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Research Article

An L^p-Estimate for Weak Solutions of Elliptic Equations

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We prove an L^p-a priori bound, $p > 2$, for solutions of second order linear elliptic partial differential equations in divergence form with discontinuous coefficients in unbounded domains.

1. Introduction

The Dirichlet problem for second order linear elliptic partial differential equations in divergence form and with discontinuous coefficients in bounded open subsets of $\mathbb{R}^n$, $n \geq 2$, is a classical problem that has been widely studied by several authors (we refer, e.g., to [1–6]).

In this paper, we want to analyze certain aspects of the same kind of problem, but in the framework of unbounded domains.

More precisely, given an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, we are interested in the study of the elliptic second order linear differential operator in variational form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i} + d_j \right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c, \tag{1.1}$$

with coefficients $a_{ij} \in L^\infty(\Omega)$, and in the following associated Dirichlet problem

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega). \end{gathered} \tag{1.2}$$

Starting from a work of Bottaro and Marina (see [7]), who proved an existence and uniqueness theorem for the solution of problem (1.2), for $n \geq 3$, assuming that

$$b_i, d_i \in L^n(\Omega), \quad i = 1, \dots, n, \quad c \in L^{n/2}(\Omega) + L^\infty(\Omega),$$
$$c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \quad \mu \in \mathbb{R}_+, \tag{1.3}$$

analogous results have been successively obtained weakening the hypotheses on the lower order terms coefficients. First generalizations in this direction have been carried on in [8], where $n \geq 2$ and b_i, d_i, and c satisfy assumptions similar to those in (1.3), but only locally. While in [9], for $n \geq 3$, these results have been further improved, since b_i, d_i, and c are assumed to belong to opportune Morrey type functional spaces with lower summability.

In the above-mentioned works ([7–9]), the authors also provide the estimate

$$\|u\|_{W^{1,2}(\Omega)} \leq C \|f\|_{W^{-1,2}(\Omega)}, \tag{1.4}$$

where the dependence of the constant C on the data of the problem is completely described.

Here we suppose that the lower order terms coefficients are as in [9] for $n \geq 3$ and as in [8] for $n = 2$ and we prove an L^p-a priori bound, $p > 2$. More precisely, for a sufficiently regular set Ω and given a datum $f \in L^2(\Omega) \cap L^\infty(\Omega)$, we show that there exists a constant C such that

$$\|u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}, \tag{1.5}$$

for any bounded solution u of (1.2) and for every $p \in]2, +\infty[$. We point out that also in our analysis the dependence of the constant C is fully determined.

We note that bound (1.5) can be also useful when dealing with certain nonvariational problems that, by means of the existence of the derivatives of the a_{ij}, can be rewritten in variational form.

Among the authors who studied the Dirichlet problem for second order linear elliptic equations in divergence form with discontinuous coefficients in unbounded domains, we quote here also Lions in [10, 11] and Chicco and Venturino in [12].

The proof of (1.5) is developed as follows. In Section 2 we extend a known result by Stampacchia (see [1], or [13] for details), obtained within the framework of the generalization of the study of certain elliptic equations in divergence form with discontinuous coefficients on a bounded open subset of $\mathbb{R}^n$ to some problems arising for harmonic or subharmonic functions in the theory of potential.

This is done in order to obtain a preliminary lemma, proved in Section 3, that permits to consider some particular test functions in the variational formulation of our problem. This allows us to prove a technical result (Lemma 4.1), that is the main point in the proof of the claimed L^p-estimate.

2. A Generalization of a Result by Stampacchia

Let

$$G : t \in \mathbb{R} \longrightarrow G(t) \tag{2.1}$$

be a uniformly Lipschitz real function, such that there exists a positive constant K such that for every $t', t'' \in \mathbb{R}$ one has

$$\left|G(t') - G(t'')\right| \leq K\left|t' - t''\right|, \tag{2.2}$$

and suppose that

$$G_{|_{[-k,k]}} = 0, \quad \text{for a } k \in \mathbb{R}_+ \tag{2.3}$$

and that its derivative G' has a finite number of discontinuity points.

A known result by Stampacchia, see Lemma 1.1 in [1] (or in [13], for details), guarantees that given a function u, defined in an open bounded subset of $\mathbb{R}^n$ and belonging to $\overset{\circ}{W}{}^{1,2}$, also the composition between G and u is in $\overset{\circ}{W}{}^{1,2}$ and gives an explicit expression for the derivative of this composition, up to sets of null Lebesgue measure.

