

Research Article

On Antiperiodic Boundary Value Problems for Higher-Order Fractional Differential Equations

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We study an antiperiodic boundary value problem of nonlinear fractional differential equations of order $q \in (4, 5]$. Some existence results are obtained by applying some standard tools of fixed-point theory. We show that solutions for lower-order anti-periodic fractional boundary value problems follow from the solution of the problem at hand. Our results are new and generalize the existing results on anti-periodic fractional boundary value problems. The paper concludes with some illustrating examples.

1. Introduction

In the preceding years, there has been a great advancement in the study of fractional calculus. A variety of results on initial and boundary value problems of fractional order, ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions, have appeared in the literature. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data [1–5]. For an updated account of mathematical tools for fractional models and methods of solutions for fractional differential equations, we refer the reader to a recent text [6] by Baleanu et al. Fractional derivatives are also regarded as an excellent tool for the description of memory and hereditary properties of various materials and processes [7]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. For more details and examples, see [8–20].

Antiperiodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have received a considerable attention. Examples include

antiperiodic trigonometric polynomials in the study of interpolation problems, antiperiodic wavelets, antiperiodic boundary conditions in physics, and so forth (for details, see [21] and the references therein). Some recent work on antiperiodic boundary value problems of fractional-order can be found in [21–27] and references therein.

In this paper, we consider an antiperiodic boundary value problems of fractional differential equations of order $q \in (4, 5]$ given by

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 4 < q \leq 5, \\ x(0) &= -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T), \\ x'''(0) &= -x'''(T), \quad x^{(iv)}(0) = -x^{(iv)}(T), \end{aligned} \quad (1.1)$$

where ${}^c D^q$ denotes the Caputo fractional derivative of order q and f is a given continuous function.

The main objective of the present work is to develop the existence theory for problem (1.1) and relate problem (1.1) with lower-order fractional antiperiodic boundary value problems. Our results are new and give further insight into the characteristics of fractional-order antiperiodic boundary value problems.

2. Preliminaries

Definition 2.1 (see [4]). The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \quad (2.1)$$

provided the integral exists.

Definition 2.2 (see [4]). For at least n -times continuously differentiable function $g : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1, \quad (2.2)$$

where $[q]$ denotes the integer part of the real number q .

To study the nonlinear problem (1.1), we need the following lemma, which deals with a linear variant of problem (1.1).

Lemma 2.3. For any $y \in C[0, T]$, the unique solution of the boundary value problem:

$$\begin{aligned} {}^c D^q x(t) &= y(t), \quad t \in [0, T], \quad 4 < q \leq 5, \\ x(0) &= -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T), \\ x'''(0) &= -x'''(T), \quad x^{(iv)}(0) = -x^{(iv)}(T) \end{aligned} \tag{2.3}$$

is

$$x(t) = \int_0^T G(t, s)y(s)ds, \tag{2.4}$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \frac{2(t-s)^{q-1} - (T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - t^4 - tT^3)(T-s)^{q-5}}{48\Gamma(q-4)}, & 0 < s < t < T, \\ -\frac{(T-s)^{q-1}}{2\Gamma(q)} + \frac{(T-2t)(T-s)^{q-2}}{4\Gamma(q-1)} + \frac{t(T-t)(T-s)^{q-3}}{4\Gamma(q-2)} \\ + \frac{(6t^2T - 4t^3 - T^3)(T-s)^{q-4}}{48\Gamma(q-3)} + \frac{(2Tt^3 - t^4 - tT^3)(T-s)^{q-5}}{48\Gamma(q-4)}, & 0 < t < s < T. \end{cases} \tag{2.5}$$

Proof. It is well known [4] that the solution of ${}^c D^q x(t) = y(t)$ can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds - b_0 - b_1t - b_2t^2 - b_3t^3 - b_4t^4, \tag{2.6}$$

where $b_0, b_1, b_2, b_3,$ and $b_4 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions for problem (2.3) in (2.6), we find that

$$\begin{aligned} b_0 &= \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)ds - \frac{T}{4} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)ds \\ &+ \frac{T^3}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)ds, \end{aligned}$$

$$\begin{aligned}
b_1 &= \frac{1}{2} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s) ds - \frac{T}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s) ds + \frac{T^3}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds, \\
b_2 &= \frac{1}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s) ds - \frac{T}{8} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s) ds, \\
b_3 &= \frac{1}{12} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s) ds - \frac{T}{24} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds, \\
b_4 &= \frac{1}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds.
\end{aligned} \tag{2.7}$$

