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Research Article

Existence of Solutions for a Quasilinear Reaction Diffusion System

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The degenerate reaction diffusion system has been applied to a variety of physical and engineering problems. This paper is extended the existence of solutions from the quasimonotone reaction functions (e.g., inhibitor-inhibitor mechanism) to the mixed quasimonotone reaction functions (e.g., activator-inhibitor mechanism). By Schauder fixed point theorem, it is shown that the system admits at least one positive solution if there exist a coupled of upper and lower solutions. This result is applied to a Lotka-Volterra predator-prey model.

1. Introduction

We consider a quasilinear reaction diffusion system in a bounded domain under coupled nonlinear boundary conditions. The system of equations is given in the form

$$\begin{gathered}\frac{\partial u_i}{\partial t} - \nabla \cdot (a_i D_i(u_i)\nabla u_i) + \mathbf{b}_i \cdot (D_i(u_i)\nabla u_i) = f_i(t,x,\mathbf{u}) \quad (t>0, x\in\Omega),\\ D_i(u_i)\frac{\partial u_i}{\partial \nu} = g_i(t,x,\mathbf{u}) \quad (t>0, x\in\partial\Omega),\\ u_i(0,x) = \psi_i(x)(x\in\Omega), \quad i=1,\dots,N,\end{gathered} \tag{1.1}$$

where $\mathbf{u} \equiv (u_1,\dots,u_N)$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. It is assumed that the boundary $\partial\Omega$ is of class $C^{1+\alpha}$. It is also assumed that, for each $i = 1,\dots,N$, the functions $a_i \equiv a_i(t,x)$, $\mathbf{b}_i \equiv \mathbf{b}_i(t,x) \equiv (b_i^{(1)},\dots,b_i^{(n)})$, $f_i(t,x,\cdot)$ and $g_i(t,x,\cdot)$ are Hölder continuous in $[0,\infty)\times\overline{\Omega}$. The density-dependent diffusion coefficient $D_i(u_i)$ may have the property $D_i(0) = 0$, which means that the elliptic operators are degenerate.

The quasilinear reaction diffusion system has been investigated extensively in the literature [1–3]. Recently by use of upper and lower solutions and its associated monotone iterations, [4, 5] deal with the scalar equation and the system endowed with the nonlinear Neumann-Robin boundary conditions, respectively. The paper in [6] is concerned with the existence, uniqueness, and asymptotic behavior for the quasilinear parabolic systems with the Dirichlet boundary condition. However, the requirement of the reaction functions in [4–6] are monotone nondecreasing. This paper relaxed the condition to mixed quasimonotone reaction functions, which leads to the difficult point that the ordered upper and lower solutions do not exist. To overcome it, we construct the coupled upper and lower solutions.

The purpose of this paper is to study the existence for the system (1.1) by the Schauder fixed point theorem. The rest of this paper is organized as follows. In Section 2 we show the existence by the method of upper and lower solutions and the Schauder fixed point theorem. An application is given in Section 3 to the Lotka-Volterra predator-prey model. The paper ends with Section 4 for some discussions.

2. Existence of Solutions

To the simplicity, throughout this paper, we denote

$$Q = (0,T] \times \Omega, \qquad S = (0,T] \times \partial\Omega, \qquad \overline{Q} = [0,T] \times \overline{\Omega}, \tag{2.1}$$

and let $C^m(Q)$ and $C^\alpha(Q)$ be the respective space of m-times differentiable and Hölder continuous functions in Q, where Q represents a domain or a section between two functions. For vector functions with N-components we denote the above function space by $\mathcal{C}^m(Q)$ and $\mathcal{C}^\alpha(Q)$, respectively.

In this paper, we make the following hypothesis.

(H) For each $i = 1, \ldots, N$, the following conditions hold:

(i) $a_i(t,x)$, $b_i^{(l)}(t,x)(l = 1,\ldots,n)$ and $f_i(t,x,\cdot)$ are in $C^{\alpha/2,\alpha}(\overline{Q})$ with $a_i \ge a_i^* > 0$, $g_i(t,x,\cdot) \in C^{1+\alpha/2,2+\alpha}(\overline{Q})$;

(ii) $D_i(u_i) \in C^{1+\alpha/2,1+\alpha}(\Lambda_i)$ and $D_i(u_i) > 0$ for $u_i > 0$ and $D_i(0) \ge 0$;

(iii) $\mathbf{f}(\cdot,\mathbf{u}), \mathbf{g}(\cdot,\mathbf{u})$ are mixed quasimonotone $\mathcal{C}^1$-functions in Λ.

In the above hypothesis, Λ and Λ_i are the sectors between a pair of coupled upper and lower solutions given by (2.8) below. It is allowed that $D_i(0) = 0$ for some i and $D_i(0) > 0$ for a different i. Particularly, if $D_i(u)$ is a positive constant for all i then system (1.1) becomes the standard coupled system of semilinear parabolic equations. Recall that a vector function $\mathbf{f}(\cdot,\mathbf{u})$ is said to be mixed quasimonotone in Λ if for each $i = 1,\ldots,N$, there exist nonnegative integers a_i and b_i with $a_i + b_i = N - 1$ such that the function $f_i(\cdot,\mathbf{u}) \equiv f_i(\cdot,u_i,[\mathbf{u}]_{a_i},[\mathbf{u}]_{b_i})$ is nondecreasing with respect to all component $[\mathbf{u}]_{a_i}$ and is nonincreasing with respect to all component $[\mathbf{u}]_{b_i}$, where $\mathbf{u} \equiv (u_i,[\mathbf{u}]_{a_i},[\mathbf{u}]_{b_i}) \in \Lambda$. Similarly, $g_i(\cdot,\mathbf{u}) \equiv g_i(\cdot,u_i,[\mathbf{u}]_{c_i},[\mathbf{u}]_{d_i})$. Our approach to the existence problem is by the method of coupled upper and lower solutions which are defined as follows.

