

Research Article

Existence and Uniqueness of Solutions to Neutral Stochastic Functional Differential Equations with Poisson Jumps

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A class of neutral stochastic functional differential equations with Poisson jumps (NSFDEwPJs), $d[x(t) - G(x_t)] = f(x_t, t)dt + g(x_t, t)dW(t) + h(x_t, t)dN(t), t \in [t_0, T]$, with initial value $x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$, is investigated. First, we consider the existence and uniqueness of solutions to NSFDEwPJs under the uniform Lipschitz condition, the linear growth condition, and the contractive mapping. Then, the uniform Lipschitz condition is replaced by the local Lipschitz condition, and the existence and uniqueness theorem for NSFDEwPJs is also derived.

1. Introduction

Neutral stochastic functional differential equations (NSFDEs) have recently been studied intensively, for instance, Mao [1] and Kolmanovskii [2], Mao et al. [3–6], Luo et al. [7], Zhou and Hu [8], and Luo [9]. Poisson jumps are becoming increasingly used to model real-world phenomena in different fields such as economics, finance, biology, and physics. There is an extensive literature concerned with Poisson jumps, for example, Wang et al. [10, 11], Ronghua et al. [12, 13], Luo [14], and Tan and Wang [15]. Therefore, it is natural and necessary to incorporate jumps in the neutral stochastic functional differential equations. However, the study of NSFDEwPJs is limited by far. Liu et al. [16] studied the stability of NSFDEwPJs by using fixed point theory, Luo and Taniguchi [17] proved the existence and uniqueness of non-Lipschitz stochastic neutral delay evolution equations driven by Poisson jumps. However, no theory for the existence and uniqueness of solutions to NSFDEwPJs has

been established yet. Therefore, in this paper, we first prove the existence and uniqueness of solutions to NSFDEwPJs.

The outline of the paper is as follows. In Section 2 we will introduce some necessary notations and assumptions. In Section 3, we will present several useful lemmas, and then we prove the existence and uniqueness of NSFDEwPJs under the uniform Lipschitz condition, the linear growth condition, and the contractive mapping. Furthermore, the uniform Lipschitz condition is replaced by the local Lipschitz condition, and the existence and uniqueness theorem is also derived.

2. Preliminaries

Let (Ω, \mathcal{F}, P) be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which satisfies the usual conditions, that is, the filtration is continuous on the right and \mathcal{F}_0 contains all P -null sets. Moreover, $C([a, b]; R^n)$ denotes the family of functions φ from $[a, b]$ to R^n that are right-continuous and have limits on the left, $C([a, b]; R^n)$ is equipped with the norm $\|\varphi\| = \sup_{a < t < b} |\varphi(t)|$, where $|\cdot|$ is the Euclidean norm in R^n , that is, $|x| = \sqrt{x^T x}$ ($x \in R^n$). Let $\mathcal{L}^p([a, b]; R^n)$ denote the family of all R^n -valued \mathcal{F}_t -adapted process $f(t)$, $t \in [a, b]$, such that $\int_a^b |f(t)|^p dt < \infty$. We denote by $\mathcal{M}^2([-\tau, 0]; R^n)$ the family of the processes $\xi(t)$ in $\mathcal{L}^2([-\tau, 0]; R^n)$ such that $E \int_{-\tau}^0 |\xi(t)|^2 dt < \infty$. $a \vee b$ means the maximum of a and b , and $a \wedge b$ means the minimum of a and b .

In this paper, we consider the n -dimensional neutral stochastic functional differential equations with Poisson jumps

$$d[x(t) - G(x_t)] = f(x_t, t)dt + g(x_t, t)dW(t) + h(x_t, t)dN(t), \quad t \in [t_0, T], \quad (2.1)$$

where $x_t = \{x(t + \theta) : -\tau \leq \theta \leq 0\}$ can be regarded as a $C([-\tau, 0]; R^n)$ -valued stochastic process, $W(t)$ is an m -dimensional standard Wiener process that is \mathcal{F}_t -adapted, and $N(t)$ is a scalar Poisson process with intensity λ . Assume that $W(t)$ and $N(t)$ are independent of \mathcal{F}_{t_0} . Moreover, the functions $G : C([-\tau, 0]; R^n) \rightarrow R^n$, $f : C([-\tau, 0]; R^n) \times [t_0, T] \rightarrow R^n$, $g : C([-\tau, 0]; R^n) \times [t_0, T] \rightarrow R^{n \times m}$, $h : C([-\tau, 0]; R^n) \times [t_0, T] \rightarrow R^n$.

The stochastic integral is defined in the Itô sense, and the integral version of (2.1) is frequently expressed as

$$x(t) - G(x_t) = x(t_0) - G(x_{t_0}) + \int_{t_0}^t f(x_s, s)ds + \int_{t_0}^t g(x_s, s)dW(s) + \int_{t_0}^t h(x_s, s)dN(s). \quad (2.2)$$

The initial value is followed:

$$\begin{aligned} x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \text{ is } \mathcal{F}_{t_0} \text{-measurable } C([-\tau, 0]; R^n) \\ \text{- valued random variable such that } \xi \in \mathcal{M}^2([-\tau, 0]; R^n). \end{aligned} \quad (2.3)$$

3. The Existence and Uniqueness Theorem

Definition 3.1. An R^n -valued stochastic process $x(t)$ on $t_0 - \tau \leq t \leq T$ is called a solution to (2.1) with initial data (2.3) if it has the following properties:

- (i) $x(t)$ is continuous and $\{x_t\}_{t_0 \leq t \leq T}$ is \mathcal{F}_t -adapted;
- (ii) $f(x_t, t) \in \mathcal{L}^1([t_0, T]; R^n)$ and $g(x_t, t) \in \mathcal{L}^2([t_0, T]; R^{n \times m})$;
- (iii) $x_{t_0} = \xi$ and (2.2) hold for each $t_0 \leq t \leq T$.

A solution $x(t)$ is said to be unique if any other solution $\bar{x}(t)$ is indistinguishable from it, that is,

$$P\{x(t) = \bar{x}(t) \forall t_0 - \tau \leq t \leq T\} = 1. \quad (3.1)$$

Let us now begin to establish the theory of the existence and uniqueness of the solution to (2.1) with initial data (2.3).