Later on, in [7], Bottaro and Marina explicitly observed that, up to few modifications, the proof of these results remains valid also for an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$. More precisely,

$$u \in \overset{\circ}{W}{}^{1,2}(\Omega) \Longrightarrow G(u) = G \circ u \in \overset{\circ}{W}{}^{1,2}(\Omega), \tag{2.4}$$

and moreover

$$(G(u))_{x_i} = G'(u)u_{xi}, \quad \text{a. e. in } \Omega,\ i = 1, \ldots, n. \tag{2.5}$$

In Lemma 2.2 below, we show a further generalization of (2.4), always in the case of unbounded domains.

In order to prove Lemma 2.2, we need the following convergence results.

Lemma 2.1. *If Ω has the uniform C^1-regularity property, then for every $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(\Phi_h)_{h \in \mathbb{N}}$ of functions such that*

$$\Phi_h \in C_o^\infty(\Omega), \quad \Phi_h \longrightarrow u \text{ in } \overset{\circ}{W}{}^{1,2}(\Omega), \quad \sup_{h \in \mathbb{N}} \|\Phi_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}. \tag{2.6}$$

If G is a uniformly Lipschitz function as in (2.1), (2.2), *and* (2.3) *and such that its derivative G' has a finite number of discontinuity points,*

$$G(\Phi_h) \longrightarrow G(u) \quad \text{in } L^2(\Omega), \tag{2.7}$$

$$G(\Phi_h) \rightharpoonup G(u) \quad \text{weakly in } \overset{\circ}{W}{}^{1,2}(\Omega). \tag{2.8}$$

Moreover, there exists a sequence $(g_h)_{h\in\mathbb{N}}$ such that

$$g_h \longrightarrow G(u) \quad \text{in } \overset{\circ}{W}{}^{1,2}(\Omega), \tag{2.9}$$

where $g_h = \sum_{j=1}^{h} c_j\, G(\Phi_j)$ with $c_j \geq 0$ and $\sum_{j=1}^{h} c_j = 1$.

Proof. The statement in (2.6) has been proved in [14].

The L^2-convergence in (2.7) easily follows by (2.2) and by the convergence in (2.6). The $\overset{\circ}{W}{}^{1,2}$-convergences in (2.8) and (2.9) can be obtained as in the proof of Lemma 1.1 of [13], with opportune modifications due to the fact that the set Ω is unbounded (see also [7]). □

We point out that next lemma is a fundamental tool in our analysis since it is the core of the proof of Lemma 3.3 that will allow us to take some specific test functions in the variational formulation of our problem.

This will consent to show a technical result (see Lemma 4.1), which is the main point in the proof of our L^p-a priori bound.

Lemma 2.2. *Let G be a uniformly Lipschitz function as in* (2.1), (2.2), *and* (2.3) *and such that its derivative G' has a finite number of discontinuity points. If Ω has the uniform C^1-regularity property, then for every $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ one has*

$$|u|^{p-2}G(u) \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad \forall p \in\,]2, +\infty[. \tag{2.10}$$

Proof. Fix $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$; to show (2.10) we need different arguments according to different values of p.

For $2 < p < 3$ we need to verify that there exists a positive constant c such that

$$\left| \int_\Omega |u|^{p-2}G(u)\varphi_{x_i}dx \right| \leq c\|\varphi\|_{L^2(\Omega)}, \quad \forall\varphi \in C^1_o(\mathbb{R}^n),\ \forall i = 1,\ldots,n, \tag{2.11}$$

this ends the proof of our lemma as a consequence of a characterization of the space $\overset{\circ}{W}{}^{1,2}(\Omega)$ (see, e.g., Proposition IX.18 of [15]).

In order to prove (2.11), we consider the sequence $(\Phi_h)_{h\in\mathbb{N}}$ introduced in Lemma 2.1 and observe that, given $\varphi \in C^1_o(\mathbb{R}^n)$, one has

$$\int_\Omega |u|^{p-2}G(u)\varphi_{x_i}dx = \lim_{h\to+\infty}\int_\Omega |\Phi_h|^{p-2}G(\Phi_h)\varphi_{x_i}dx, \tag{2.12}$$

for $i = 1,\ldots,n$.