Definition 2.1. A pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1, \ldots, \tilde{u}_N), \hat{\mathbf{u}} = (\hat{u}_1, \ldots, \hat{u}_N) \in C(\overline{Q}) \cap C^{1,2}(Q)$ are called coupled upper and lower solutions of (1.1) if $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ and if

$$\begin{gathered}
\frac{\partial \tilde{u}_i}{\partial t} - \nabla \cdot (a_i D_i(\tilde{u}_i) \nabla \tilde{u}_i) + \mathbf{b_i} \cdot (D_i(\tilde{u}_i) \nabla \tilde{u}_i) \geq f_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q, \\
\frac{\partial \hat{u}_i}{\partial t} - \nabla \cdot (a_i D_i(\hat{u}_i) \nabla \hat{u}_i) + \mathbf{b_i} \cdot (D_i(\hat{u}_i) \nabla \hat{u}_i) \leq f_i\left(\cdot, \hat{u}_i, [\hat{\mathbf{u}}]_{a_i}, [\tilde{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q, \\
D_i(\tilde{u}_i) \frac{\partial \tilde{u}_i}{\partial \nu} \geq g_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{c_i}, [\hat{\mathbf{u}}]_{d_i}\right) \quad \text{on } S, \\
D_i(\hat{u}_i) \frac{\partial \hat{u}_i}{\partial \nu} \leq g_i\left(\cdot, \hat{u}_i, [\hat{\mathbf{u}}]_{c_i}, [\tilde{\mathbf{u}}]_{d_i}\right) \quad \text{on } S, \\
\tilde{u}_i(0,x) \geq \psi_i(x), \qquad \hat{u}_i(0,x) \leq \psi_i(x) \quad \text{in } \Omega,\ i = 1, \ldots, N.
\end{gathered} \tag{2.2}$$

Define

$$w_i = I_i(u_i) = \int_0^{u_i} D_i(s)\,ds \quad \text{for } u_i \geq 0,\ i = 1, \ldots, N, \tag{2.3}$$

it follows from the following Hypothesis (H) that

$$I_i'(u_i) = \frac{dI_i}{du_i} = D_i(u_i) > 0, \tag{2.4}$$

then the inverse $u_i \equiv q_i(w_i)$ exists and is an increasing function of $w_i > 0$. In view of

$$\frac{\partial w_i}{\partial t} = D_i(u_i) \frac{\partial u_i}{\partial t}, \qquad \nabla w_i = D_i(u_i) \nabla u_i, \qquad \frac{\partial w_i}{\partial \nu} = D_i(u_i) \frac{\partial u_i}{\partial \nu} \tag{2.5}$$

we may write (1.1) in the equivalent form

$$\begin{gathered}
(D_i(u_i))^{-1} \frac{\partial w_i}{\partial t} - \nabla \cdot (a_i \nabla w_i) + \mathbf{b_i} \cdot \nabla w_i = f_i\left(t, x, u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) \quad \text{in } Q, \\
\frac{\partial w_i}{\partial \nu} = g_i\left(t, x, u_i, [\mathbf{u}]_{c_i}, [\mathbf{u}]_{d_i}\right) \quad \text{on } S, \\
w_i(0,x) = \eta_i(x) \quad \text{in } \Omega, \\
u_i = q_i(w_i) \quad i = 1, \ldots, N \text{ in } \overline{\Omega},
\end{gathered} \tag{2.6}$$

where $\eta_i(x) = I_i(\psi_i(x))$. Thus the pair $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ and $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$, where $\tilde{w}_i = I_i(\tilde{u}_i)$ and $\hat{w}_i = I(\hat{u}_i)$, satisfy the inequalities

$$\begin{gathered}
(D_i(\tilde{u}_i))^{-1}\frac{\partial \tilde{w}_i}{\partial t} - \nabla\cdot(a_i\nabla\tilde{w}_i) + \mathbf{b_i}\cdot(\nabla\tilde{w}_i) \ge f_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q,\\
(D_i(\hat{u}_i))^{-1}\frac{\partial \hat{w}_i}{\partial t} - \nabla\cdot(a_i\nabla\hat{w}_i) + \mathbf{b_i}\cdot(\nabla\hat{w}_i) \le f_i\left(\cdot, \hat{u}_i, [\hat{\mathbf{u}}]_{a_i}, [\tilde{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q,\\
\frac{\partial \tilde{w}_i}{\partial \nu} \ge g_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{c_i}, [\hat{\mathbf{u}}]_{d_i}\right) \quad \text{on } S,\\
\frac{\partial \hat{w}_i}{\partial \nu} \le g_i\left(\cdot, \hat{u}_i, [\hat{\mathbf{u}}]_{c_i}, [\tilde{\mathbf{u}}]_{d_i}\right) \quad \text{on } S,\\
\tilde{w}_i(0,x) \ge \eta_i(x) \qquad \hat{w}_i(0,x) \le \eta_i(x) \quad \text{in } \Omega,\ i = 1,\ldots,N,
\end{gathered} \tag{2.7}$$

is referred to as coupled upper and lower solutions of (2.6). For a given pair of coupled upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ we set

$$\begin{gathered}
\Lambda_i = \left\{u_i \in C\left(\overline{Q}\right) : \hat{u}_i \le u \le \tilde{u}_i\right\}, \qquad \Lambda = \left\{\mathbf{u} \in C\left(\overline{Q}\right) : \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\right\},\\
\Lambda \times \overline{\Lambda} = \left\{(\mathbf{u}, \mathbf{w}) \in C\left(\overline{Q}\right) \times C\left(\overline{Q}\right) : (\hat{\mathbf{u}}, \hat{\mathbf{w}}) \le (\mathbf{u}, \mathbf{w}) \le (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})\right\}.
\end{gathered} \tag{2.8}$$

In Hypothesis (H)-(ii) we allow $D_i(0) = 0$ which leads to a degenerate diffusion coefficient. If $D_i(0) = 0$, we set $\hat{u}_i \ge \delta_i > 0$, which ensures that $D_i(u_i)$ has a positive lower bound. Since (H)-(iii), there exist smooth nonnegative functions $c_i^{(l)} \equiv c_i^{(l)}(t,x), l = 1, 2$, such that

$$c_i^{(1)} D_i(u_i) + \frac{\partial f_i}{\partial u_{\text{i}}}(\cdot, \mathbf{u}) \ge 0, \qquad c_i^{(2)} D_i(u_i) + \frac{\partial g_i}{\partial u_i}(\cdot, \mathbf{u}) \ge 0 \quad \text{for } \mathbf{u} \in \Lambda. \tag{2.9}$$