Theorem 3.2. Assume that there exist two positive constants \bar{K} and K such that

(H1) (the Lipschitz condition) For all $\phi, \varphi \in C([- \tau, 0]; R^n)$ and $t \in [t_0, T]$,

$$|f(\phi, t) - f(\varphi, t)|^2 \vee |g(\phi, t) - g(\varphi, t)|^2 \vee |h(\phi, t) - h(\varphi, t)|^2 \leq \bar{K} \|\phi - \varphi\|^2. \quad (3.2)$$

(H2) (the linear growth condition) for all $(\phi, t) \in C([- \tau, 0]; R^n) \times [t_0, T]$,

$$|f(\phi, t)|^2 \vee |g(\phi, t)|^2 \vee |h(\phi, t)|^2 \leq K(1 + \|\phi\|^2). \quad (3.3)$$

(H3) (the contractive mapping) there is a positive constant $\kappa \in (0, 1)$ such that, for all $\phi, \varphi \in C([- \tau, 0]; R^n)$,

$$|G(\phi) - G(\varphi)| \leq \kappa \|\phi - \varphi\|. \quad (3.4)$$

Then, there exists a unique solution $x(t)$ to (2.1) with initial data (2.3). Moreover, the solution belongs to $\mathcal{M}^2([t_0 - \tau, T]; R^n)$.

To prove Theorem 3.2, we give the following useful Lemmas 3.3 and 3.4 first.

Lemma 3.3. If $p \geq 2$, $g \in \mathcal{M}^2([t_0, T]; R^{n \times m})$ such that

$$E \int_{t_0}^t |g(s)|^p ds < \infty, \quad (3.5)$$

then $E \left| \int_{t_0}^T g(s) dW(s) \right|^p \leq (p(p-1)/2)^{p/2} T^{(p-2)/2} E \int_{t_0}^T |g(s)|^p ds$.

In particular, for $p = 2$, there is equality.

The proof of Lemma 3.3 can be found in [6].

Lemma 3.4. For any $a, b > 0$ and $0 < \alpha < 1$ one has

$$(a + b)^2 \leq \frac{a^2}{\alpha} + \frac{b^2}{1 - \alpha}. \quad (3.6)$$

The proof of Lemma 3.4 can also be found in [6].

To prove the uniqueness of the solution for (2.1), we will establish Lemma 3.5.

Lemma 3.5. Let (3.3) and (3.4) hold. If $x(t)$ is a solution to (2.1) with initial data (2.3), then

$$\begin{aligned} E \left(\sup_{t_0 - \tau \leq t \leq T} |x(t)|^2 \right) &\leq \left(1 + \frac{5 + \kappa\sqrt{\kappa}}{(1 - \kappa)(1 - \sqrt{\kappa})} E \|\xi\|^2 \right) \\ &\times \exp \left[\frac{4K(T - t_0)(T - t_0 + 4K + 8\lambda + 2\lambda^2(T - t_0))}{(1 - \kappa)(1 - \sqrt{\kappa})} \right], \end{aligned} \quad (3.7)$$

where κ is the contraction constant in (3.4). In particular, $x(t)$ belongs to $\mathcal{M}^2([t_0 - \tau, T]; \mathbb{R}^n)$.

Proof. For every integer $n \geq 1$, define the stopping time

$$\tau_n = T \wedge \inf \{ t \in [t_0, T] : \|x_t\| \geq n \}. \quad (3.8)$$

Clearly, $\tau_n \uparrow T$ a.s. Set $x^n(t) = x(t \wedge \tau_n)$ for $t \in [t_0, T]$. Then, for $t_0 \leq t \leq T$

$$\begin{aligned} x^n(t) &= G(x_t^n) - G(\xi) + \xi(0) + \int_{t_0}^t f(x_s^n, s) I_{[t_0, \tau_n]}(s) ds \\ &+ \int_{t_0}^t g(x_s^n, s) I_{[t_0, \tau_n]}(s) dW(s) + \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]}(s) dN(s), \end{aligned} \quad (3.9)$$

where I_A is the indicator function of a set A , that is,

$$I_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \quad (3.10)$$

Set

$$\begin{aligned} J^n(t) &= \xi(0) + \int_{t_0}^t f(x_s^n, s) I_{[t_0, \tau_n]} ds + \int_{t_0}^t g(x_s^n, s) I_{[t_0, \tau_n]} dW(s) \\ &+ \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]} dN(s). \end{aligned} \quad (3.11)$$

Applying Lemma 3.4 twice and (3.4), one derives that

$$\begin{aligned}
 |x^n(t)|^2 &\leq \frac{1}{\kappa} |G(x_t^n) - G(\xi)|^2 + \frac{1}{1-\kappa} |J^n(t)|^2 \\
 &\leq \frac{1}{\kappa} \kappa^2 \|x_t^n - \xi\|^2 + \frac{1}{1-\kappa} |J^n(t)|^2 \\
 &= \kappa \|x_t^n - \xi\|^2 + \frac{1}{1-\kappa} |J^n(t)|^2 \\
 &\leq \kappa \left(\frac{1}{\sqrt{\kappa}} \|x_t^n\|^2 + \frac{1}{1-\sqrt{\kappa}} \|\xi\|^2 \right) + \frac{1}{1-\kappa} |J^n(t)|^2 \\
 &= \sqrt{\kappa} \|x_t^n\|^2 + \frac{\kappa}{1-\sqrt{\kappa}} \|\xi\|^2 + \frac{1}{1-\kappa} |J^n(t)|^2.
 \end{aligned} \tag{3.12}$$

Hence,

$$\begin{aligned}
 E \left(\sup_{t_0 \leq s \leq t} |x^n(s)|^2 \right) &\leq \sqrt{\kappa} E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) + \frac{\kappa}{1-\sqrt{\kappa}} E \|\xi\|^2 \\
 &\quad + \frac{1}{1-\kappa} E \left(\sup_{t_0 \leq s \leq t} |J^n(s)|^2 \right).
 \end{aligned} \tag{3.13}$$

Noting that $E(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2) \leq E\|\xi\|^2 + E(\sup_{t_0 \leq s \leq t} |x^n(s)|^2)$, one sees that

$$\begin{aligned}
 E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) &\leq \sqrt{\kappa} E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) + \frac{1 + \kappa - \sqrt{\kappa}}{1 - \sqrt{\kappa}} E \|\xi\|^2 \\
 &\quad + \frac{1}{1-\kappa} E \left(\sup_{t_0 \leq s \leq t} |J^n(s)|^2 \right).
 \end{aligned} \tag{3.14}$$

Consequently,

$$E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) \leq \frac{1 + \kappa - \sqrt{\kappa}}{(1 - \sqrt{\kappa})^2} E \|\xi\|^2 + \frac{1}{(1 - \kappa)(1 - \sqrt{\kappa})} E \left(\sup_{t_0 \leq s \leq t} |J^n(s)|^2 \right). \tag{3.15}$$

