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Research Article

Relations between Solutions of Differential Equations and Small Functions

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We investigate relations between solutions, their derivatives of differential equation $f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0$, and functions of small growth, where A_j $(j = 0, 1, \ldots, k-1)$ are entire functions of finite order. By these relations, we see that every transcendental solution and its derivative of above equation have infinitely many fixed points.

1. Introduction and Results

In this paper, we use the standard notations of the Nevanlinna's value distribution theory ([1–3]). We use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote exponents of convergence of the zero sequence and the sequence of distinct zeros of a meromorphic function $f(z)$, and $\sigma(f)$ to denote the order of growth of $f(z)$.

In 2000, Chen [4] considered fixed points of solutions of second-order linear differential equations and obtained precise estimation of the number of fixed points of solutions. Recently, a number of papers (including [5–11]) considered relations between solutions, their derivatives of some differential equations, and functions of small growth.

In 2006, Chen and Shon [7] proved the following theorem.

Theorem A. *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions of $\sigma(A_j) < 1$, a, b be complex constants such that $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(o < c < 1)$. Let $\varphi(z)$ $(\not\equiv 0)$ be an entire function of finite order. Then, every solution f $(\not\equiv 0)$ of the equation*

$$f'' + A_1 e^{az} f' + A_0 e^{bz} f = 0, \tag{1.1}$$

satisfies

$$\overline{\lambda}(f-\varphi)=\overline{\lambda}(f'-\varphi)=\overline{\lambda}(f''-\varphi)=\infty. \tag{1.2}$$

In 2010, Xu and Yi [11] proved the following theorem.

Theorem B. *Let $A_j\ (z)\ (\not\equiv 0)\ \ (j=0,1)$ be entire functions of $\sigma\ (A_j)<1$, a, b be complex constants such that $ab\neq 0$ and $a/b\notin\{1,2\}$. Let $\varphi\ (z)\ (\not\equiv 0)$ be an entire function of $\sigma\ (\varphi)<1$. Then, every solution $f\ (\not\equiv 0)$ of* (1.1) *satisfies* (1.2).

In [5, 6, 8–10], authors considered similar problems in Theorems A and B. For relations between solutions, their derivatives of some differential equations, and functions of small growth, particularly, relations between derivatives and functions of small growth are difficult problems. Such problems on higher-order differential equations are more difficult.

In this paper, we consider the higher-order differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=0, \tag{1.3}$$

and prove the following results.

Theorem 1.1. *Let $A_j\ (j=0,1,\ldots,k-1)$ be entire functions of finite order, not all identically equal to zero, such that if $A_j\not\equiv 0$, then $\lambda\ (A_j)<\sigma\ (A_j)$; if $i\neq j$, then $\sigma\ (A_i/A_j)=\max\{\sigma(A_i),\sigma(A_j)\}$. Suppose that $\varphi(z)$ is a finite-order transcendental entire function. Then, every transcendental solution f of* (1.3) *satisfies $\overline{\lambda}\ (f-\varphi)=\sigma\ (f)=\infty$. Furthermore, if $\lambda\ (\varphi)<\lambda\ (A_0)$, then every solution $f\ (\not\equiv 0)$ of* (1.3) *satisfies $\overline{\lambda}\ (f'-\varphi)=\sigma\ (f)=\infty$.*

Theorem 1.2. *Let $A_j\ (j=0,1,\ldots,k-1)$ satisfy conditions of Theorem 1.1 and $A_0\not\equiv 0$. Suppose that H is a nonzero polynomial. Then, every solution $f(\not\equiv 0)$ of* (1.3) *satisfies $\overline{\lambda}(f'-H)=\overline{\lambda}(f-H)=\sigma(f)=\infty$.*

Corollary 1.3. *Let $A_j\ (j=0,1,\ldots,k-1)$ satisfy all conditions of Theorem 1.2. Then, every solution $f(\not\equiv 0)$ and its derivative of* (1.3) *have infinitely many fixed points.*

To prove Theorems 1.1 and 1.2, we use a new method. Our method is different from methods before (including methods applied in [4–13]) which cannot be applied to prove our Theorems 1.1 and 1.2.