Indeed, by Hölder inequality we get

$$\begin{aligned}
&\left|\int_\Omega |u|^{p-2}G(u)\varphi_{x_i}dx-\int_\Omega |\Phi_h|^{p-2}\ G(\Phi_h)\varphi_{x_i}dx\right|\\
&\quad\le\left|\int_\Omega |u|^{p-2}(G(u)-G(\Phi_h))\varphi_{x_i}\,dx\right|\\
&\qquad+\left[\int_\Omega \left(|u|^{p-2}-|\Phi_h|^{p-2}\right)G(\Phi_h)\varphi_{x_i}\,dx\right]\\
&\quad\le \|u\|^{p-2}_{L^\infty(\Omega)}\|G(u)-G(\Phi_h)\|_{L^2(\Omega)}\left\|\varphi_x\right\|_{L^2(\Omega)}\\
&\qquad+\|u-\Phi_h\|^{p-2}_{L^2(\Omega)}\|G(\Phi_h)\|_{L^2(\Omega)}\left\|\varphi_x\right\|_{L^{2/(3-p)}},
\end{aligned}\tag{2.13}$$

and this quantity vanishes letting $h\to+\infty$, as a consequence of (2.6) and (2.7).

On the other hand,

$$\begin{aligned}
\int_\Omega |\Phi_h|^{p-2}G(\Phi_h)\varphi_{x_i}dx&=-\int_\Omega\left(|\Phi_h|^{p-2}G(\Phi_h)\right)_{x_i}\varphi\,dx\\
&=-(p-2)\int_\Omega |\Phi_h|^{p-4}\Phi_h(\Phi_h)_{x_i}G(\Phi_h)\varphi\,dx-\int_\Omega |\Phi_h|^{p-2}(G(\Phi_h))_{x_i}\varphi\,dx\\
&=-(p-2)\int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h)\left((\Phi_h)_{x_i}-u_{x_i}\right)\varphi\,dx\\
&\quad-(p-2)\int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h)u_{x_i}\varphi\,dx-\int_\Omega |\Phi_h|^{p-2}(G(\Phi_h))_{x_i}\varphi\,dx.
\end{aligned}\tag{2.14}$$

Having in mind (2.12), we want to pass to the limit as $h\to+\infty$ in the right-hand side of this equality.

Concerning the first term, by (2.2), Hölder inequality, and using the last relation in (2.6), we obtain

$$\left|\int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h)\left((\Phi_h)_{x_i}-u_{x_i}\right)\varphi\,dx\right|\le K\ \|u\|^{p-2}_{L^\infty(\Omega)}\left\|(\Phi_h)_{x_i}-u_{x_i}\right\|_{L^2(\Omega)}\left\|\varphi\right\|_{L^2(\Omega)}.\tag{2.15}$$

Thus, by the convergence in (2.6), the quantity on the left-hand side goes to zero, letting $h\to+\infty$, and therefore

$$\lim_{h\to+\infty}\int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h)\left((\Phi_h)_{x_i}-u_{x_i}\right)\varphi\,dx=0.\tag{2.16}$$

For the last term we have

$$\lim_{h\to+\infty}\int_\Omega |\Phi_h|^{p-2}(G(\Phi_h))_{x_i}\varphi\,dx=\int_\Omega |u|^{p-2}\ (G(u))_{x_i}\varphi\,dx.\tag{2.17}$$

Indeed,

$$\begin{aligned}
&\left|\int_\Omega |u|^{p-2}(G(u))_{x_i}\varphi\, dx - \int_\Omega |\Phi_h|^{p-2}\ (G(\Phi_h))_{x_i}\varphi\, dx\right| \\
&\quad \le \left|\int_\Omega |u|^{p-2}\big((G(u))_{x_i} - (G(\Phi_h))_{x_i}\big)\varphi\, dx\right| + \left|\int_\Omega \Big(|u|^{p-2} - |\Phi_h|^{p-2}\Big)(G(\Phi_h))_{x_i}\varphi\, dx\right|.
\end{aligned} \tag{2.18}$$

Moreover, by the weak convergence in (2.8) the first term on the right-hand side vanishes letting $h \to +\infty$. Concerning the second one, we get

$$\left|\int_\Omega \Big(|u|^{\ p-2} - |\Phi_h|^{p-2}\Big)(G(\Phi_h))_{x_i}\varphi\, dx\right| \le \|u - \Phi_h\|_{L^2(\Omega)}^{p-2}\|(G(\Phi_h))_x\|_{L^2(\Omega)}\|\varphi\|_{L^{2/(3-p)}}, \tag{2.19}$$

and, by (2.6) and (2.8), also this quantity is null passing to the limit as $h \to +\infty$.

It remains to treat the second term of the right-hand side of (2.14). To this aim let us introduce the sets

$$D_h = \{x : |\Phi_h(x)| > k\}, \qquad D = \{x \in \Omega : |u(x)| > k\}, \tag{2.20}$$

where k is that of (2.3).

We observe that, in view of (2.6), there exists $h_0 \in \mathbb{N}$ such that, up to sets of null Lebesgue measure,

$$D_h \subseteq D, \quad \forall h \ge h_0, \tag{2.21}$$

and we can assume, without loss of generality, that $h_0 = 1$.