In fact, it suffices to choose any $\mathbf{c^{(1)}} \equiv (c_1^{(1)}, \ldots, c_N^{(1)})$, $\mathbf{c^{(2)}} \equiv (c_1^{(2)}, \ldots, c_N^{(2)})$ satisfying

$$\begin{gathered}
c_i^{(1)}(t,x) \ge \max\left\{-\frac{\partial f_i/\partial u_i(t,x,\mathbf{u})}{D_i(u_i)} : \mathbf{u} \in \Lambda\right\},\\
c_i^{(2)}(t,x) \ge \max\left\{-\frac{\partial g_i/\partial u_i(t,x,\mathbf{u})}{D_i(u_i)} : \mathbf{u} \in \Lambda\right\}.
\end{gathered} \tag{2.10}$$

Define for each $i = 1, \ldots, N$,

$$\begin{gathered}
F_i(t,x,\mathbf{u}) = c_i^{(1)}(t,x) I_i(u_i) + f_i(t,x,\mathbf{u}), \qquad G_i(t,x,\mathbf{u}) = c_i^{(2)}(t,x) I_i(u_i) + g_i(t,x,\mathbf{u}),\\
L_i w_i = \nabla\cdot(a_i\nabla w_i) - \mathbf{b_i}\nabla\cdot w_i - c_i^{(1)}(t,x) w_i, \qquad B_i w_i = \frac{\partial \mathrm{w}_i}{\partial \nu} + c_i^{(2)}(t,x) w_i.
\end{gathered} \tag{2.11}$$

Since (2.9), (H) and $I_i'(u_i) = D_i(u_i)$, $F_i(\cdot, \mathbf{u})$ and $G_i(\cdot, \mathbf{u})$ possess the property

$$\begin{gathered}
F_i\left(\cdot, v_i, [\mathbf{v}]_{a_i}, [\mathbf{u}]_{b_i}\right) \le F_i\left(\cdot, u_i, [\mathbf{u}]_{a_i}, [\mathbf{v}]_{b_i}\right)\\
G_i\left(\cdot, v_i, [\mathbf{v}]_{c_i}, [\mathbf{u}]_{d_i}\right) \le G_i\left(\cdot, u_i, [\mathbf{u}]_{c_i}, [\mathbf{v}]_{d_i}\right), \quad \text{whenever } \hat{\mathbf{u}} \le \mathbf{v} \le \mathbf{u} \le \tilde{\mathbf{u}}.
\end{gathered} \tag{2.12}$$

Moreover, (2.6) is equivalent to

$$\begin{gathered}(D_i(u_i))^{-1}\frac{\partial w_i}{\partial t} - L_i w_i = F_i\left(\cdot, u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right) \quad \text{in } Q,\\ B_i w_i = G_i\left(\cdot, u_i, [\mathbf{u}]_{c_i}, [\mathbf{u}]_{d_i}\right) \quad \text{on } S,\\ w_i(0,x) = \eta_i(x) \quad \text{in } \Omega,\\ u_i = q_i(w_i), \quad i = 1,\dots,N, \text{ in } \overline{\Omega}.\end{gathered} \tag{2.13}$$

Thus the pair $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ and $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$, where $\tilde{w}_i = I_i(\tilde{u}_i)$ and $\hat{w}_i = I_i(\hat{u}_i)$, satisfies the inequalities

$$\begin{gathered}(D_i(\tilde{u}_i))^{-1}\frac{\partial \tilde{w}_i}{\partial t} - L_i\tilde{w}_i \geq F_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q,\\ (D_i(\hat{u}_i))^{-1}\frac{\partial \hat{w}_i}{\partial t} - L_i\hat{w}_i \leq F_i\left(\cdot, \hat{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}\right) \quad \text{in } Q,\\ B_i\tilde{w}_i \geq G_i\left(\cdot, \tilde{u}_i, [\tilde{\mathbf{u}}]_{c_i}, [\hat{\mathbf{u}}]_{d_i}\right) \quad \text{on } S,\\ B_i\hat{w}_i \leq G_i\left(\cdot, \hat{u}_i, [\hat{\mathbf{u}}]_{c_i}, [\tilde{\mathbf{u}}]_{d_i}\right) \quad \text{on } S,\\ \tilde{w}_i(0,x) \geq \eta_i(x) \qquad \hat{w}_i(0,x) \leq \eta_i(x) \quad \text{in } \Omega,\end{gathered} \tag{2.14}$$

are referred to coupled upper and lower solutions of (2.13).

The property (2.12) is quite useful for the construction of monotone convergent sequences. To ensure the existence of the sequence to be constructed in the iteration process (2.16) below we assume that either $D_i(0) > 0$ or $D_i(0) = 0$ for $\hat{u}_i \geq \delta_i > 0$. Define a modified function $\overline{D}_i(u_i)$ by

$$\overline{D}_i(u_i) = \begin{cases} D_i(u_i) + (u_i - \tilde{u}_i) & \text{if } u_i > \tilde{u}_i,\\ D_i(u_i) & \text{if } \hat{u}_i \leq u_i \leq \tilde{u}_i,\\ D_i(u_i) + (\hat{u}_i - u_i) & \text{if } u_i < \hat{u}_i.\end{cases} \tag{2.15}$$

Then by the above assumption, there exists $d_0 > 0$ such that $\overline{D}_i(u) \geq d_0$ for all $u \in \mathbb{R}$.