2. Auxiliary Lemmas

Lemma 2.1 (see [12]). *Let $A_j\ (j=0,1,\ldots,k-1)$ be entire functions of finite order, not all identically zero. Suppose that if $A_j\not\equiv 0$, then $\lambda\ (A_j)<\sigma(A_j)$; if $i\neq j$, then $\sigma\ (A_i/A_j)=\max\{\sigma(A_i),\sigma(A_j)\}$. Then, every transcendental solution f of* (1.3) *satisfies $\sigma\ (f)=\infty$. Furthermore, according to the order of $A_0,A_1,\ldots,A_{k-1}$, if A_j is the first coefficient satisfying $A_j\not\equiv 0$, then* (1.3) *may at most have polynomial solutions of degree $\leq j-1$, and all other solutions are of infinite order. If $A_0\not\equiv 0$, then every nonzero solution f of* (1.3) *has infinite order.*

Lemma 2.2 (see [13]). *Let A_j $(j = 0, 1, \ldots, k-1)$, F $(\not\equiv 0)$ be meromorphic functions of finite order. Then, every meromorphic solution of*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F, \tag{2.1}$$

satisfies $\overline{\lambda}\,(f) = \lambda\,(f) = \sigma\,(f)$.

Lemma 2.3 (see [14]). *Let f be a transcendental meromorphic function of $\sigma\,(f) = \sigma < \infty$. Let $H = \{(k_1, j_1), (k_2, j_2), \ldots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, 2, \ldots, q$. Also, let $\varepsilon > 0$ be a given constant. Then,*

(*i*) *there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that, if $\psi \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \geqslant R_0$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma - 1 + \varepsilon)}, \tag{2.2}$$

(*ii*) *there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, such that, for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma - 1 + \varepsilon)}, \tag{2.3}$$

(*iii*) *there exists a set $E \subset (0, \infty)$ of finite linear measure such that, for all z satisfying $|z| \notin E$ and for all $(k, j) \in H$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leqslant |z|^{(k-j)(\sigma + \varepsilon)}. \tag{2.4}$$

Lemma 2.4 (see [7]). *Let $g(z)$ be a meromorphic function of $\sigma\,(g) = \beta < \infty$. Then, for any given $\varepsilon > 0$, there is a set $E \subset [0, 2\pi)$ that has linear measure zero, such that, if $\psi \in [0, 2\pi) \setminus E$, there is a constant $R = R\,(\psi) > 1$ such that, for all z satisfying $\arg z = \psi$ and $|z| = r \geq R$, we have*

$$\exp\left\{-r^{\beta+\varepsilon}\right\} \leq \left|g(z)\right| \leq \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{2.5}$$

Lemma 2.5 (see [12, 15]). *Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ be a polynomial with degree $n \geq 1$, where α, β are real numbers satisfying $|\alpha| + |\beta| \neq 0$. Let $\omega\,(z) \not\equiv 0$ be an entire function with $\sigma\,(\omega) < n$. Set $g = \omega e^P$, $z = re^{i\theta}$, $\delta\,(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then, for any given ε $(0 < \varepsilon < 1)$, there exists a set $H_1 \subset [0, 2\pi)$ of linear measure zero such that, for $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is a constant $R > 0$ such that, for $|z| = r > R$, we have*

(i) *if $\delta\,(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P, \theta) r^n\} \leq \left| g\left(re^{i\theta}\right) \right| \leq \exp\{(1+\varepsilon)\delta(P, \theta) r^n\}, \tag{2.6}$$

(ii) *if* $\delta\ (P,\theta) < 0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \leq \left|g\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.7}$$

where $H_2 = \{\theta \in [0,2\pi); \delta(P,\theta) = 0\}$ *is a finite set.*

3. Proof

Proof of Theorem 1.1. Suppose that $f\ (z)$ is a transcendental solution of (1.3). By Lemma 2.1, we know that $\sigma\ (f) = \infty$. Set $g_0 = f - \varphi$. Then, $\sigma\ (g_0) = \sigma\ (f) = \infty$ and $\overline{\lambda}(g_0) = \overline{\lambda}(f - \varphi)$. Substituting $f = g_0 + \varphi$ into (1.3), we obtain

$$g_0^{(k)} + A_{k-1}g_0^{(k-1)} + \cdots + A_1 g_0' + A_0 g_0 = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi\right]. \tag{3.1}$$

Since all transcendental solutions of (1.3) have infinite order and φ is a transcendental entire function of finite order, we see that $\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_1\varphi' + A_0\varphi \not\equiv 0$. So that, by Lemma 2.2, we obtain $\overline{\lambda}(g_0) = \sigma(g_0) = \infty$, that is, $\overline{\lambda}(f - \varphi) = \sigma(f) = \infty$.