Therefore, by (2.3) and (2.21), one has

$$\begin{aligned}
\int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi\, dx &= \int_{D_h} |\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi \\
&= \int_D |\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi.
\end{aligned} \tag{2.22}$$

On the other hand, always using (2.6), we can also deduce, with no loss of generality, that

$$|\Phi_h(x)| > \frac{k}{2}, \quad \text{for a. e. } x \in D, \quad \forall h \in \mathbb{N}. \tag{2.23}$$

This, together with (2.6) and (2.7), and by definition of D, gives, up to a subsequence,

$$|\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi \longrightarrow |u|^{p-4} u G(u) u_{x_i}\varphi, \quad \text{for a. e. } x \in D. \tag{2.24}$$

Moreover, by (2.2) and (2.6),

$$\left| |\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi \right| \leq K \ \|u\|^{p-2}_{L^\infty(\Omega)} |u_{x_i}| |\varphi|, \quad \text{for a. e. } x \in D, \tag{2.25}$$

$\forall h \in \mathbb{N}$.

Therefore, (2.24) and (2.25) being true, the bounded convergence theorem applies giving, up to a subsequence,

$$\lim_{h\to+\infty} \int_\Omega |\Phi_h|^{p-4}\Phi_h G(\Phi_h) u_{x_i}\varphi \, dx = \int_D |u|^{p-4} u G(u) u_{x_i}\varphi \, dx. \tag{2.26}$$

Combining (2.12), (2.14), (2.16), (2.17), and (2.26), we conclude, by (2.2) and Hölder inequality, that

$$\left| \int_\Omega |u|^{p-2} G(u)\varphi_{x_i} dx \right| \leq \Big((p-2)K \ \|u_x\|_{L^2(\Omega)} + \|(G(u))_x\|_{L^2(\Omega)} \Big) \|u\|^{p-2}_{L^\infty(\Omega)} \|\varphi\|_{L^2(\Omega)}, \tag{2.27}$$

for $i = 1, \ldots, n$, that is (2.11).

For $p \geq 3$, let us consider the sequence $(g_h)_{h\in\mathbb{N}}$ introduced in Lemma 2.1 and put

$$u_h = |u|^{p-2}(g_h - G(u)). \tag{2.28}$$

Simple calculations give

$$\|u_h\|^2_{W^{1,2}(\Omega)} \leq c_1 \left(\|u\|^{2(p-2)}_{L^\infty(\Omega)} \|g_h - G(u)\|^2_{W^{1,2}(\Omega)} + \|u\|^{2(p-3)}_{L^\infty(\Omega)} \int_\Omega (g_h - G(u))^2 u_x^2 \, dx \right), \tag{2.29}$$

with c_1 positive constant depending only on p.

We want to pass to the limit in the right-hand side of this inequality. For the first term it is easily seen that it goes to zero, in view of (2.9).

For the last one, again from (2.9), we get, up to a subsequence,

$$(g_h - G(u))^2 u_x^2 \longrightarrow 0, \quad \text{for a. e. } x \in \Omega. \tag{2.30}$$

Moreover, by (2.2) it follows that

$$(g_h - G(u))^2 u_x^2 \leq 4K^2 \|u\|^2_{L^\infty(\Omega)} u_x^2, \quad \text{for a. e. } x \in \Omega, \ \ \forall h \in \mathbb{N}. \tag{2.31}$$

Hence, from these last considerations and using the bounded convergence theorem we obtain, up to a subsequence,

$$\lim_{h\to+\infty} \int_\Omega (g_h - G(u))^2 u_x^2 \ dx = 0. \tag{2.32}$$

Therefore, by (2.29), up to a subsequence, we have

$$|u|^{p-2}g_h \longrightarrow |u|^{p-2}G(u) \quad \text{in } W^{1,2}(\Omega). \tag{2.33}$$

Now, observe that $|u|^{p-2}g_h \in \overset{\circ}{W}{}^{1,2}(\Omega)$, because of its compact support, then for any $h \in \mathbb{N}$ there exists a sequence $(\psi_{h_m})_{m\in\mathbb{N}} \subset C_o^\infty(\Omega)$ such that

$$\psi_{h_m} \longrightarrow |u|^{p-2}g_h \quad \text{in } \overset{\circ}{W}{}^{1,2}(\Omega), \tag{2.34}$$

this means that there exists $m_h \in \mathbb{N}$ such that

$$\left\| \psi_{h_{m_h}} - |u|^{p-2}g_h \right\|_{W^{1,2}(\Omega)} \leq \frac{1}{h}. \tag{2.35}$$

By (2.33) and (2.35) we deduce that

$$\psi_{h_{m_h}} \longrightarrow |u|^{p-2}G(u) \quad \text{in } W^{1,2}(\Omega), \tag{2.36}$$

this ends the proof of our lemma. □

3. Tools

We recall the definitions of the Morrey type spaces where the lower order terms coefficients of the operator will be chosen. These functional spaces were introduced for the first time in [9] in order to generalize to the case of unbounded domains of the classical notion of Morrey spaces.