Using the basic inequality $|a + b + c + d|^2 \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|d|^2$ and the Hölder inequality, and for the jump integral, we can transform to the compensated Poisson process $\widetilde{N}(t) = N(t) - \lambda t$, and we can get

$$\begin{aligned}
 |J^n(t)|^2 &\leq 4|\xi(0)|^2 + 4 \left| \int_{t_0}^t f(x_s^n, s) I_{[t_0, \tau_n]}(s) ds \right|^2 + 4 \left| \int_{t_0}^t g(x_s^n, s) I_{[t_0, \tau_n]}(s) dW(s) \right|^2 \\
 &\quad + 4 \left| \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]}(s) dN(s) \right|^2
 \end{aligned}$$

$$\begin{aligned}
&\leq 4|\xi(0)|^2 + 4(t-t_0) \int_{t_0}^t |f(x_s^n, s)|^2 ds + 4 \left| \int_{t_0}^t g(x_s^n, s) I_{[t_0, \tau_n]}(s) dW(s) \right|^2 \\
&\quad + 4 \left| \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]}(s) d\widetilde{N}(s) + \lambda \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]}(s) ds \right|^2 \\
&\leq 4|\xi(0)|^2 + 4(t-t_0) \int_{t_0}^t |f(x_s^n, s)|^2 ds + 4 \left| \int_{t_0}^t g(x_s^n, s) I_{[t_0, \tau_n]}(s) dW(s) \right|^2 \\
&\quad + 8 \left| \int_{t_0}^t h(x_s^n, s) I_{[t_0, \tau_n]}(s) d\widetilde{N}(s) \right|^2 + 8\lambda^2(t-t_0) \int_{t_0}^t |h(x_s^n, s)|^2 ds.
\end{aligned} \tag{3.16}$$

Since both the Wiener process $W(t)$ and compensated Poisson process $\widetilde{N}(t)$ are martingales, using the Doob martingale inequality, we have

$$\begin{aligned}
E \left(\sup_{t_0 \leq s \leq t} |J^n(s)|^2 \right) &\leq 4E|\xi(0)|^2 + 4(t-t_0)E \int_{t_0}^t |f(x_s^n, s)|^2 ds \\
&\quad + 4 \times 4E \int_{t_0}^t |g(x_s^n, s)|^2 ds \\
&\quad + 4(2 \times 4\lambda + 2\lambda^2(t-t_0))E \int_{t_0}^t |h(x_s^n, s)|^2 ds \\
&\leq 4E\|\xi\|^2 + 4K(T-t_0 + 4 + 8\lambda + 2\lambda^2(T-t_0)) \int_{t_0}^t (1 + E\|x_s^n\|^2) ds.
\end{aligned} \tag{3.17}$$

Substituting this into (3.15),

$$\begin{aligned}
E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) &\leq \frac{5 + \kappa\sqrt{\kappa}}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 \\
&\quad + \frac{4K(T-t_0 + 4 + 8\lambda + 2\lambda^2(T-t_0))}{(1-\kappa)(1-\sqrt{\kappa})} \int_{t_0}^t (1 + E\|x_s^n\|^2) ds.
\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}
1 + E \left(\sup_{t_0 - \tau \leq s \leq t} |x^n(s)|^2 \right) &\leq 1 + \frac{5 + \kappa\sqrt{\kappa}}{(1-\kappa)(1-\sqrt{\kappa})} E\|\xi\|^2 \\
&\quad + \frac{4K(T-t_0 + 4 + 8\lambda + 2\lambda^2(T-t_0))}{(1-\kappa)(1-\sqrt{\kappa})} \int_{t_0}^t \left(1 + E \left(\sup_{t_0 - \tau \leq r \leq s} |x^n(r)|^2 \right) \right) ds.
\end{aligned} \tag{3.19}$$

Now the Gronwall inequality yields that

$$1 + E \left(\sup_{t_0 - \tau \leq t \leq T} |x^n(t)|^2 \right) \leq \left(1 + \frac{5 + \kappa\sqrt{\kappa}}{(1 - \kappa)(1 - \sqrt{\kappa})} E \|\xi\|^2 \right) \times \exp \left[\frac{4K(T - t_0)(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{(1 - \kappa)(1 - \sqrt{\kappa})} \right]. \tag{3.20}$$

Consequently

$$E \left(\sup_{t_0 - \tau \leq t \leq \tau_n} |x(t)|^2 \right) \leq \left(1 + \frac{5 + \kappa\sqrt{\kappa}}{(1 - \kappa)(1 - \sqrt{\kappa})} E \|\xi\|^2 \right) \times \exp \left[\frac{4K(T - t_0)(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{(1 - \kappa)(1 - \sqrt{\kappa})} \right]. \tag{3.21}$$

Letting $n \rightarrow \infty$, it then implies the following inequality:

$$E \left(\sup_{t_0 - \tau \leq t \leq T} |x(t)|^2 \right) \leq \left(1 + \frac{5 + \kappa\sqrt{\kappa}}{(1 - \kappa)(1 - \sqrt{\kappa})} E \|\xi\|^2 \right) \times \exp \left[\frac{4K(T - t_0)(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{(1 - \kappa)(1 - \sqrt{\kappa})} \right]. \tag{3.22}$$

The proof is complete. □

Proof of Theorem 3.2. Uniqueness. Let $x(t)$ and $\bar{x}(t)$ be two solutions to (2.1). By Lemma 3.5 both of them belong to $\mathcal{M}^2([t_0 - \tau, T]; R^n)$. Note that

$$x(t) - \bar{x}(t) = G(x_t) - G(\bar{x}_t) + J(t), \tag{3.23}$$

where

$$J(t) = \int_{t_0}^t [f(x_s, s) - f(\bar{x}_s, s)] ds + \int_{t_0}^t [g(x_s, s) - g(\bar{x}_s, s)] dW(s) + \int_{t_0}^t [h(x_s, s) - h(\bar{x}_s, s)] dN(s). \tag{3.24}$$

By Lemma 3.4 and condition (3.4), one sees easily that

$$\begin{aligned} |x(t) - \bar{x}(t)|^2 &\leq \frac{1}{\kappa} |G(x_t) - G(\bar{x}_t)|^2 + \frac{1}{1 - \kappa} |J(t)|^2 \\ &\leq \kappa \|x_t - \bar{x}_t\|^2 + \frac{1}{1 - \kappa} |J(t)|^2. \end{aligned} \tag{3.25}$$

Therefore,

$$E \sup_{t_0 \leq s \leq t} |x(s) - \bar{x}(s)|^2 \leq \kappa E \sup_{t_0 \leq s \leq t} |x(s) - \bar{x}(s)|^2 + \frac{1}{1 - \kappa} E \sup_{t_0 \leq s \leq t} |J(s)|^2, \quad (3.26)$$

which implies that

$$E \sup_{t_0 \leq s \leq t} |x(s) - \bar{x}(s)|^2 \leq \frac{1}{(1 - \kappa)^2} E \sup_{t_0 \leq s \leq t} |J(s)|^2. \quad (3.27)$$