Now suppose that $\lambda\ (\varphi) < \lambda\ (A_0)$. Thus, $A_0 \not\equiv 0$. In what follows, we prove that $\overline{\lambda}(f' - \varphi) = \sigma\ (f) = \infty$.

Set $g_1 = f' - \varphi$. Then, $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$ and $\overline{\lambda}(g_1) = \overline{\lambda}(f' - \varphi)$. Differentiating both sides of (1.3), we obtain

$$f^{(k+1)} + A_{k-1}f^{(k)} + \left(A_{k-1}' + A_{k-2}\right)f^{(k-1)} + \cdots + \left(A_1' + A_0\right)f' + A_0' f = 0. \tag{3.2}$$

By (1.3), we obtain

$$f = -\frac{1}{A_0}\left[f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f'\right]. \tag{3.3}$$

Substituting (3.3) into (3.2), we deduce that

$$\begin{aligned} &f^{(k+1)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)f^{(k)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)f^{(k-1)} \\ &\quad + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)f' = 0. \end{aligned} \tag{3.4}$$

Substituting $f' = g_1 + \varphi,\ f'' = g_1' + \varphi', \dots, f^{(k+1)} = g_1^{(k)} + \varphi^{(k)}$ into (3.4), we obtain

$$\begin{aligned} &g_1^{(k)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)g_1^{(k-1)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)g_1^{(k-2)} \\ &\quad + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)g_1 = h, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} -h = \varphi^{(k)} &+ \left(A_{k-1} - \frac{A_0'}{A_0}\right)\varphi^{(k-1)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)\varphi^{(k-2)} \\ &+ \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)\varphi. \end{aligned} \tag{3.6}$$

Since when $A_j \not\equiv 0,\ \lambda(A_j) < \sigma(A_j)$, by Hadamard-Borel theorem, we know that $A_j(z) = h_j(z)e^{P_j(z)}$ where $h_j(z)$ is nonzero entire function, $P_j(z)$ is a nonzero polynomial, such that $\sigma(h_j) = \lambda(A_j) < \sigma(A_j) = \deg P_j$. By $A_j(z) = h_j(z)e^{P_j(z)}$, we obtain

$$\begin{aligned} \frac{A_j'(z)}{A_j(z)} &= \frac{h_j'(z)}{h_j(z)} + P_j'(z), \\ A_j'(z) &= \left(h_j'(z) + P_j'(z)h_j(z)\right)e^{P_j(z)}. \end{aligned} \tag{3.7}$$

Next we prove $h \not\equiv 0$. Suppose to the contrary $h \equiv 0$. Then,

$$\begin{aligned} &\varphi^{(k)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)\varphi^{(k-1)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)\varphi^{(k-2)} \\ &\quad + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)\varphi = 0. \end{aligned} \tag{3.8}$$

Dividing φ into both sides of (3.8) and substituting (3.7) into (3.8), we obtain

$$B_{k-1}(z)e^{P_{k-1}(z)} + B_{k-2}(z)e^{P_{k-2}(z)} + \cdots + B_1(z)e^{P_1(z)} + B_0(z)e^{P_0(z)} + B(z) = 0, \tag{3.9}$$

where

$$\begin{aligned} B_0 &= h_0, \\ B_1 &= h_1\frac{\varphi'}{\varphi} + \left(h_1' + h_1P_1' - \frac{A_0'}{A_0}h_1\right), \end{aligned}$$

$$
\begin{aligned}
B_2 &= h_2 \frac{\varphi''}{\varphi} + \left(h_2' + h_2 P_2' - \frac{A_0'}{A_0} h_2 \right) \frac{\varphi'}{\varphi}, \\
&\ \vdots \\
B_j &= h_j \frac{\varphi^{(j)}}{\varphi} + \left(h_j' + h_j P_j' - \frac{A_0'}{A_0} h_j \right) \frac{\varphi^{(j-1)}}{\varphi}, \\
&\ \vdots \\
B_{k-1} &= h_{k-1} \frac{\varphi^{(k-1)}}{\varphi} + \left(h_{k-1}' + h_{k-1} P_{k-1}' - \frac{A_0'}{A_0} h_{k-1} \right) \frac{\varphi^{(k-2)}}{\varphi}, \\
B &= \frac{\varphi^{(k)}}{\varphi} - \frac{A_0'}{A_0} \frac{\varphi^{(k-1)}}{\varphi}.
\end{aligned}
\tag{3.10}
$$

Since $\lambda\ (h_0) = \lambda\ (A_0) > \lambda\ (\varphi)$, we see that $B_0 = h_0 \not\equiv 0$. It is obviously that not all $B_{k-1}, B_{k-2}, \ldots, B_0$ are equal to zero. Without loss of generality, we may suppose that all $B_j\ (j = 0, 1, \ldots, k-1)$ are not identically zero. In fact, if there exists some $B_j \equiv 0$, we can remove it and rewrite the subscript of each function in (3.9).