We start with some notation. Given any Lebesgue measurable subset F of $\mathbb{R}^n$, we denote by $\Sigma(F)$ the σ-algebra of all Lebesgue measurable subsets of F. For any $E \in \Sigma(F)$, χ_E is its characteristic function and $E(x,r)$ is the intersection $E \cap B(x,r)$ $(x \in \mathbb{R}^n,\ r \in \mathbb{R}_+)$, where $B(x,r)$ is the open ball centered in x and with radius r.

For $q \in [1,+\infty[$ and $\lambda \in [0,n[$, the space of Morrey type $M^{q,\lambda}(\Omega)$ is the set of all the functions g in $L^q_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega)} = \sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \tau^{-\lambda/q} \|g\|_{L^q(\Omega(x,\tau))} < +\infty, \tag{3.1}$$

endowed with the norm just defined. Moreover, $M_\circ^{q,\lambda}(\Omega)$ denotes the closure of $C_\circ^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$.

For reader's convenience, we state here a result of [16], adapted to our needs, providing the boundedness and an embedding estimate for the multiplication operator

$$u \in \overset{\circ}{W}{}^{1,2}(\Omega) \longrightarrow gu \in L^2(\Omega), \tag{3.2}$$

where the function g belongs to a suitable space of Morrey type $M^{q,\lambda}(\Omega)$.

Lemma 3.1. *If $g \in M^{q,\lambda}(\Omega)$, with $q > 2$, $\lambda = 0$ if $n = 2$, and $q \in]2, n]$, $\lambda = n - q$ if $n > 2$, then the operator in* (3.2) *is bounded. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^2(\Omega)} \leq c\|g\|_{M^{q,\lambda}(\Omega)}\|u\|_{W^{1,2}(\Omega)}, \quad \forall u \in \overset{\circ}{W}{}^{1,2}(\Omega), \tag{3.3}$$

with $c = c(n, q)$.

Now, we recall a lemma, proved in [9], describing the main properties of some functions u_s, introduced in [7], that will be of crucial relevance in the proof of our main result.

Let $h \in \mathbb{R}_+ \cup \{+\infty\}$ and $k \in \mathbb{R}$, with $0 \leq k \leq h$. For each $t \in \mathbb{R}$ we set

$$G_{kh}(t) = \begin{cases} t - k & \text{if } t > k, \\ 0 & \text{if } -k \leq t \leq k, \quad \text{if } h = +\infty, \\ t + k & \text{if } t < -k, \end{cases} \tag{3.4}$$

$$G_{kh}(t) = G_{k\infty}(t) - G_{h\infty}(t), \quad \text{if } h \in \mathbb{R}_+. \tag{3.5}$$

Lemma 3.2. *Let $g \in M_o^{q,\lambda}(\Omega)$, $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ and $\varepsilon \in \mathbb{R}_+$. Then there exist $r \in \mathbb{N}$ and $k_1, \ldots, k_r \in \mathbb{R}$, with $0 = k_r < k_{r-1} < \cdots < k_1 < k_0 = +\infty$, such that, setting*

$$u_s = G_{k_s\, k_{s-1}}(u), \qquad s = 1, \ldots, r, \tag{3.6}$$

one has $u_1, \ldots, u_r \in \overset{\circ}{W}{}^{1,2}(\Omega)$ and

$$\|g\chi_{\operatorname{supp}(u_s)_x}\|_{M^{q,\lambda}(\Omega)} \leq \varepsilon, \quad s = 1, \ldots, r, \tag{3.7}$$

$$uu_s \geq u_s^2, \quad s = 1, \ldots, r, \tag{3.8}$$

$$u_{x_i}(u_s)_{x_j} = (u_s)_{x_i}(u_s)_{x_j}, \quad s = 1, \ldots, r, \; i, j = 1, \ldots, n, \tag{3.9}$$

$$(u_1 + \cdots + u_s)_{x_i} u_s = u_{x_i} u_s, \quad s = 1, \ldots, r, \; i = 1, \ldots, n, \tag{3.10}$$

$$u_1 + \cdots + u_r = u, \tag{3.11}$$

$$r \leq c, \tag{3.12}$$

with $c = c\ (\varepsilon, q, \|g\|_{M^{q,\lambda}(\Omega)})$ positive constant.