On the other hand,

$$\begin{aligned} E \sup_{t_0 \leq s \leq t} |J(s)|^2 &\leq 3\bar{K} \left(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0) \right) \int_{t_0}^t E \|x_s - \bar{x}_s\|^2 ds \\ &= 3\bar{K} \left(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0) \right) \int_{t_0}^t E \sup_{t_0 - \tau \leq r \leq s} |x(r) - \bar{x}(r)|^2 ds. \end{aligned} \quad (3.28)$$

Noting the fact that $x(r) = \bar{x}(r) = \xi$, for $r \in [t_0 - \tau, t_0]$, we have

$$\sup_{t_0 - \tau \leq r \leq s} |x(r) - \bar{x}(r)|^2 = \sup_{t_0 \leq r \leq s} |x(r) - \bar{x}(r)|^2. \quad (3.29)$$

Then

$$E \sup_{t_0 \leq s \leq t} |J(s)|^2 \leq 3\bar{K} \left(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0) \right) \int_{t_0}^t E \sup_{t_0 \leq r \leq s} |x(r) - \bar{x}(r)|^2 ds. \quad (3.30)$$

Therefore,

$$\begin{aligned} E \sup_{t_0 \leq s \leq t} |x(s) - \bar{x}(s)|^2 \\ \leq \frac{3\bar{K} \left(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0) \right)}{(1 - \kappa)^2} \int_{t_0}^t E \sup_{t_0 \leq r \leq s} |x(r) - \bar{x}(r)|^2 ds. \end{aligned} \quad (3.31)$$

The Gronwall inequality then yields that

$$E \sup_{t_0 \leq t \leq T} |x(t) - \bar{x}(t)|^2 = 0. \quad (3.32)$$

Therefore the uniqueness has been proved.

Existence. We divide the whole proof of the existence into two steps.

Step 1. We impose an additional condition: $T - t_0$ is sufficiently small so that

$$\delta := \kappa + \frac{3\bar{K} \left(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0) \right)}{1 - \kappa} (T - t_0) < 1. \quad (3.33)$$

Define $x_{t_0}^0 = \xi$ and $x^0(t) = \xi(0)$ for $t_0 \leq t \leq T$. For each $n = 1, 2, \dots$, set $x_{t_0}^n = \xi$ and define, by the Picard iterations

$$\begin{aligned} x^n(t) - G(x_t^{n-1}) &= \xi(0) - G(\xi) + \int_{t_0}^t f(x_s^{n-1}, s) ds + \int_{t_0}^t g(x_s^{n-1}, s) dW(s) \\ &+ \int_{t_0}^t h(x_s^{n-1}, s) dN(s). \end{aligned} \quad (3.34)$$

Obviously, $x^0(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$, and by induction $x^n(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$. In fact,

$$\begin{aligned} x^n(t) &= G(x_t^{n-1}) - G(\xi) + \xi(0) + \int_{t_0}^t f(x_s^{n-1}, s) ds + \int_{t_0}^t g(x_s^{n-1}, s) dW(s) \\ &+ \int_{t_0}^t h(x_s^{n-1}, s) dN(s). \end{aligned} \quad (3.35)$$

Let

$$P^{n-1}(t) = \xi(0) + \int_{t_0}^t f(x_s^{n-1}, s) ds + \int_{t_0}^t g(x_s^{n-1}, s) dW(s) + \int_{t_0}^t h(x_s^{n-1}, s) dN(s). \quad (3.36)$$

Then by Lemma 3.4 and (3.4) we have

$$\begin{aligned} |x^n(t)|^2 &\leq \frac{1}{\kappa} |G(x_t^n) - G(\xi)|^2 + \frac{1}{1-\kappa} |P^{n-1}(t)|^2 \\ &\leq \kappa \|x_t^n - \xi\|^2 + \frac{1}{1-\kappa} |P^{n-1}(t)|^2 \\ &\leq \kappa \left(\frac{1}{\sqrt{\kappa}} \|x_t^n\|^2 + \frac{1}{1-\sqrt{\kappa}} \|\xi\|^2 \right) + \frac{1}{1-\kappa} |P^{n-1}(t)|^2 \\ &= \sqrt{\kappa} \|x_t^n\|^2 + \frac{\kappa}{1-\sqrt{\kappa}} \|\xi\|^2 + \frac{1}{1-\kappa} |P^{n-1}(t)|^2 \\ &\leq \sqrt{\kappa} (\|\xi\|^2 + |x^n(t)|^2) + \frac{\kappa}{1-\sqrt{\kappa}} \|\xi\|^2 + \frac{1}{1-\kappa} |P^{n-1}(t)|^2. \end{aligned} \quad (3.37)$$

Then,

$$|x^n(t)|^2 \leq \frac{\sqrt{\kappa}}{(1-\sqrt{\kappa})^2} \|\xi\|^2 + \frac{1}{(1-\sqrt{\kappa})(1-\kappa)} |P^{n-1}(t)|^2. \quad (3.38)$$

Therefore,

$$E|x^n(t)|^2 \leq \frac{\sqrt{\kappa}}{(1-\sqrt{\kappa})^2} E\|\xi\|^2 + \frac{1}{(1-\sqrt{\kappa})(1-\kappa)} E|P^{n-1}(t)|^2. \quad (3.39)$$

By the basic inequality $|a + b + c + d|^2 \leq 4|a|^2 + 4|b|^2 + 4|c|^2 + 4|d|^2$, the Hölder inequality, Lemma 3.3, and martingale isometry, we have

$$\begin{aligned}
E \left| P^{n-1}(t) \right|^2 &\leq 4E|\xi(0)|^2 + 4E \left| \int_{t_0}^t f(x_s^{n-1}, s) ds \right|^2 \\
&\quad + 4E \left| \int_{t_0}^t g(x_s^{n-1}, s) dW(s) \right|^2 + 4E \left| \int_{t_0}^t h(x_s^{n-1}, s) dN(s) \right|^2 \\
&\leq 4E|\xi(0)|^2 + 4(t - t_0)E \int_{t_0}^t |f(x_s^{n-1}, s)|^2 ds + 4E \int_{t_0}^t |g(x_s^{n-1}, s)|^2 ds \\
&\quad + 4(2\lambda + 2\lambda^2(t - t_0))E \int_{t_0}^t |h(x_s^{n-1}, s)|^2 ds \\
&\leq 4E\|\xi\|^2 + 4K(T - t_0 + 1) \int_{t_0}^t \left(1 + E\|x_s^{n-1}\|^2\right) ds \\
&\quad + 4K(2\lambda + 2\lambda^2(T - t_0)) \int_{t_0}^t \left(1 + E\|x_s^{n-1}\|^2\right) ds \\
&\leq 4E\|\xi\|^2 + 4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0))(T - t_0) \\
&\quad + 4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0))E \int_{t_0}^t \sup_{t_0 \leq r \leq s} |x^{n-1}(r)|^2 ds \\
&\leq 4E\|\xi\|^2 + 4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0))(T - t_0) \\
&\quad + 4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0)) \int_{t_0}^t E|x^{n-1}(s)|^2 ds.
\end{aligned} \tag{3.40}$$