Since $\sigma(\varphi) < +\infty$ and $\sigma(A_0) < +\infty$, by Lemma 2.3, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero, such that, if $\theta \in [0, 2\pi) \setminus E_1$, there is a contant $R = R(\theta) > 1$, such that, for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$
\begin{aligned}
&\left| \frac{\varphi^{(j)}(z)}{\varphi(z)} \right| \leq |z|^{j \cdot \sigma(\varphi)} \quad (j = 1, 2, \ldots, k-1), \\
&\left| \frac{A_0'(z)}{A_0(z)} \right| \leq |z|^{\sigma(A_0)}.
\end{aligned}
\tag{3.11}
$$

By (3.10) and (3.11), we obtain

$$
\begin{aligned}
|B(z)| &\leq \left| \frac{\varphi^{(k)}(z)}{\varphi(z)} \right| + \left| \frac{A_0'(z)}{A_0(z)} \right| \left| \frac{\varphi^{(k-1)}(z)}{\varphi(z)} \right| \\
&\leq |z|^{k\sigma(\varphi)} + |z|^{\sigma(A_0)} |z|^{k\sigma(\varphi)} \leq 2r^{2k\sigma},
\end{aligned}
\tag{3.12}
$$

where $\sigma = \max\{\sigma(\varphi), \sigma(A_0)\}$.

Since $\varphi,\ A_0$ are entire functions of finite order, then we obtain

$$\begin{aligned} &m\left(r,\frac{A_0'}{A_0}\right)=O(\log r),\\ &m\left(r,\frac{\varphi^{(j)}}{\varphi}\right)=O(\log r)\quad (j=1,\dots,k-1). \end{aligned} \tag{3.13}$$

By (3.10), (3.13), for sufficiently large r, we obtain

$$\begin{aligned} m(r,B_j)&\le 3m(r,h_j)+m\left(r,\frac{\varphi^{(j)}}{\varphi}\right)+m\left(r,h_j'\right)+m\left(r,P_j'\right)+m\left(r,\frac{A_0'}{A_0}\right)\\ &\quad+m\left(r,\frac{\varphi^{(j-1)}}{\varphi}\right)+O(1)\le 4T(r,h_j)+O(\log r)\quad (j=2,\dots,k-1),\\ N(r,B_j)&\le N\left(r,\frac{1}{\varphi}\right)+N\left(r,\frac{1}{A_0}\right)\quad (j=1,\dots,k-1). \end{aligned} \tag{3.14}$$

By (3.14), we obtain

$$\begin{aligned} T(r,B_j)=m(r,B_j)+N(r,B_j)&\le 4T(r,h_j)+N\left(r,\frac{1}{\varphi}\right)+N\left(r,\frac{1}{A_0}\right)\\ &\quad+O(\log r)\quad (j=2,\dots,k-1). \end{aligned} \tag{3.15}$$

Since $\sigma(h_j)=\lambda(A_j),\lambda(\varphi)<\lambda(A_0)$, by (3.15), we obtain

$$\sigma(B_j)\le\max\{\lambda(\varphi),\lambda(A_0),\sigma(h_j)\}=\max\{\lambda(A_0),\lambda(A_j)\}\quad (j=2,\dots,k-1). \tag{3.16}$$

Using the same method as above, we obtain

$$\sigma(B_1)\le\max\{\lambda(A_0),\lambda(A_1)\}. \tag{3.17}$$

Clearly,

$$\sigma(B_0)=\sigma(h_0)=\lambda(A_0). \tag{3.18}$$

By (3.16)–(3.18), we obtain

$$\sigma(B_s)\le\max\{\lambda(A_j)\mid 0\le j\le k-1\}\quad (s=0,1,\dots,k-1). \tag{3.19}$$

Set

$$d = \max\{\deg P_j \mid j = 0, 1, \dots, k-1\},$$
$$\widetilde{d} = \max\{\deg P_j, \lambda(A_i) \mid i = 0, \dots, k-1, \deg P_j < d,\ j \in \{0, 1, \dots, k-1\}\}. \tag{3.20}$$