As already mentioned, the next lemma will allow us, in the last section, to take the products $|u|^{p-2}u_s$ as test functions in the variational formulation of our problem.

Lemma 3.3. *If Ω has the uniform C^1-regularity property, then for every $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ and for any $p \in]2, +\infty[$ one has*

$$|u|^{p-2} u_s \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad s = 1, \dots, r, \tag{3.13}$$

where u_s, for $s = 1, \dots, r$, are the functions of Lemma 3.2.

Proof. If $r = 1$, then $u_1 = G_{0\,\infty}(u) = u$; therefore, by Lemma 3.2 in [14], one has $|u|^{p-2}u \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

If $r > 1$ and $s < r$, then $u_s = G_{k_s\, k_{s-1}}(u)$, therefore $|u|^{p-2}u_s = |u|^{p-2}G(u)$, for the choice $k = k_s$ in (2.3). This entails that $|u|^{p-2}u_s \in \overset{\circ}{W}{}^{1,2}(\Omega)$, by means of Lemma 2.2.

In view of these considerations and (3.11) being true, we also get $|u|^{p-2}u_r = |u|^{p-2}u - \sum_{s=1}^{r-1} |u|^{p-2}u_s \in \overset{\circ}{W}{}^{1,2}(\Omega)$. □

4. An A Priori Bound

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$, such that

$$\Omega \text{ has the uniform } C^1\text{-regularity property.} \tag{h_0}$$

We consider in Ω the second order linear differential operator in variational form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial}{\partial x_i} + d_j \right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c, \tag{4.1}$$

with the following conditions on the coefficients:

$$\begin{aligned} & a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \dots, n \\ & \exists\, \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a. e. in } \Omega, \ \forall \xi \in \mathbb{R}^n, \end{aligned} \tag{h_1}$$

$$\begin{aligned} & b_i, d_i \in M^{2t,\lambda}(\Omega), \quad b_i - d_i \in M_o^{2t,\lambda}(\Omega), \quad i = 1, \dots, n, \\ & c \in M^{t,\lambda}(\Omega), \\ & \text{with } t > 1, \ \lambda = 0 \quad \text{if } n = 2, \\ & \text{with } t \in \left]1, \frac{n}{2}\right], \ \lambda = n - 2t \quad \text{if } n > 2, \end{aligned} \tag{h_2}$$

$$\begin{aligned} & c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \quad \mu = \text{constant} > 0, \\ & \text{in the distributional sense on } \Omega. \end{aligned} \tag{h_3}$$

We also associate to L the bilinear form

$$a(u,v)=\int_{\Omega}\left(\sum_{i,j=1}^{n}\left(a_{ij}u_{x_i}+d_j\ u\right)v_{x_j}+\left(\sum_{i=1}^{n}b_iu_{x_i}+cu\right)v\right)dx, \tag{4.2}$$

$u,v\in \overset{\circ}{W}{}^{1,2}(\Omega)$.

We point out that, as a consequence of Lemma 3.1, a is continuous on $\overset{\circ}{W}{}^{1,2}(\Omega)\times\overset{\circ}{W}{}^{1,2}(\Omega)$ and so the operator $L:\overset{\circ}{W}{}^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ is continuous as well.

We start showing a technical lemma.

Let u_s be the functions of Lemma 3.2 obtained in correspondence of a given $u\in\overset{\circ}{W}{}^{1,2}(\Omega)\cap L^{\infty}(\Omega)$, of $g=\sum_{i=1}^{n}|b_i-d_i|$ and of a positive real number ε specified in the proof of Lemma 4.1. The following result holds true.

Lemma 4.1. *Let a be the bilinear form defined in* (4.2). *Under hypotheses (21)–(24), there exists a constant $C\in\mathbb{R}_+$ such that*

$$\int_{\Omega}|u|^{p-2}\left((u_s)_x^2+u_s^2\right)\,dx\le C\sum_{h=1}^{s}a\left(u,|u|^{p-2}u_h\right),\quad s=1,\dots,r,\ \forall p\in\,]2,+\infty[, \tag{4.3}$$

with $C=C(s,\nu,\mu)$.

Proof. Let u, g, ε, and u_s, for $s=1,\dots,r$, be as above specified and $p>2$. We start observing that in view of Lemma 3.3 one has $|u|^{p-2}u_s\in\overset{\circ}{W}{}^{1,2}(\Omega)$, for $s=1,\dots,r$.