Therefore,

$$\begin{aligned}
E|x^n(t)|^2 &\leq \frac{\sqrt{\kappa}}{(1 - \sqrt{\kappa})^2} E\|\xi\|^2 + \frac{1}{(1 - \sqrt{\kappa})(1 - \kappa)} E \left| P^{n-1}(t) \right|^2 \\
&\leq \frac{\sqrt{\kappa}}{(1 - \sqrt{\kappa})^2} E\|\xi\|^2 + \frac{4}{(1 - \sqrt{\kappa})(1 - \kappa)} E\|\xi\|^2 \\
&\quad + \frac{4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0))(T - t_0)}{(1 - \sqrt{\kappa})(1 - \kappa)} \\
&\quad + \frac{4K(T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0))}{(1 - \sqrt{\kappa})(1 - \kappa)} \int_{t_0}^t E|x^{n-1}(s)|^2 ds.
\end{aligned} \tag{3.41}$$

Hence, for any $k \geq 1$, we have

$$\begin{aligned} \max_{1 \leq n \leq k} E|x^n(t)|^2 &\leq \left[\frac{\sqrt{\kappa}}{(1-\sqrt{\kappa})^2} + \frac{4}{(1-\sqrt{\kappa})(1-\kappa)} \right] E\|\xi\|^2 \\ &\quad + \frac{4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0))(T-t_0)}{(1-\sqrt{\kappa})(1-\kappa)} \\ &\quad + \frac{4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0))}{(1-\sqrt{\kappa})(1-\kappa)} \int_{t_0}^t \max_{1 \leq n \leq k} E|x^{n-1}(s)|^2 ds. \end{aligned} \tag{3.42}$$

Note that

$$\begin{aligned} \max_{1 \leq n \leq k} E|x^{n-1}(t)|^2 &= \max \left\{ E|\xi(0)|^2, E|x^1(s)|^2, \dots, E|x^{k-1}(s)|^2 \right\} \\ &\leq \max \left\{ E\|\xi\|^2, E|x^1(s)|^2, \dots, E|x^{k-1}(s)|^2 \right\} \\ &= \max \left\{ E\|\xi\|^2, \max_{1 \leq n \leq k} E|x^n(t)|^2 \right\} \\ &\leq E\|\xi\|^2 + \max_{1 \leq n \leq k} E|x^n(t)|^2. \end{aligned} \tag{3.43}$$

Therefore,

$$\begin{aligned} \max_{1 \leq n \leq k} E|x^n(t)|^2 &\leq \left[\frac{\sqrt{\kappa}}{(1-\sqrt{\kappa})^2} + \frac{4}{(1-\sqrt{\kappa})(1-\kappa)} \right] E\|\xi\|^2 \\ &\quad + \frac{4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0))(T-t_0)}{(1-\sqrt{\kappa})(1-\kappa)} \\ &\quad + \frac{4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0))}{(1-\sqrt{\kappa})(1-\kappa)} \int_{t_0}^t \left(E\|\xi\|^2 + \max_{1 \leq n \leq k} E|x^n(t)|^2 \right) ds \\ &\leq c_1 + c_2 \int_{t_0}^t \max_{1 \leq n \leq k} E|x^n(s)|^2 ds, \end{aligned} \tag{3.44}$$

where $c_1 = [(\sqrt{\kappa}/(1-\sqrt{\kappa})^2) + (4/(1-\sqrt{\kappa})(1-\kappa)) + (4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0)))/((1-\sqrt{\kappa})(1-\kappa))]E\|\xi\|^2 + (4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0))(T-t_0)/((1-\sqrt{\kappa})(1-\kappa)))$, $c_2 = (4K(T-t_0+1+2\lambda+2\lambda^2(T-t_0)))/((1-\sqrt{\kappa})(1-\kappa))$.

By the Gronwall inequality, we have

$$\max_{1 \leq n \leq k} E|x^n(t)|^2 \leq c_1 e^{c_2(T-t_0)}. \tag{3.45}$$

Because k is arbitrary, we have

$$E|x^n(t)|^2 \leq c_1 e^{c_2(T-t_0)}. \tag{3.46}$$

Hence, for all $n \geq 0$, $x^n(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$.

Note that, for $t_0 \leq t \leq T$,

$$\begin{aligned} x^1(t) - x^0(t) &= x^1(t) - \xi(0) \\ &= G(x_t^0) - G(\xi) + \int_{t_0}^t f(x_s^0, s) ds + \int_{t_0}^t g(x_s^0, s) dW(s) \\ &\quad + \int_{t_0}^t h(x_s^0, s) dN(s). \end{aligned} \quad (3.47)$$

In a similar way to that in the proof of the existence, we get that

$$\begin{aligned} E \sup_{t_0 \leq t \leq T} |x^1(t) - x^0(t)|^2 &\leq \kappa E \sup_{t_0 \leq t \leq T} \|x_t^0 - \xi\|^2 \\ &\quad + \frac{3K(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{1 - \kappa} \int_{t_0}^t (1 + \|x_t^0\|)^2 ds \\ &\leq 2\kappa E \|\xi\|^2 + \frac{3K(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{1 - \kappa} (1 + E\|\xi\|^2)(T - t_0) := C. \end{aligned} \quad (3.48)$$

Note also that, for $n \geq 1$ and $t_0 \leq t \leq T$,

$$\begin{aligned} x^{n+1}(t) - x^n(t) &= G(x_t^n) - G(x_t^{n-1}) + \int_{t_0}^t [f(x_s^n, s) - f(x_s^{n-1}, s)] ds \\ &\quad + \int_{t_0}^t [g(x_s^n, s) - g(x_s^{n-1}, s)] dW(s) \\ &\quad + \int_{t_0}^t [h(x_s^n, s) - h(x_s^{n-1}, s)] dN(s). \end{aligned} \quad (3.49)$$