According to definitions of d and $\widetilde{d}$ and (3.19), we obtain $\widetilde{d} < d$ and

$$\sigma(B_s) \le \max\{\lambda(A_j) \mid 0 \le j \le k-1\} \le \widetilde{d} \quad (s = 0, 1, \dots, k-1). \tag{3.21}$$

Next, we discuss functions $B_j e^{P_j}$ $(j = 0, 1, 2 \dots, k-1)$. We divide them into two cases:

$$\mathrm{I} = \left\{B_j e^{P_j} \mid \deg P_j < d,\ j \in \{0, 1, \dots, k-1\}\right\},$$
$$\mathrm{II} = \left\{B_j e^{P_j} \mid \deg P_j = d,\ j \in \{0, 1, \dots, k-1\}\right\}. \tag{3.22}$$

Firstly, we consider $B_j e^{P_j} \in \mathrm{I}$. By the definition of $\widetilde{d}$ and (3.21), for any $B_j e^{P_j} \in \mathrm{I}$, we have

$$\sigma\left(B_j e^{P_j}\right) \le \widetilde{d}. \tag{3.23}$$

By Lemma 2.4, for any given ε_1 $(0 < \varepsilon_1 < d - \widetilde{d})$, there is a set E_2 of linear measure zero, such that, if $\theta \in [0, 2\pi) \setminus E_2$, there is a constant $R = R\ (\theta) > 1$ such that, for all z satisfying $\arg z = \theta$ and $|z| = r \ge R$, we have

$$\left|B_j(z) e^{P_j(z)}\right| \le \exp\left\{r^{\widetilde{d}+\varepsilon_1}\right\}. \tag{3.24}$$

As $r \to \infty$, we have $r^{\widetilde{d}+\varepsilon_1}/r^d \to 0$, that is, $r^{\widetilde{d}+\varepsilon_1} \le \varepsilon_1 r^d$. Then, inequality (3.24) can be rewritten as form

$$\left|B_j(z) e^{P_j(z)}\right| \le \exp\left\{\varepsilon_1 r^d\right\}. \tag{3.25}$$

Secondly, we consider $B_j e^{P_j} \in \mathrm{II}$. By the definition of II, for every $B_j e^{P_j}$, we have $\deg P_j = d$. By (3.21), we obtain $\sigma\ (B_j) \le \widetilde{d} < d$. So, for any $B_j e^{P_j} \in \mathrm{II}$, we have $\sigma\ (B_j) < d = \sigma\ (P_j)$. By Lemma 2.5, there is a set $E_3 \subset [0, 2\pi)$ which has the linear measure zero, such that, for any given ε_2 $(0 < \varepsilon_2 < 1)$, and we have that for all z satisfying $\arg z = \theta \in [0, 2\pi) \setminus E_3$ and $|z| = r \ge R$, if $\delta\ (P_j, \theta) > 0$, then

$$\exp\left\{(1-\varepsilon)\delta(P_j, \theta) r^d\right\} \le \left|B_j e^{P_j}\right| \le \exp\left\{(1+\varepsilon)\delta(P_j, \theta) r^d\right\}, \tag{3.26}$$

if $\delta\ (P_j, \theta) < 0$, then

$$\exp\left\{(1+\varepsilon)\delta\left(P_j, \theta_0\right) r^d\right\} \le \left|B_j e^{P_j}\right| \le \exp\left\{(1-\varepsilon)\delta\left(P_j, \theta_0\right) r^d\right\}. \tag{3.27}$$

Now, we further consider $B_j e^{P_j}$ $(j = 0, 1, \ldots, k-1)$. Take a fixed polynomial $P_s \in \mathrm{II}$. Thus, $\deg P_s = d$. Set

$$\begin{gathered} E = \{\theta \in [0, 2\pi) \mid \delta(P_s, \theta) > 0\}, \\ E_4 = \left\{\theta \in [0, 2\pi) \mid \delta\left(P_i - P_j, \theta\right) = 0,\ 0 \le i < j \le k-1\right\}, \\ \bigcup\left\{\theta \in [0, 2\pi) \mid \delta\left(P_j, \theta\right) = 0,\ j = 0, 1, \ldots, k-1\right\}. \end{gathered} \tag{3.28}$$