This allows us to take $|u|^{p-2}u_s$ as test function in (4.2). Hence, simple calculations together with (3.9) and (3.10) give

$$\begin{aligned}
a\left(u,|u|^{p-2}u_s\right)&=\int_{\Omega}\Bigg[(p-2)\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}|u|^{p-4}uu_s\\
&\quad+\sum_{i,j=1}^{n}a_{ij}u_{x_i}(u_s)_{x_j}|u|^{p-2}+\sum_{i=1}^{n}d_i\left(|u|^{p-2}uu_s\right)_{x_i}\\
&\quad+c|u|^{p-2}uu_s+\sum_{i=1}^{n}(b_i-d_i)|u|^{p-2}u_{x_i}u_s\Bigg]dx\\
&=\int_{\Omega}\Bigg[(p-2)\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}|u|^{p-4}uu_s+\sum_{i,j=1}^{n}a_{ij}(u_s)_{x_i}(u_s)_{x_j}|u|^{p-2}\\
&\quad+\sum_{i=1}^{n}d_i\left(|u|^{p-2}uu_s\right)_{x_i}+c|u|^{p-2}uu_s+\sum_{i=1}^{n}\left((b_i-d_i)|u|^{p-2}\sum_{h=1}^{s}(u_h)_{x_i}u_s\right)\Bigg]dx.
\end{aligned} \tag{4.4}$$

From this last equality, (3.8), and hypotheses (21) and (24) we get

$$\begin{aligned} a\left(u,|u|^{p-2}u_s\right) &\geq \int_\Omega \Bigg[\nu(p-2)|u|^{p-4}u_x^2u_s^2+\nu|u|^{p-2}(u_s)_x^2 \\ &\quad +\mu|u|^{p-2}u_s^2-\sum_{i=1}^n|b_i-d_i||u|^{p-2}\sum_{h=1}^s(u_h)_x|u_s|\Bigg]dx \\ &\geq \min\{\nu,\mu\}\int_\Omega\left[(p-2)|u|^{p-4}u_x^2u_s^2+|u|^{p-2}\left((u_s)_x^2+u_s^2\right)\right]dx \\ &\quad -\sum_{h=1}^s\int_\Omega\sum_{i=1}^n|b_i-d_i||u|^{p-2}(u_h)_x|u_s|dx. \end{aligned} \tag{4.5}$$

Hence, setting

$$\mu_0=\min\{\nu,\mu\},\quad g=\sum_{i=1}^n|b_i-d_i|, \tag{4.6}$$

$$F_s(u)=\left[(p-2)|u|^{p-4}u_x^2u_s^2+|u|^{p-2}\left((u_s)_x^2+u_s^2\right)\right], \tag{4.7}$$

we obtain

$$\mu_0\int_\Omega F_s(u)dx\leq a\left(u,|u|^{p-2}u_s\right)+\sum_{h=1}^s\int_\Omega g|u|^{p-2}(u_h)_x|u_s|dx. \tag{4.8}$$

On the other hand, by the Hölder inequality, Lemmas 3.2 and 3.3, the embedding results contained in Lemma 3.1 and using hypothesis (23) and (3.7), one has that there exists a constant $c_0\in\mathbb{R}_+$, such that

$$\begin{aligned} \sum_{h=1}^s\int_\Omega g|u|^{p-2}(u_h)_x|u_s|\ dx &\leq \sum_{h=1}^s\left\|g|u|^{p/(2-1)}u_s\right\|_{L^2(\operatorname{supp}(u_h)_x)}\left\||u|^{p/(2-1)}(u_h)_x\right\|_{L^2(\Omega)} \\ &\leq c_0\left\||u|^{p/(2-1)}u_s\right\|_{W^{1,2}(\Omega)}\sum_{h=1}^s\left\|g\chi_{\operatorname{supp}(u_h)_x}\right\|_{M^{2t,\lambda}(\Omega)}\left\||u|^{p/(2-1)}(u_h)_x\right\|_{L^2(\Omega)} \\ &\leq c_0\varepsilon\left\||u|^{p/(2-1)}u_s\right\|_{W^{1,2}(\Omega)}\sum_{h=1}^s\left\||u|^{p/(2-1)}(u_h)_x\right\|_{L^2(\Omega)}, \end{aligned} \tag{4.9}$$

with $c_0=c_0(n,t)$.

Now, we observe that explicit computations give

$$\left\||u|^{p/(2-1)}u_h\right\|_{W^{1,2}(\Omega)}^2\leq c_1\int_\Omega F_h(u)dx,\quad h=1,\ldots,s, \tag{4.10}$$

with $c_1=c_1(n,p)$.