In the same way as in the proof of the uniqueness, one derives that

$$\begin{aligned} E \sup_{t_0 \leq t \leq T} |x^{n+1}(t) - x^n(t)|^2 &\leq \kappa E \sup_{t_0 \leq t \leq T} |x^n(t) - x^{n-1}(t)|^2 \\ &\quad + \frac{3\bar{K}(T - t_0 + 4 + 8\lambda + 2\lambda^2(T - t_0))}{1 - \kappa} \int_{t_0}^T E \sup_{t_0 \leq s \leq t} |x^n(s) - x^{n-1}(s)|^2 dt \\ &\leq \delta E \sup_{t_0 \leq t \leq T} |x^n(t) - x^{n-1}(t)|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \delta^n E \sup_{t_0 \leq t \leq T} \left| x^1(t) - x^0(t) \right|^2 \\
&\leq C\delta^n.
\end{aligned} \tag{3.50}$$

Next we verify that $\{x^n(t)\}$ converges to $x(t)$ in the sense of L^2 and probability 1 on $\mathcal{M}^2([t_0 - \tau, T]; R^n)$. Moreover, $x(t)$ is the solution to (2.1) with initial data (2.3). Hence, by the Chebyshev inequality,

$$P \left\{ \sup_{t_0 \leq t \leq T} \left| x^{n+1}(t) - x^n(t) \right|^2 > \frac{1}{2^n} \right\} \leq C(4\delta)^n. \tag{3.51}$$

Since $\sum_{n=0}^{\infty} C(4\delta)^n \leq \infty$, the Borel-Cantelli lemma yields that, for almost all $\omega \in \Omega$, there exist a positive integer $n_0 = n_0(\omega)$ such that

$$\sup_{t_0 \leq t \leq T} \left| x^{n+1}(t) - x^n(t) \right|^2 \leq \frac{1}{2^n}, \quad n \geq n_0. \tag{3.52}$$

It follows that, with probability 1, the partial sum

$$x^0(t) + \sum_{i=0}^{n-1} \left[x^{i+1}(t) - x^i(t) \right] = x^n(t) \tag{3.53}$$

is the partial sum of function series

$$x^0(t) + \left| x^1(t) - x^0(t) \right| + \cdots + \left| x^n(t) - x^{n-1}(t) \right| + \cdots. \tag{3.54}$$

By the second item of series (3.54), the absolute value of every item (3.54) is less than the corresponding item of positive

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots. \tag{3.55}$$

Moreover, the positive series is convergent; further, by the Weierstrass criterion, series (3.54) is convergent on $[t_0, T]$. Furthermore, it is uniformly on $[t_0, T]$. Let the sum function be $x(t)$. Therefore, the approximate sequence $\{x^n(t)\}$ uniformly converges to $x(t)$ on $[t_0, T]$ and is \mathcal{F}_t -adapted, hence $x(t)$ is also continuous and \mathcal{F}_t -adapted. On the other hand, (3.50) implies that, for each t , sequence $\{x^n(t)\}$ is a Cauchy sequence in $\mathcal{M}^2([t_0 - \tau, T]; R^n)$ as well. Hence, we also have $x^n(t) \rightarrow x(t)$ in $\mathcal{M}^2([t_0 - \tau, T]; R^n)$. Letting $n \rightarrow \infty$ in (3.46) gives

$$E|x(t)|^2 \leq c_1 e^{c_2(T-t_0)}, \quad t_0 \leq t \leq T. \tag{3.56}$$

Therefore, by the use of the above result, we get that

$$\begin{aligned} E \int_{t_0-\tau}^T |x(s)|^2 ds &= E \int_{t_0-\tau}^{t_0} |x(s)|^2 ds + E \int_{t_0}^T |x(s)|^2 ds \\ &\leq E \int_{t_0-\tau}^{t_0} |\xi(s)|^2 ds + E \int_{t_0-\tau}^T c_1 e^{c_2(T-t_0)} ds < \infty. \end{aligned} \quad (3.57)$$

That is, $x(t) \in \mathcal{M}^2([t_0 - \tau, T]; \mathbb{R}^n)$.

Now we prove that $x(t)$ satisfies (2.1):

$$\begin{aligned} &E \left| \int_{t_0}^t [f(x_s^n, s) - f(x_s, s)] ds \right|^2 + E \left| \int_{t_0}^t [g(x_s^n, s) - g(x_s, s)] dW(s) \right|^2 \\ &\quad + E \left| \int_{t_0}^t [h(x_s^n, s) - h(x_s, s)] dN(s) \right|^2 \\ &\leq (t - t_0) E \int_{t_0}^t |f(x_s^n, s) - f(x_s, s)|^2 ds \\ &\quad + E \int_{t_0}^t |g(x_s^n, s) - g(x_s, s)|^2 ds \\ &\quad + (2\lambda + 2\lambda^2(t - t_0)) E \int_{t_0}^t |h(x_s^n, s) - h(x_s, s)|^2 ds \\ &\leq (t - t_0) \bar{K} E \int_{t_0}^t \|x_s^n - x_s\|^2 ds + \bar{K} E \int_{t_0}^t \|x_s^n - x_s\|^2 ds \\ &\quad + (2\lambda + 2\lambda^2(t - t_0)) \bar{K} E \int_{t_0}^t \|x_s^n - x_s\| ds \\ &\leq \bar{K} (T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0)) E \int_{t_0}^t \|x_s^n - x_s\| ds \\ &\leq \bar{K} (T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0)) \int_{t_0}^t E \sup_{t_0 \leq r \leq s} |x^n(r) - x(r)|^2 ds \\ &\leq \bar{K} (T - t_0 + 1 + 2\lambda + 2\lambda^2(T - t_0)) \int_{t_0}^T E |x^n(s) - x(s)|^2 ds \longrightarrow 0 \end{aligned} \quad (n \longrightarrow \infty). \quad (3.58)$$

Hence, we can let $n \rightarrow \infty$ in (3.34) to obtain that

$$\begin{aligned} x(t) - G(x_t) &= \xi(0) - G(\xi) + \int_{t_0}^t f(x_s, s) ds + \int_{t_0}^t g(x_s, s) dW(s) \\ &\quad + \int_{t_0}^t h(x_s, s) dN(s). \end{aligned} \tag{3.59}$$

So $x(t)$ is the solution to (2.1).