By using $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ and $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ as the initial iteration we can construct sequences $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ and $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$ from the nonlinear iteration process

$$\begin{gathered}\left(\overline{D}_i\left(\overline{u}_i^{(m)}\right)\right)^{-1}\frac{\partial \overline{w}_i^{(m)}}{\partial t} - L_i\overline{w}_i^{(m)} = F_i\left(\cdot, \overline{u}_i^{(m-1)}, \left[\overline{\mathbf{u}}^{(m-1)}\right]_{a_i}, \left[\underline{\mathbf{u}}^{(m-1)}\right]_{b_i}\right) \quad \text{in } Q,\\ \left(\overline{D}_i\left(\underline{u}_i^{(m)}\right)\right)^{-1}\frac{\partial \underline{w}_i^{(m)}}{\partial t} - L_i\underline{w}_i^{(m)} = F_i\left(\cdot, \underline{u}_i^{(m-1)}, \left[\underline{\mathbf{u}}^{(m-1)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(m-1)}\right]_{b_i}\right) \quad \text{in } Q,\\ B_i\overline{w}_i^{(m)} = G_i\left(\cdot, \overline{u}_i^{(m-1)}, \left[\overline{\mathbf{u}}^{(m-1)}\right]_{c_i}, \left[\underline{\mathbf{u}}^{(m-1)}\right]_{d_i}\right) \quad \text{on } S,\\ B_i\underline{w}_i^{(m)} = G_i\left(\cdot, \underline{u}_i^{(m-1)}, \left[\underline{\mathbf{u}}^{(m-1)}\right]_{c_i}, \left[\overline{\mathbf{u}}^{(m-1)}\right]_{d_i}\right) \quad \text{on } S,\\ \overline{w}_i^{(m)}(0,x) = \eta_i(x), \qquad \underline{w}_i^{(m)}(0,x) = \eta_i(x) \quad \text{in } \Omega.\end{gathered} \tag{2.16}$$

The sequences $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ and $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$ are well defined by the existence theorem of [1]. The following lemma gives the monotone property of these sequences.

Lemma 2.2. *The sequences* $\{\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\}$ *governed by* (2.16) *possess the monotone property*

$$\begin{aligned}(\hat{\mathbf{u}}, \hat{\mathbf{w}}) &\leq \left(\underline{\mathbf{u}}^{(\mathbf{m})}, \underline{\mathbf{w}}^{(\mathbf{m})}\right) \leq \left(\underline{\mathbf{u}}^{(\mathbf{m+1})}, \underline{\mathbf{w}}^{(\mathbf{m+1})}\right) \leq \left(\overline{\mathbf{u}}^{(\mathbf{m+1})}, \overline{\mathbf{w}}^{(\mathbf{m+1})}\right) \\ &\leq \left(\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{w}}^{(\mathbf{m})}\right) \leq (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \quad \text{for } m = 1, 2, \ldots. \end{aligned} \tag{2.17}$$

Moreover, for each $m = 1, 2, \ldots,$ $\overline{\mathbf{u}}^{(\mathbf{m})}$ *and* $\underline{\mathbf{u}}^{(\mathbf{m})}$ *are coupled upper and lower solutions of* (1.1).

Proof. Let $\underline{z}_i^{(1)} = \underline{w}_i^{(1)} - \underline{w}_i^{(0)}$, $i = 1, \ldots, N$. Then by (2.14) and (2.16), $\underline{z}_i^{(1)}$ satisfies

$$\begin{aligned}\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial \underline{z}_i^{(1)}}{\partial t} - L_i\underline{z}_i^{(1)} &= F_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{b_i}\right) \\ &\quad - \left[\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial \underline{w}_i^{(0)}}{\partial t} - L_i\underline{w}_i^{(0)}\right] \\ &= F_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{b_i}\right) - \left[\left(\overline{D}_i\left(\underline{u}_i^{(0)}\right)\right)^{-1}\frac{\partial \underline{w}_i^{(0)}}{\partial t - L_i\underline{w}_i^{(0)}}\right] \\ &\quad - \left[\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1} - \left(\overline{D}_i\left(\underline{u}_i^{(0)}\right)\right)^{-1}\right]\frac{\partial \underline{w}_i^{(0)}}{\partial t} \\ &\geq -\left[\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1} - \left(\overline{D}_i\left(\underline{u}_i^{(0)}\right)\right)^{-1}\right]\frac{\partial \underline{w}_i^{(0)}}{\partial t}.\end{aligned} \tag{2.18}$$

Since by the mean value theorem,

$$\begin{aligned}\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1} - \left(\overline{D}_i\left(\underline{u}_i^{(0)}\right)\right)^{-1} &= -\left[\frac{\overline{D}_i'(\xi^{(0)})}{\left(\overline{D}_i(\xi^{(0)})\right)^2}\right]\left(\underline{u}_i^{(1)} - \underline{u}_i^{(0)}\right) \\ &= -\left[\frac{\overline{D}_i'(\xi^{(0)})}{\left(\overline{D}_i(\xi^{(0)})\right)^3}\right]\left(\underline{w}_i^{(1)} - \underline{w}_i^{(0)}\right),\end{aligned} \tag{2.19}$$

for some intermediate value $\xi^{(0)} \equiv \xi^{(0)}(t, x)$ between $\underline{u}_i^{(0)}$ and $\underline{u}_i^{(1)}$, we have

$$\left(\overline{D}_i\left(\underline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial \underline{z}_i^{(1)}}{\partial t} - L_i\underline{z}_i^{(1)} + \gamma^{(0)}\underline{z}_i^{(1)} \geq 0, \tag{2.20}$$

where

$$\gamma^{(0)} = -\left[\frac{\overline{D}'_i(\xi^{(0)})}{\left(\overline{D}_i(\xi^{(0)})\right)^3}\right]\frac{\partial \underline{w}_i^{(0)}}{\partial t}. \tag{2.21}$$

Since (2.14), the boundary and initial inequalities

$$\begin{gathered} B_i\underline{z}_i^{(1)} = G_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{c_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{d_i}\right) - B_i\widehat{w}_i \geq 0 \quad \text{on } S, \\ \underline{z}_i^{(1)}(0,x) = \eta_i(x) - \eta_i(x) = 0 \quad \text{in } \Omega. \end{gathered} \tag{2.22}$$

In view of the definition of $\overline{D}_i$ in (2.15), the function $\overline{D}_i(\underline{u}_i^{(1)})\gamma^{(0)}$ of (2.20) is bounded. From the weak maximum principle, it follows $\underline{z}_i^{(1)} \geq 0$ on $\overline{Q}$. This gives $\underline{w}_i^{(1)} \geq \underline{w}_i^{(0)}$ and thus $\underline{u}_i^{(1)} \geq \underline{u}_i^{(0)}$. A similar argument yields $\overline{w}_i^{(1)} \leq \overline{w}_i^{(0)}$ and $\overline{u}_i^{(1)} \leq \overline{u}_i^{(0)}$.