Therefore, combining (4.8), (4.9), and (4.10) we get

$$\int_\Omega F_s(u)dx \le \frac{1}{\mu_0}\ a\left(u,|u|^{p-2}u_s\right) + \frac{c_2}{\mu_0}\varepsilon\left(\int_\Omega F_s(u)dx\right)^{1/2}\sum_{h=1}^{s}\left(\int_\Omega F_h(u)dx\right)^{1/2}, \tag{4.11}$$

with $c_2 = c_2(n,t,p)$.

Thus,

$$\begin{aligned}\int_\Omega F_s(u)dx &\le \frac{1}{\mu_0}\ a\left(u,|u|^{p-2}u_s\right) + \frac{c_3}{\mu_0}\varepsilon\left(\int_\Omega F_s(u)dx\right)^{1/2}\left(\sum_{h=1}^{s}\int_\Omega F_h(u)dx\right)^{1/2} \\ &\le \frac{1}{\mu_0}\ a\left(u,|u|^{p-2}u_s\right) + \frac{c_3}{\mu_0}\left(\frac{\eta}{2}\int_\Omega F_s(u)dx + \frac{\varepsilon^2}{2\eta}\sum_{h=1}^{s}\int_\Omega F_h(u)dx\right),\end{aligned} \tag{4.12}$$

with $c_3 = c_3(n,t,p,r)$.

Choosing $\eta = \mu_0/c_3$ and $\varepsilon = \mu_0/(c_3\sqrt{2})$, we have

$$\int_\Omega F_s(u)dx \le \frac{2}{\mu_0}\ a\left(u,|u|^{p-2}u_s\right) + \frac{1}{2}\sum_{h=1}^{s}\int_\Omega F_h(u)dx. \tag{4.13}$$

Finally we conclude by (4.7) and (4.13) that

$$\int_\Omega |u|^{p-2}\left((u_s)_x^2 + u_s^2\right)dx \le \int_\Omega F_s(u)dx \le C\sum_{h=1}^{s} a\left(u,|u|^{p-2}u_h\right), \tag{4.14}$$

with $C = C(s,\mu_0)$. This ends the proof of (4.3). □

Finally, we consider the Dirichlet problem

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}\ (\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega), \end{gathered} \tag{4.15}$$

and we prove the following L^p-a priori bound.

Theorem 4.2. *Under the hypotheses (21)–(24) and if f is in $L^2(\Omega)\cap L^\infty(\Omega)$ and the solution u of (4.15) is in $\overset{\circ}{W}{}^{1,2}(\Omega)\cap L^\infty(\Omega)$, then one has*

$$\|u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \quad \forall p \in\ ]2,+\infty[, \tag{4.16}$$

where C is a constant depending on $n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1,\dots,n$.

Proof. Fix $p \in]2, +\infty[$. We firstly prove that

$$\int_{\Omega} |u|^{p-2}\left(u_x^2 + u^2\right)\, dx \leq C a\left(u, |u|^{p-2}u\right), \tag{4.17}$$

with $C = C(n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)})$.

Indeed, if we consider the functions u_s, $s = 1, \ldots, r$, obtained in correspondence with the solution u, of g and ε as in Lemma 4.1, by (3.11) we get

$$\int_{\Omega} |u|^{p-2}\left(u_x^2 + u^2\right) dx \leq c_0 \int_{\Omega} |u|^{p-2} \sum_{s=1}^{r} \left((u_s)_x^2 + u_s^2\right) dx, \tag{4.18}$$

with $c_0 = c_0(r)$.

Thus, taking into account (4.3),

$$\int_{\Omega} |u|^{p-2}\left(u_x^2 + u^2\right)\, dx \leq c_0 \sum_{s=1}^{r} C_s \sum_{h=1}^{s} a\left(u, |u|^{p-2}u_h\right) \leq C \sum_{s=1}^{r} a\left(u, |u|^{p-2}u_s\right), \tag{4.19}$$

with $C_s = C_s(s, \nu, \mu)$ and $C = C(r, \nu, \mu)$.

The linearity of a together with (3.11) and (3.12) then give (4.17).

Now, using (4.17) and Hölder inequality we end the proof, since

$$\begin{aligned}\|u\|_{L^p(\Omega)}^p &\leq \int_{\Omega} |u|^{p-2}\left(u_x^2 + u^2\right)\, dx \leq C\, a\left(u, |u|^{p-2}u\right) \\ &= C \int_{\Omega} f|u|^{p-2}u\, dx \leq C \int_{\Omega} \left|f\right| |u|^{p-1}\, dx \leq C \left\|f\right\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}.\end{aligned} \tag{4.20}$$

□

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