Step 2. We need to remove the additional condition (3.33). Let $\sigma > 0$ be sufficiently small for

$$\delta := \kappa + \frac{3\bar{K}(\sigma + 4 + 8\lambda + 2\lambda^2\sigma)}{1 - \kappa} \sigma < 1. \tag{3.60}$$

By Step 1, there is a solution to (2.1) on $[t_0 - \tau, t_0 + \sigma]$. Now consider (2.1) on $[t_0 + \sigma, t_0 + 2\sigma]$ with initial data $x_{t_0 + \sigma}$. By Step 1 again, there is a solution to (2.1) on $[t_0 + \sigma, t_0 + 2\sigma]$. Repeating this procedure we see that there is a solution to (2.1) on the entire interval $[t_0 - \tau, T]$. The proof is complete. \square

For NSFDEwPJs, we know that the global Lipschitz condition imposed on Theorem 3.2 is a big restriction; now we will replace the global Lipschitz condition by the local Lipschitz condition. Then Theorem 3.6 follows.

Theorem 3.6. *Assume that there exist two positive constants K_n and K such that*

- (i) *(the local Lipschitz condition) for all $\phi, \varphi \in C([- \tau, 0]; R^n)$ with $\|\phi\| \vee \|\varphi\| \leq n$ and $t \in [t_0, T]$,*

$$\begin{aligned} &|f(\phi, t) - f(\varphi, t)|^2 \vee |g(\phi, t) - g(\varphi, t)|^2 \vee |h(\phi, t) - h(\varphi, t)|^2 \\ &\leq K_n \|\phi - \varphi\|^2. \end{aligned} \tag{3.61}$$

- (ii) *(the linear growth condition) for all $(\phi, t) \in C([- \tau, 0]; R^n) \times [t_0, T]$,*

$$|f(\phi, t)|^2 \vee |g(\phi, t)|^2 \vee |h(\phi, t)|^2 \leq K(1 + \|\phi\|^2). \tag{3.62}$$

- (iii) *(the contractive mapping) there is a positive constant $\kappa \in (0, 1)$ such that, for all $\phi, \varphi \in C([- \tau, 0]; R^n)$,*

$$|G(\phi) - G(\varphi)| \leq \kappa \|\phi - \varphi\|. \tag{3.63}$$

Then there exists a unique solution $x(t)$ to (2.1) with initial data (2.3). Moreover, the solution belongs to $\mathcal{M}^2([t_0 - \tau, T]; R^n)$.

Proof. For each $n \geq 1$, define truncation functions f_n , g_n , and h_n as follows:

$$\begin{aligned} f_n(x_t, t) &= \begin{cases} f(x_t, t) & \|x_t\| \leq n, \\ f\left(\frac{nx_t}{\|x_t\|}, t\right) & \|x_t\| > n, \end{cases} \\ g_n(x_t, t) &= \begin{cases} g(x_t, t) & \|x_t\| \leq n, \\ g\left(\frac{nx_t}{\|x_t\|}, t\right) & \|x_t\| > n, \end{cases} \\ h_n(x_t, t) &= \begin{cases} h(x_t, t) & \|x_t\| \leq n, \\ h\left(\frac{nx_t}{\|x_t\|}, t\right) & \|x_t\| > n, \end{cases} \end{aligned} \quad (3.64)$$

Then f_n , g_n , and h_n satisfy conditions (3.2) and (3.3). By Theorem 3.2, there is a unique solution $x^n(t)$ and $x^n(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$ to the equation

$$\begin{aligned} x^n(t) - G(x_t^n) &= \xi(0) - G(\xi) + \int_{t_0}^t f_n(x_s^n, s) ds \\ &+ \int_{t_0}^t g_n(x_s^n, s) dW(s) \\ &+ \int_{t_0}^t h_n(x_s^n, s) dN(s), \quad t \in [t_0, T]. \end{aligned} \quad (3.65)$$

Certainly $x^{n+1}(t)$ is a unique solution to the equation

$$\begin{aligned} x^{n+1}(t) - G(x_t^{n+1}) &= \xi(0) - G(\xi) + \int_{t_0}^t f_{n+1}(x_s^{n+1}, s) ds \\ &+ \int_{t_0}^t g_{n+1}(x_s^{n+1}, s) dW(s) \\ &+ \int_{t_0}^t h_{n+1}(x_s^{n+1}, s) dN(s), \quad t \in [t_0, T] \end{aligned} \quad (3.66)$$

and $x^{n+1}(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$ to the equation.

Now define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : \|x_t^n\| \geq n\}. \quad (3.67)$$

We can show that $x^n(t) = x^{n+1}(t)$ for $t \in [t_0, T]$. Then by Lemma 3.4, basic inequality, we have

$$\begin{aligned}
 |x^{n+1}(t) - x^n(t)|^2 &\leq \frac{1}{\kappa} \left| G(x_t^{n+1}) - G(x_t^n) \right|^2 \\
 &\quad + \frac{3}{1-\kappa} \left| \int_{t_0}^t [f_{n+1}(x_s^{n+1}, s) - f_n(x_s^n, s)] ds \right|^2 \\
 &\quad + \frac{3}{1-\kappa} \left| \int_{t_0}^t [g_{n+1}(x_s^{n+1}, s) - g_n(x_s^n, s)] dW(s) \right|^2 \\
 &\quad + \frac{3}{1-\kappa} \left| \int_{t_0}^t [h_{n+1}(x_s^{n+1}, s) - h_n(x_s^n, s)] dN(s) \right|^2.
 \end{aligned} \tag{3.68}$$

Taking the expectation, by the Hölder inequality and Lemma 3.3, we transform the Poisson process to the compensated Poisson process $\widetilde{N}(t) = N(t) - \lambda t$, and, by martingale isometry, we have

$$\begin{aligned}
 E|x^{n+1}(t) - x^n(t)|^2 &\leq \kappa E\|x_t^{n+1} - x_t^n\|^2 + \frac{3}{1-\kappa}(t-t_0)E \int_{t_0}^t |f_{n+1}(x_s^{n+1}, s) - f_n(x_s^n, s)|^2 ds \\
 &\quad + \frac{3}{1-\kappa} E \int_{t_0}^t |g_{n+1}(x_s^{n+1}, s) - g_n(x_s^n, s)|^2 ds \\
 &\quad + \frac{3}{1-\kappa} (2\lambda + 2\lambda^2(t-t_0)) E \int_{t_0}^t |h_{n+1}(x_s^{n+1}, s) - h_n(x_s^n, s)|^2 ds \\
 &\leq \kappa E\|x_t^{n+1} - x_t^n\|^2 + \frac{6}{1-\kappa}(t-t_0)E \int_{t_0}^t |f_{n+1}(x_s^{n+1}, s) - f_{n+1}(x_s^n, s)|^2 ds \\
 &\quad + \frac{6}{1-\kappa}(t-t_0)E \int_{t_0}^t |f_{n+1}(x_s^n, s) - f_n(x_s^n, s)|^2 ds \\
 &\quad + \frac{6}{1-\kappa} E \int_{t_0}^t |g_{n+1}(x_s^{n+1}, s) - g_{n+1}(x_s^n, s)|^2 ds \\
 &\quad + \frac{6}{1-\kappa} E \int_{t_0}^t |g_{n+1}(x_s^n, s) - g_n(x_s^n, s)|^2 ds \\
 &\quad + \frac{6}{1-\kappa} (2\lambda + 2\lambda^2(t-t_0)) E \int_{t_0}^t |h_{n+1}(x_s^{n+1}, s) - h_{n+1}(x_s^n, s)|^2 ds \\
 &\quad + \frac{6}{1-\kappa} (2\lambda + 2\lambda^2(t-t_0)) E \int_{t_0}^t |h_{n+1}(x_s^n, s) - h_n(x_s^n, s)|^2 ds.
 \end{aligned} \tag{3.69}$$