Moreover, letting $z_i^{(1)} = \overline{w}_i^{(1)} - \underline{w}_i^{(1)}$, by (2.12), (2.16), and after the similar above argument

$$\begin{gathered} \left(\overline{D}_i\left(\overline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial z_i^{(1)}}{\partial t} - L_i z_i^{(1)} + \gamma_i^{(0)} z_i^{(1)} = F_i\left(\cdot, \overline{u}_i^{(0)}, \left[\overline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\underline{\mathbf{u}}^{(0)}\right]_{b_i}\right) \\ -F_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{b_i}\right) \geq 0 \quad \text{in } Q, \\ B_i z_i^{(1)} = G_i\left(\cdot, \overline{u}_i^{(0)}, \left[\overline{\mathbf{u}}^{(0)}\right]_{c_i}, \left[\underline{\mathbf{u}}^{(0)}\right]_{d_i}\right) - G_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{c_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{d_i}\right) \geq 0 \quad \text{on } S, \\ z_i^{(1)}(0,x) = \eta_i(x) - \eta_i(x) = 0 \quad \text{in } \Omega, \end{gathered} \tag{2.23}$$

where

$$\gamma_i^{(0)} = -\left[\frac{\overline{D}'_i\left(\xi_i^{(0)}\right)}{\left(\overline{D}_i\left(\xi_i^{(0)}\right)\right)^3}\right]\frac{\partial \underline{w}_i^{(0)}}{\partial t}, \tag{2.24}$$

for some intermediate value $\xi_i^{(0)} \equiv \xi_i^{(0)}(t,x)$ between $\underline{u}_i^{(0)}$ and $\overline{u}_i^{(1)}$. It follows again from the weak maximum principle that $\overline{\mathbf{w}}^{(1)} \geq \underline{\mathbf{w}}^{(1)}$ and thus $\overline{\mathbf{u}}^{(1)} \geq \underline{\mathbf{u}}^{(1)}$. The above conclusions show that

$$\left(\underline{\mathbf{u}}^{(0)}, \underline{\mathbf{w}}^{(0)}\right) \leq \left(\underline{\mathbf{u}}^{(1)}, \underline{\mathbf{w}}^{(1)}\right) \leq \left(\overline{\mathbf{u}}^{(1)}, \overline{\mathbf{w}}^{(1)}\right) \leq \left(\overline{\mathbf{u}}^{(0)}, \overline{\mathbf{w}}^{(0)}\right). \tag{2.25}$$

Now we show that $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ are coupled upper and lower solutions of (1.1). Since (2.25), $\overline{D}_i(\overline{u}_i^{(1)}) = D_i(\overline{u}_i^{(1)})$ for $i = 1, \ldots, N$. It suffices to show that $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ satisfy (2.14). Since (2.12) and (2.16), we have

$$\begin{gathered}
\left(D_i\left(\overline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial \overline{w}_i^{(1)}}{\partial t} - L_i\overline{w}_i^{(1)} = F_i\left(\cdot, \overline{u}_i^{(0)}, \left[\overline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\underline{\mathbf{u}}^{(0)}\right]_{b_i}\right) \geq F_i\left(\cdot, \overline{u}_i^{(1)}, \left[\overline{\mathbf{u}}^{(1)}\right]_{a_i}, \left[\underline{\mathbf{u}}^{(1)}\right]_{b_i}\right), \\
\left(D_i\left(\underline{u}_i^{(1)}\right)\right)^{-1}\frac{\partial \underline{w}_i^{(1)}}{\partial t} - L_i\underline{w}_i^{(1)} = F_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{b_i}\right) \leq F_i\left(\cdot, \underline{u}_i^{(1)}, \left[\underline{\mathbf{u}}^{(1)}\right]_{a_i}, \left[\overline{\mathbf{u}}^{(1)}\right]_{b_i}\right), \\
B_i\overline{u}_i^{(1)} = G_i\left(\cdot, \overline{u}_i^{(0)}, \left[\overline{\mathbf{u}}^{(0)}\right]_{c_i}, \left[\underline{\mathbf{u}}^{(0)}\right]_{d_i}\right) \geq G_i\left(\cdot, \overline{u}_i^{(1)}, \left[\overline{\mathbf{u}}^{(1)}\right]_{c_i}, \left[\underline{\mathbf{u}}^{(1)}\right]_{d_i}\right), \\
B_i\underline{u}_i^{(1)} = G_i\left(\cdot, \underline{u}_i^{(0)}, \left[\underline{\mathbf{u}}^{(0)}\right]_{c_i}, \left[\overline{\mathbf{u}}^{(0)}\right]_{d_i}\right) \leq G_i\left(\cdot, \underline{u}_i^{(1)}, \left[\underline{\mathbf{u}}^{(1)}\right]_{c_i}, \left[\overline{\mathbf{u}}^{(1)}\right]_{d_i}\right), \\
\overline{u}_i^{(1)}(0,x) = \eta_i(x) \qquad \underline{u}_i^{(1)}(0,x) = \eta_i(x).
\end{gathered} \tag{2.26}$$

Next we use an induction method. We assume that $\overline{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ are coupled upper and lower solutions of (1.1) and satisfying the following relation:

$$(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \leq \left(\underline{\mathbf{u}}^{(\mathbf{m})}, \underline{\mathbf{w}}^{(\mathbf{m})}\right) \leq \left(\overline{\mathbf{u}}^{(\mathbf{m})}, \overline{\mathbf{w}}^{(\mathbf{m})}\right) \leq (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}). \tag{2.27}$$

Then by choosing $\overline{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ as the coupled upper and lower solutions $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, after the similar above argument, we have

$$\left(\underline{\mathbf{u}}^{(m)}, \underline{\mathbf{w}}^{(m)}\right) \leq \left(\underline{\mathbf{u}}^{(m+1)}, \underline{\mathbf{w}}^{(m+1)}\right) \leq \left(\overline{\mathbf{u}}^{(m+1)}, \overline{\mathbf{w}}^{(m+1)}\right) \leq \left(\overline{\mathbf{u}}^{(m)}, \overline{\mathbf{w}}^{(m)}\right), \tag{2.28}$$

$\overline{\mathbf{u}}^{(m+1)}$ and $\underline{\mathbf{u}}^{(m+1)}$ are coupled upper and lower solutions of (1.1). The conclusion of the lemma follows from the induction principle. □