Substituting the values of b_0 , b_1 , b_2 , b_3 , and b_4 in (2.6), we obtain

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} y(s) ds \\
&\quad + \frac{(T-2t)}{4} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s) ds + \frac{t(T-t)}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s) ds \\
&\quad + \frac{(6t^2T - 4t^3 - T^3)}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s) ds + \frac{(2Tt^3 - t^4 - tT^3)}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s) ds \\
&= \int_0^T G(t,s) y(s) ds,
\end{aligned} \tag{2.8}$$

where $G(t,s)$ is given by (2.5). This completes the proof. \square

2.1. Relationship with Lower-Order Problems

We observe that the first term in expressions for $G(t,s)$ given by (2.5) corresponds to the Green's function for the problem:

$$\begin{aligned}
{}^c D^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 0 < q \leq 1, \\
x(0) &= -x(T);
\end{aligned} \tag{2.9}$$

the first two terms in (2.5) form Green's function for the problem [21]:

$$\begin{aligned}
{}^c D^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 1 < q \leq 2, \\
x(0) &= -x(T), \quad x'(0) = -x'(T);
\end{aligned} \tag{2.10}$$

the first three terms in (2.5) give the Green's function for the problem [22]:

$$\begin{aligned} {}^cD^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 2 < q \leq 3, \\ x(0) &= -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T); \end{aligned} \tag{2.11}$$

while the first four terms in (2.5) yield the Green's function for the antiperiodic problem [23]:

$$\begin{aligned} {}^cD^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad T > 0, \quad 3 < q \leq 4, \\ x(0) &= -x(T), \quad x'(0) = -x'(T), \quad x''(0) = -x''(T), \quad x'''(0) = -x'''(T). \end{aligned} \tag{2.12}$$

From the above deductions, it can easily be concluded that Green's function (2.5) for an antiperiodic boundary value problem of fractional order $q \in (4, 5]$ contains Green's function (or solution) for lower-order fractional antiperiodic problems. We can further interpret that the last term in expressions for Green's function (2.5) arises due to consideration of the order $q \in (4, 5]$, whereas the remaining terms correspond to the lower-order problems. This observation gives a useful insight into the study of antiperiodic fractional boundary value problems that a unit-increase in the fractional order of the problem gives rise to a new term in expressions for Green's function, preserving the terms corresponding to lower-order antiperiodic problems. In other words, one can say that Green's function (or solution) for a higher-order antiperiodic fractional boundary value problem inherits all the characteristics of lower-order fractional antiperiodic problems. Hence, our results generalize the existing results on antiperiodic fractional boundary value problems ([21–23]).

3. Existence Results

Let $\mathcal{X} := C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions defined on $[0, T] \times \mathbb{R}$ endowed with a topology of uniform convergence with the norm $\|x\| = \sup_{t \in [0, T]} |x(t)|$.

To prove the existence results for problem (1.1), we need the following known results [28].

Theorem 3.1. *Let X be a Banach space. Assume that $T : X \rightarrow X$ is completely continuous operator and the set*

$$V = \{u \in X \mid u = \mu Tu, 0 < \mu < 1\} \tag{3.1}$$

is bounded. Then T has a fixed point in X .

Theorem 3.2. *Let X be a Banach space. Assume that Ω is an open-bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \rightarrow X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega. \tag{3.2}$$

Then T has a fixed point in $\overline{\Omega}$.

By Lemma 2.3, we define an operator $\mathcal{U} : \mathcal{X} \rightarrow \mathcal{X}$ as

$$\begin{aligned}
(\mathcal{U}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \\
&+ \frac{(T-2t)}{4} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds + \frac{t(T-t)}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s, x(s)) ds \\
&+ \frac{(6t^2T - 4t^3 - T^3)}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s, x(s)) ds \\
&+ \frac{(2Tt^3 - t^4 - tT^3)}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s, x(s)) ds, \quad t \in [0, T].
\end{aligned} \tag{3.3}$$

Observe that the problem (1.1) has a solution if and only if the operator equation $\mathcal{U}x = x$ has a fixed point.

Theorem 3.3. *Assume that there exists a positive constant L_1 such that $|f(t, x)| \leq L_1$ for $t \in [0, T], x \in \mathbb{R}$. Then the problem (1.1) has at least one solution.*

Proof. First of all, we show that the operator \mathcal{U} is completely continuous. Note that the operator \mathcal{U} is continuous in view of the continuity of f . Let $\mathcal{B} \subset \mathcal{X}$ be a bounded set. By the assumption that $|f(t, x)| \leq L_1$, for $x \in \mathcal{B}$, we have

$$\begin{aligned}
|(\mathcal{U}x)(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\
&+ \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\
&+ \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s, x(s))| ds \\
&+ \frac{|6t^2T - 4t^3 - T^3|}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s, x(s))| ds \\
&+ \frac{|2Tt^3 - t^4 - tT^3|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |f(s, x(s))| ds \\
&\leq L_1 \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\
&\quad \left. + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \right]
\end{aligned}$$