For $t_0 < t < t_n$, we have know that

$$\begin{aligned} f_{n+1}(x_s^n, s) &= f_n(x_s^n, s) = f(x_s^n, s), \\ g_{n+1}(x_s^n, s) &= g_n(x_s^n, s) = g(x_s^n, s), \\ h_{n+1}(x_s^n, s) &= h_n(x_s^n, s) = h(x_s^n, s). \end{aligned} \quad (3.70)$$

Again by $x_{t_0}^{n+1} = x_{t_0}^n = \xi$, we get that

$$\begin{aligned} & E \left| x^{n+1}(s) - x^n(s) \right|^2 \\ & \leq \kappa E \left\| x_t^{n+1} - x_t^n \right\|^2 + \frac{6}{1-\kappa} (t-t_0) E \int_{t_0}^t \left| f_{n+1}(x_s^{n+1}, s) - f_{n+1}(x_s^n, s) \right|^2 ds \\ & \quad + \frac{6}{1-\kappa} E \int_{t_0}^t \left| g_{n+1}(x_s^{n+1}, s) - g_{n+1}(x_s^n, s) \right|^2 ds \\ & \quad + \frac{6}{1-\kappa} (2\lambda + 2\lambda^2(t-t_0)) E \int_{t_0}^t \left| h_{n+1}(x_s^{n+1}, s) - h_{n+1}(x_s^n, s) \right|^2 ds, \\ & = \kappa E \sup_{t_0-\tau \leq s \leq t} \left| x^{n+1}(s) - x^n(s) \right|^2 + \frac{6}{1-\kappa} (t-t_0) E \int_{t_0}^t \left| f(x_s^{n+1}, s) - f(x_s^n, s) \right|^2 ds \\ & \quad + \frac{6}{1-\kappa} E \int_{t_0}^t \left| g(x_s^{n+1}, s) - g(x_s^n, s) \right|^2 ds \\ & \quad + \frac{6}{1-\kappa} (2\lambda + 2\lambda^2(t-t_0)) E \int_{t_0}^t \left| h(x_s^{n+1}, s) - h(x_s^n, s) \right|^2 ds. \end{aligned} \quad (3.71)$$

Noting the fact that $x^{n+1}(s) = x^n(s) = \xi$, for $s \in [t_0 - \tau, t_0]$, we get

$$\begin{aligned} & E \sup_{t_0 \leq s \leq t} \left| x^{n+1}(s) - x^n(s) \right|^2 \\ & \leq \frac{6[(t-t_0) + 1 + 2\lambda + 2\lambda^2(t-t_0)]}{(1-\kappa)^2} E \int_{t_0}^t K_n \left\| x_s^{n+1} - x_s^n \right\|^2 ds \\ & \leq \frac{6[(t-t_0) + 1 + 2\lambda + 2\lambda^2(t-t_0)]}{(1-\kappa)^2} K_n E \int_{t_0}^t \sup_{t_0 \leq s \leq t} \left| x^{n+1}(s) - x^n(s) \right|^2 ds. \end{aligned} \quad (3.72)$$

From the Gronwall inequality one sees that

$$E \sup_{t_0 \leq s \leq t} \left| x^{n+1}(s) - x^n(s) \right|^2, \quad t_0 < t < \tau_n. \quad (3.73)$$

This means that, for $t_0 < t < \tau_n$, we always have

$$x^n(t) = x^{n+1}(t). \tag{3.74}$$

It is then deduced that τ_n is increasing, that is as $n \rightarrow \infty$, $\tau_n \uparrow T$ a.s. By the linear growth condition, for all almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$ such that $\tau_n = T$ as $n > n_0$. Now define $x(t)$ by $x(t) = x_{n_0}(t)$, $t \in [t_0, T]$. Next to verify that $x(t)$ is the solution to (2.1), by (3.74) $x(t \wedge \tau_n) = x_n(t \wedge \tau_n)$, it follows that

$$\begin{aligned} x(t \wedge \tau_n) - G(x_{t \wedge \tau_n}^n) &= \xi(0) - G(\xi) + \int_{t_0}^{t \wedge \tau_n} f_n(x_s, s) ds \\ &+ \int_{t_0}^{t \wedge \tau_n} g_n(x_s, s) dW(s) + \int_{t_0}^{t \wedge \tau_n} h_n(x_s, s) dN(s), \end{aligned} \tag{3.75}$$

that is,

$$\begin{aligned} x(t \wedge \tau_n) - G(x_{t \wedge \tau_n}^n) &= \xi(0) - G(\xi) + \int_{t_0}^{t \wedge \tau_n} f(x_s, s) ds \\ &+ \int_{t_0}^{t \wedge \tau_n} g(x_s, s) dW(s) + \int_{t_0}^{t \wedge \tau_n} h(x_s, s) dN(s). \end{aligned} \tag{3.76}$$

Letting $n \rightarrow \infty$

$$\begin{aligned} x(t \wedge T) - G(x_t^t \wedge T) &= \xi(0) - G(\xi) + \int_{t_0}^{t \wedge T} f(x_s, s) ds \\ &+ \int_{t_0}^{t \wedge T} g(x_s, s) dW(s) + \int_{t_0}^{t \wedge T} h(x_s, s) dN(s), \end{aligned} \tag{3.77}$$

that is

$$\begin{aligned} x(t) - G(x_t) &= \xi(0) - G(\xi) + \int_{t_0}^t f(x_s, s) ds \\ &+ \int_{t_0}^t g(x_s, s) dW(s) + \int_{t_0}^t h(x_s, s) dN(s). \end{aligned} \tag{3.78}$$

We can see that $x(t)$ is the solution to (2.1) and $x(t) \in \mathcal{M}^2([t_0 - \tau, T]; R^n)$. The proof of existence is complete. By stopping our process, uniqueness is obtained. This completes the proof. \square

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