Clearly, the linear measure of $E \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$ is greater than zero. Now, we take ray $\arg z = \theta_0 \in E \setminus (E_1 \cup E_2 \cup E_3 \cup E_4)$, then $\delta\ (P_s, \theta_0) > 0$. When $j \ne s$, we have $\delta\ (P_j, \theta_0) \ne 0$; when $i < j$ and $\deg P_i = \deg P_j$, we have $\delta(P_i, \theta_0) \ne \delta(P_j, \theta_0)$. Set

$$\delta = \max\left\{\delta\left(P_j, \theta_0\right) \mid B_j e^{P_j} \in \mathrm{II}\right\}. \tag{3.29}$$

It is clearly $\delta > 0$. Since $\delta\ (P_i, \theta_0) \ne \delta(P_j, \theta_0)$ $(i < j$ and $\deg P_i = \deg P_j)$, there exists a unique integer t $(0 \le t \le k-1)$, such that $\delta\ (P_t, \theta_0) = \delta$. Suppose that P_s satisfies $\delta\ (P_s, \theta_0) = \delta$. On the ray $\arg z = \theta_0$, we have that

$$\left|B_s(z) e^{P_s(z)}\right| \ge \exp\left\{(1-\varepsilon)\delta r^d\right\}. \tag{3.30}$$

Set

$$\tilde{\delta} = \max\left\{\delta\left(P_j, \theta_0\right) \mid B_j e^{p_j} \in \mathrm{II} \setminus \left\{B_s e^{P_s}\right\}\right\}. \tag{3.31}$$

Thus, $\tilde{\delta} < \delta$. For any $B_j e_j^P \in \mathrm{II} \setminus \{B_s e^{P_s}\}$, by (3.26) and (3.27), we see that, on the ray $\arg z = \theta_0$, if $\delta\ (P_j, \theta_0) > 0$, we have

$$\left|B_j e^{P_j}\right| \le \exp\left\{(1+\varepsilon)\delta\left(P_j, \theta_0\right) r^d\right\}, \tag{3.32}$$

if $\delta(P_j, \theta_0) < 0$, we have

$$\left|B_j(z) e^{P_j(z)}\right| \le \exp\left\{(1-\varepsilon)\delta\left(P_j, \theta_0\right) r^d\right\} < 1. \tag{3.33}$$

Hence, if $B_j e_j^P \in \mathrm{II} \setminus \{B_s e^{P_s}\}$, then we have

$$\left|B_j(z) e^{P_j(z)}\right| \le \exp\left\{(1+\varepsilon)\delta\left(P_j, \theta_0\right) r^d\right\} + 1 \le \exp\left\{(1+\varepsilon)\tilde{\delta} r^d\right\} + 1. \tag{3.34}$$

Hence, (3.9) can be rewritten as form

$$B_s(z)e^{P_s(z)} = \Sigma_{j \neq s} B_j(z)e^{P_j(z)} + B(z). \tag{3.35}$$

By (3.12), (3.25), (3.30), (3.34), and (3.35), for above ε, set

$$\varepsilon = \frac{1}{2}\min\left\{\frac{\delta}{1+\delta}, \frac{\delta - \tilde{\delta}}{\delta + \tilde{\delta}}, \varepsilon_1, \varepsilon_2\right\}, \tag{3.36}$$

then, for all z satisfying $\arg z = \theta_0$ and sufficiently large r, we have

$$\begin{aligned} \exp\left\{(1-\varepsilon)\delta r^d\right\} &\leq \left|B_s(z)e^{P_s(z)}\right| \leq \Sigma_{j \neq s}\left|B_j(z)e^{P_j(z)}\right| + |B(z)| \\ &\leq O(1)\exp\left\{\varepsilon r^d\right\} + O(1)\exp\left\{(1+\varepsilon)\tilde{\delta} r^d\right\} + O(1) + 2r^{2k\sigma}. \end{aligned} \tag{3.37}$$

Thus, we obtain $1 \leq 0$. This is a contradiction which shows $h \not\equiv 0$.

Since $h \not\equiv 0$ and $\sigma\ (g_1) = \infty$, by Lemma 2.2 and (3.5), we obtain $\overline{\lambda}\ (g_1) = \sigma(g_1) = \infty$, that is, $\overline{\lambda}\ (f' - \varphi) = \sigma(f) = \infty$.

Thus, Theorem 1.1 is proved. □

Proof of Theorem 1.2. Using the same method as in the proof of Theorem 1.1, we can prove Theorem 1.2. □

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