Theorem 2.3. *Let $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ be a pair of coupled upper and lower solutions of* (1.1), *and let hypothesis* (H) *hold. Assume that either $D_i(0) > 0$ for some i or $\hat{u}_i \geq \delta_i > 0$. Then the problem* (1.1) *has at least one solution $\mathbf{u} \in \Lambda$.*

Proof. We first consider the problem (2.13), where D_i is replaced by $\overline{D}_i$. For each $i = 1, \ldots, N$, we define operators $\mathcal{L}_i : \mathfrak{D}_i \times \overline{\mathfrak{D}}_i \to \mathcal{R}_i$ and $\mathcal{L} : \mathfrak{D} \times \overline{\mathfrak{D}} \to \mathcal{R}$ by

$$\begin{gathered}
\mathcal{L}_i(u_i, w_i) = \left(\overline{D}_i(u_i)\right)^{-1}\frac{\partial w_i}{\partial t} - L_i w_i \quad (i = 1, \ldots, N), \\
\mathcal{L}(\mathbf{u}, \mathbf{w}) = (\mathcal{L}_1(u_1, w_1), \ldots, \mathcal{L}_N(u_N, w_N)),
\end{gathered} \tag{2.29}$$

where

$$\begin{aligned}
&\mathfrak{D}_i = \left\{u_i \in C^{1+\alpha/2, 2+\alpha}(Q);\ u_i(0,x) = \psi_i(x) \quad \text{in } \Omega\right\}, \\
&\mathcal{R}_i = \left\{u_i \in C^{\alpha/2, \alpha}(Q)\right\}, \\
&\mathfrak{D} = \mathfrak{D}_1 \times \ldots \times \mathfrak{D}_N, \qquad \mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_N.
\end{aligned} \tag{2.30}$$

Define also

$$\mathcal{F}(\mathbf{u}) = \left(F_1\left(\cdot, u_1, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right), \ldots, F_N\left(\cdot, u_N, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right)\right). \tag{2.31}$$

Then the system (2.13), in which D_i is replaced by $\overline{D}_i$, may be written in the form

$$\mathcal{L}(\mathbf{u}, \mathbf{w}) = \mathcal{F}(\mathbf{u}), \qquad \mathbf{B}(\mathbf{w}) = G(\mathbf{u}), \quad \left((\mathbf{u}, \mathbf{w}) \in \mathfrak{D} \times \overline{\mathfrak{D}}\right), \tag{2.32}$$

where $\mathbf{B} = (B_1, \ldots, B_N)$ and $\mathbf{G} = (G_1, \ldots, G_N)$ are given in (2.11). Given any $\mathbf{v} \in \Lambda$ and any $i = 1, \ldots, N$, we consider the scalar problem

$$\mathcal{L}_i(u_i, w_i) = \mathcal{F}_i(\mathbf{v}) \quad \text{in } Q, \qquad B_i w_i = G_i(\mathbf{v}) \quad \text{on } S, \qquad u_i(0, x) = \psi_i(x) \quad \text{in } \Omega. \tag{2.33}$$

It follows from the existence theorem of [1] (Chapter V, Section 7) that (2.33) has a unique solution $(u_i^*, w_i^*) \in \mathfrak{D}_i \times \overline{\mathfrak{D}}_i$. In fact, the inverse $\mathcal{L}_i^{-1} : \mathcal{R}_i \to \mathfrak{D}_i \times \overline{\mathfrak{D}}_i$ exists and is a positive compact operator on $\mathcal{R}_i$. This implies that the equation

$$\mathcal{L}(\mathbf{u}, \mathbf{w}) = \mathcal{F}(\mathbf{v}) \qquad \mathbf{B}(\mathbf{w}) = G(\mathbf{v}), \tag{2.34}$$

has a unique solution $(\mathbf{u}, \mathbf{w}) = \mathcal{L}^{-1}[\mathcal{F}(\mathbf{v})], \mathbf{w} = \mathbf{B}^{-1}G(\mathbf{v})$. Let χ be the closed bounded convex sunset given by

$$\chi = \left\{(\mathbf{u}, \mathbf{w}) \in \mathcal{R} \times \overline{\mathcal{R}} : \hat{\mathbf{u}} \leq \mathbf{u} \leq \tilde{\mathbf{u}}, \hat{\mathbf{w}} \leq \mathbf{w} \leq \tilde{\mathbf{w}}\right\}. \tag{2.35}$$

By the compact property on $\mathcal{L}^{-1}$ and the hypothesis on $\mathbf{f}$ the operator $\mathcal{L}^{-1}\mathcal{F}$ is compact on χ. We show that $\mathcal{L}^{-1}\mathcal{F}$ maps χ to itself.

Let $(\mathbf{v}, \mathbf{z}) \in \chi$ be given, and $(\mathbf{u}, \mathbf{w}) = \mathcal{L}^{-1}[\mathcal{F}(\mathbf{v})]$. After the similar argument of the proof of Lemma 2.2, we conclude $\hat{\mathbf{u}} \leq \mathbf{u} \leq \tilde{\mathbf{u}}$, therefore $\mathcal{L}^{-1}\mathcal{F}$ maps χ to itself. It follows from the Schauder fixed point theorem that (2.13) with D_i being replaced by $\overline{D}_i$ has at least one solution $\mathbf{u} \in \chi$. Since $\hat{\mathbf{u}} \leq \mathbf{u} \leq \tilde{\mathbf{u}}$, it follows from (2.15) that $\overline{D}_i(u_i) = D_i(u_i)$ for $i = 1, \ldots, N$. Thus $\mathbf{u}$ is also the solution of (2.13). Therefore the existence of the solution to (1.1) is proved. □

3. Applications

As an application of the results obtained in the previous section we consider a Lotka-Volterra predator model. This model involves two species u_1 and u_2 that are governed by the system

$$
\begin{gathered}
\frac{\partial u_1}{\partial t} - \nabla \cdot (D_1(u_1)\nabla u_1) = u_1(a_1 - b_{11}u_1 - b_{12}u_2) \quad (t > 0,\ x \in \Omega) \\
\frac{\partial u_2}{\partial t} - \nabla \cdot (D_2(u_2)\nabla u_2) = u_2(a_2 + b_{21}u_1 - b_{22}u_2) \qquad (t > 0,\ x \in \Omega), \\
\frac{\partial u_1}{\partial \nu} + \beta_1 u_1 = 0, \qquad \frac{\partial u_2}{\partial \nu} + \beta_2 u_2 = 0 \quad (t > 0,\ x \in \partial\Omega), \\
u_i(0, x) = \psi_i(x) \qquad i = 1, 2, (x \in \Omega),
\end{gathered}
\tag{3.1}
$$

where a_i, b_{ij} are positive constants $\beta_i \equiv \beta_i(x) \geq 0 \in \partial\Omega$, the initial functions $\psi_i(x)$ for $i = 1, 2$ have a positive lower bound. The density-dependent diffusion coefficients $D_1(0) = D_2(0) = 0$.

