||n|^{2i+2}}, \frac{\varphi(-n^i x, \ldots, -n^i x)}{|8||n|^{2i+2}} \right\} \\
= 0
\] 

(3.16)

for all \( x \in H \). Therefore, \( T = T' \), which proves the uniqueness of \( T \).

**Corollary 3.2.** Let \( X \) and \( Y \) be non-Archimedean normed spaces over \( \mathbb{K} \) with \( |n| < 1 \). If \( Y \) is complete and \( f : X \to Y \) satisfies the inequality

\[
\| Df(x_1, x_2, \ldots, x_n) \| \leq \theta \sum_{i=1}^{n} \| x_i \|^r
\]

(3.17)

for all \( x_1, x_2, \ldots, x_n \in X \) and for some \( r > 2 \), then there exists a unique quadratic-additive function \( T : X \to Y \) such that

\[
\| f(x) - T(x) \| \leq \frac{n\theta}{|4||n|^2} \| x \|^r
\]

(3.18)

for all \( x \in X \).

**Proof.** Let \( \varphi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^r \). Since \( |n| < 1 \) and \( r - 2 > 0 \), we get

\[
\lim_{m \to \infty} |n|^{-2m} \varphi(n^m x_1, n^m x_2, \ldots, n^m x_n) = \lim_{m \to \infty} |n|^{m(r-2)} \varphi(x_1, x_2, \ldots, x_n) = 0
\]

(3.19)

for all \( x_1, x_2, \ldots, x_n \in X \). Therefore, the conditions of Theorem 3.1 are satisfied. Indeed, it is easy to see that \( \varphi(x) = n\theta(|4|^{-1}|n|^{-2}) \| x \|^r \). By Theorem 3.1, there exists a unique quadratic-additive function \( T : X \to Y \) such that (3.18) holds.

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