It is easy to verify that if $(\tilde{u}_1, \tilde{u}_2)$ and $(\hat{u}_1, \hat{u}_2)$ satisfy $(\tilde{u}_1, \tilde{u}_2) \geq (\hat{u}_1, \hat{u}_2)$ and the following inequalities:

$$
\begin{gathered}
\frac{\partial \tilde{u}_1}{\partial t} - \nabla \cdot (D_1(\tilde{u}_1)\nabla \tilde{u}_1) \geq \tilde{u}_1(a_1 - b_{11}\tilde{u}_1 - b_{12}\hat{u}_2), \\
\frac{\partial \tilde{u}_2}{\partial t} - \nabla \cdot (D_2(\tilde{u}_2)\nabla \tilde{u}_2) \geq \tilde{u}_2(a_2 + b_{21}\tilde{u}_1 - b_{22}\tilde{u}_2), \\
\frac{\partial \hat{u}_1}{\partial t} - \nabla \cdot (D_1(\hat{u}_1)\nabla \hat{u}_1) \leq \hat{u}_1(a_1 - b_{11}\hat{u}_1 - b_{12}\tilde{u}_2), \\
\frac{\partial \hat{u}_2}{\partial t} - \nabla \cdot (D_2(\hat{u}_2)\nabla \hat{u}_2) \leq \hat{u}_2(a_2 + b_{21}\hat{u}_1 - b_{22}\hat{u}_2), \\
\frac{\partial \tilde{u}_1}{\partial \nu} + \beta_1 \tilde{u}_1 \geq 0, \qquad \frac{\partial \tilde{u}_2}{\partial \nu} + \beta_2 \tilde{u}_2 \geq 0, \\
\frac{\partial \hat{u}_1}{\partial \nu} + \beta_1 \hat{u}_1 \leq 0, \qquad \frac{\partial \hat{u}_2}{\partial \nu} + \beta_2 \hat{u}_2 \leq 0, \\
\hat{u}_i(0, x) \leq \psi_i(x) \leq \tilde{u}_i(0, x), \qquad i = 1, 2,
\end{gathered}
\tag{3.2}
$$

then the pair $(\tilde{u}_1, \tilde{u}_2)$, $(\hat{u}_1, \hat{u}_2)$ are coupled upper and lower solutions of (3.1).

To guarantee (3.2), we seek such a pair in the form

$$
(\tilde{u}_1, \tilde{u}_2) = (M_1, M_2), \qquad (\hat{u}_1, \hat{u}_2) = \left(q_1(\delta_1 \phi_1), q_2(\delta_2 \phi_2)\right),
\tag{3.3}
$$

where for each $i = 1, 2$, M_i and δ_i are positive constants to be chosen, q_i is the inverse of (2.3), and ϕ_i is the (normalized) positive eigenfunction corresponding to the smallest eigenvalue of the eigenvalue problem

$$
\nabla^2 \phi_i + \lambda_i \phi = 0 \quad \text{in } \Omega, \qquad \frac{\partial \phi_i}{\partial \nu} + \gamma_i \phi_i = 0 \quad \text{on } \partial\Omega \ (i = 1, 2).
\tag{3.4}
$$

The constant $\gamma_i > 0$ will be determined in the following discussion. If we set

$$M_1 = \frac{a_1}{b_{11}} \qquad M_2 = \frac{(a_2 + b_{21} M_1)}{b_{22}}, \tag{3.5}$$

then the first and second inequalities of (3.2) are satisfied. The third and fourth inequalities become

$$\begin{aligned} -\nabla^2(\delta_1\phi_1) &\leq q_1(\delta_1\phi_1)(a_1 - b_{11}q_1(\delta_1\phi_1) - b_{12}M_2), \\ -\nabla^2(\delta_2\phi_2) &\leq q_2(\delta_2\phi_2)(a_2 + b_{21}q_1(\delta_1\phi_1) - b_{22}q_2(\delta_2\phi_2)). \end{aligned} \tag{3.6}$$

By (3.4) and $q_i(\delta_i\phi_i) > 0$, the above inequalities are satisfied by some sufficiently small $\delta_i > 0$ if

$$\begin{aligned} \lambda_1 &< \left(\frac{q_1(\delta_1\phi_1)}{\delta_1\phi_1}\right)(a_1 - b_{12}M_2)\ , \\ \lambda_2 &< \left(\frac{q_2(\delta_2\phi_2)}{\delta_2\phi_2}\right)a_2. \end{aligned} \tag{3.7}$$

Since $D_i(0) = 0$, by L'Hopital's rule,

$$\lim_{w\to 0^+}\left[\frac{q_i(w)}{w}\right] = \lim_{w\to 0^+} q_i'(w) = \lim_{z\to 0^+}\frac{1}{D_i(z)} = \infty, \tag{3.8}$$

we see that there exists $\delta_i^* > 0$ such that the inequalities in (3.7) are satisfied by every $\delta_i \leq \delta_i^*$ if we impose the condition

$$M_2 < \frac{a_1}{b_{12}}. \tag{3.9}$$

By (3.3), the fifth inequalities of (3.2) are trivially satisfied, and the sixth inequalities of (3.2) become

$$\begin{aligned} D_1(q_1(\delta_1\phi_1))\partial q_1\frac{(\delta_1\phi_1)}{\partial \nu} + \beta_1 D_1(q_1(\delta_1\phi_1))q_1(\delta_1\phi_1) \leq 0, \\ D_2(q_2(\delta_2\phi_2))\partial q_2\frac{(\delta_2\phi_2)}{\partial \nu} + \beta_2 D_2(q_2(\delta_2\phi_2))q_2(\delta_2\phi_2) \leq 0, \end{aligned} \tag{3.10}$$

Substituting (3.4) into (3.10) yields

$$\begin{aligned} -\gamma_1(\delta_1\phi_1) &\leq -\beta_1 D_1(q_1(\delta_1\phi_1))q_1(\delta_1\phi_1), \\ -\gamma_2(\delta_2\phi_2) &\leq -\beta_2 D_2(q_2(\delta_2\phi_2))q_2(\delta_2\phi_2), \end{aligned} \tag{3.11}$$

It is obvious that the above relations hold for any $\gamma_i \geq 0$ if $\beta_i(x) \equiv 0$. In the general case $\beta_i(x) \not\equiv 0$ the relations $\delta_1\phi_1 = I_1(\hat{u}_1)$ and $\delta_2\phi_2 = I_2(\hat{u}_2)$, where I_i is defined in (2.3), implies that (3.11) is satisfied if

$$\gamma_1 \geq \frac{\beta_1 \hat{u}_1 D_1(\hat{u}_1)}{I_1(\hat{u}_1)}, \qquad \gamma_2 \geq \frac{\beta_2 \hat{u}_2 D_2(\hat{u}_2)}{I_2(\hat{u}_2)}. \tag{3.12}$$

Since $I_i'(z) = D_i(z)$,

$$\lim_{z\to 0^+}\left[\frac{zD_i(z)}{I_i(z)}\right] = \lim_{z\to 0^+}\left[\frac{D_i(z) + zD_i'(z)}{D_i(z)}\right] = 1 + \lim_{z\to 0^+}\left[\frac{zD_i'(z)}{D_i(z)}\right]. \tag{3.13}$$

If we impose the condition

$$\lim_{z\to 0^+}\frac{zD_i'(z)}{D_i(z)} = \rho_i, \tag{3.14}$$

then by setting

$$\gamma_i > \beta_i(1 + \rho_i), \tag{3.15}$$

(3.12) is satisfied. If the below (3.16) holds, then (3.5) and (3.9) are satisfied. Thus all inequalities of (3.2) are satisfied. Directly applying Theorem 2.3, we have the following theorem.

Theorem 3.1. *Suppose the initial functions $\psi_i(x) \leq M_i$ in* (2.8) *for $i = 1, 2$. Let*

$$\frac{a_1}{a_2} > \frac{b_{11}b_{12}}{b_{11}b_{22} - b_{12}b_{21}}, \tag{3.16}$$

and let $\beta_i(x) \geq 0$ and $D_1(u_1), D_2(u_2)$ satisfy (H)-(ii) with $D_1(0) = D_2(0) = 0$. Assume that either $\beta_i(x) \equiv 0$ or

$$\lim_{z\to 0^+}\left[\frac{zD_i'(z)}{D_i(z)}\right] = \rho_i, \quad i = 1, 2, \tag{3.17}$$

for some constants ρ_i. Then the system (3.1) *admits at least one positive solution.*

Remark 3.2. Pao and Ruan [5] have considered a Lotka-Volterra competition model with density-dependent diffusion, where the coefficient b_{21} of the system (3.1) is negative. The difference between them is that our method does not require that the reaction functions possess the monotone nondecreasing property. The condition for the existence for the solutions of the competition model is $b_{12}/b_{22} < a_1/a_2 < b_{11}/b_{21}$, while the condition for the existence for the solutions of the predator model is $a_1/a_2 > b_{11}b_{12}/(b_{11}b_{22} - b_{12}b_{21})$.

Remark 3.3. In a special case $D_1(u_1) = (m-1)u_1^{m-1}$, $D_2(u_2) = (m-1)u_2^{m-1}$ for $m > 1$, (3.1) becomes

$$\begin{gathered}\frac{\partial u_1}{\partial t} - \nabla^2 u_1^m = u_1(a_1 - b_{11}u_1 - b_{12}u_2) \quad (t > 0,\ x \in \Omega),\\ \frac{\partial u_2}{\partial t} - \nabla^2 u_2^m = v(a_2 + b_{21}u_1 - b_{22}u_2) \quad (t > 0,\ x \in \Omega),\\ \frac{\partial u_1}{\partial \nu} + \beta_1 u_1 = 0, \qquad \frac{\partial u_2}{\partial \nu} + \beta_2 u_2 = 0 \quad (t > 0,\ x \in \partial\Omega).\end{gathered} \tag{3.18}$$

Then the condition (3.17) is trivially satisfied. The conclusions in Theorem 3.1 hold true for (3.18). In fact, if $D_1(u_1) = D_2(u_2) = d$, the condition (3.17) is also trivial true, hence Theorem 2.3 is also valid. After the similar proof as Theorem 3.1, we conclude that Theorem 3.1 holds true for semilinear parabolic system.

4. Discussions

The intension of the present paper is to demonstrate the existence of solutions for the degenerate diffusion reaction system with nonlinear boundary condition. Our method is to look for the positive solution by constructing the coupled upper and lower solutions. The virtue of the technique is that it helps to extend the results for the scalar equation to the coupled system. Our existence theorem of Theorem 2.3 in this paper is applicable to various Lotka-Volterra models, such as competition, predator-prey, or mutualism model, while the method in [6] is not applicable to predator-prey model.

Since Levin and Segel illuminated the important role of the diffusion on the patterns in [7], a number of Lotka-Volterra models with constant diffusion have been investigated in the past three decades. In fact the concern of the density-dependent diffusion is also reasonable in animal disperse model (see [8] for a review). Our study is a starting attempt to consider the role of the density-dependent diffusion on Lotka-Volterra model. In biological terms, the results of Theorems 3.1 imply that if the rate of intraspecific competition of the predator is large, the two species are coexistent. The results also have applicability to 3 species model. Note that for Lotka-Volterra predator-prey model with constant diffusion, when the rate of intraspecific competition of the prey is large, the two species are both extinct. When the density-dependent diffusion is taken into account, it is an open problem whether there exist the extinct phenomena.

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