$$\begin{aligned}
 & + \frac{|6t^2T - 4t^3 - T^3|}{48\Gamma(q-3)} \int_0^T (T-s)^{q-4} ds + \frac{|2Tt^3 - t^4 - tT^3|}{48\Gamma(q-4)} \int_0^T (T-s)^{q-5} ds \Big] \\
 \leq & L_1 \left[\frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) \right] = L_2,
 \end{aligned} \tag{3.4}$$

which implies that $\|(\mathcal{U}x)\| \leq L_2$. Further, we find that

$$\begin{aligned}
 |(\mathcal{U}x)'(t)| & = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\
 & + \frac{|T-2t|}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s, x(s))| ds + \frac{|t(T-t)|}{4} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s, x(s))| ds \\
 & + \frac{|6Tt^2 - 4t^3 - T^3|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |f(s, x(s))| ds \\
 \leq & L_1 \left[\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \right. \\
 & + \frac{|T-2t|}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} ds + \frac{|t(T-t)|}{4} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} ds \\
 & \left. + \frac{|6Tt^2 - 4t^3 - T^3|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} ds \right] \\
 \leq & L_1 \left[\frac{T^{q-1}}{2\Gamma(q)} \left(3 + \frac{(q-1)(q^2 - 2q + 12)}{24} \right) \right] = L_3.
 \end{aligned} \tag{3.5}$$

Hence, for $t_1, t_2 \in [0, T]$, we have

$$|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)| \leq \int_{t_1}^{t_2} |(\mathcal{U}x)'(s)| ds \leq L_3(t_2 - t_1). \tag{3.6}$$

This implies that \mathcal{U} is equicontinuous on $[0, T]$. Thus, by the Arzela-Ascoli theorem, the operator $\mathcal{U} : \mathcal{E} \rightarrow \mathcal{E}$ is completely continuous.

Next, we consider the set

$$V = \{x \in \mathcal{E} \mid x = \mu \mathcal{U}x, 0 < \mu < 1\}, \tag{3.7}$$

and show that the set V is bounded. Let $x \in V$, then $x = \mu \mathcal{U}x$, $0 < \mu < 1$. For any $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds - \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds \\ &\quad + \frac{(T-2t)}{4} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds + \frac{t(T-t)}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s, x(s)) ds \\ &\quad + \frac{(6t^2T - 4t^3 - T^3)}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s, x(s)) ds \\ &\quad + \frac{(2Tt^3 - t^4 - tT^3)}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s, x(s)) ds, \end{aligned} \quad (3.8)$$

$$\begin{aligned} |x(t)| &= \mu |(\mathcal{U}x)(t)| \leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\ &\quad + \frac{|T-2t|}{4} \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| ds \\ &\quad + \frac{|t(T-t)|}{4} \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} |f(s, x(s))| ds \\ &\quad + \frac{|6t^2T - 4t^3 - T^3|}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} |f(s, x(s))| ds \\ &\quad + \frac{|2Tt^3 - t^4 - tT^3|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} |f(s, x(s))| ds \\ &\leq L_1 \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\ &\quad + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \\ &\quad + \frac{|6t^2T - 4t^3 - T^3|}{48\Gamma(q-3)} \int_0^T (T-s)^{q-4} \Gamma(q-3) ds \\ &\quad \left. + \frac{|2Tt^3 - t^4 - tT^3|}{48\Gamma(q-4)} \int_0^T (T-s)^{q-5} ds \right] \\ &\leq \max_{t \in [0, T]} \left\{ \frac{2|t^q| + T^q}{2\Gamma(q+1)} + \frac{|T-2t|T^{q-1}}{4\Gamma(q)} + \frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)} + \frac{|6t^2T - 4t^3 - T^3|T^{q-3}}{48\Gamma(q-2)} \right. \\ &\quad \left. + \frac{|2Tt^3 - t^4 - tT^3|T^{q-4}}{48\Gamma(q-3)} \right\} L_1 \\ &= M_1. \end{aligned} \quad (3.9)$$

Thus, $\|x\| \leq M_1$ for any $t \in [0, T]$. So, the set V is bounded. Thus, by the conclusion of Theorem 3.1, the operator \mathcal{U} has at least one fixed point, which implies that (1.1) has at least one solution. \square

Theorem 3.4. *Let there exist a small positive number τ such that $|f(t, x)| \leq \delta|x|$ for $0 < |x| < \tau$, where $\delta > 0$ satisfies the condition*

$$\max_{t \in [0, T]} \left\{ \frac{2|t^q| + T^q}{2\Gamma(q+1)} + \frac{|T - 2t|T^{q-1}}{4\Gamma(q)} + \frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)} + \frac{|6t^2T - 4t^3 - T^3|T^{q-3}}{48\Gamma(q-2)} + \frac{|2Tt^3 - t^4 - tT^3|T^{q-4}}{48\Gamma(q-3)} \right\} \delta \leq 1. \tag{3.10}$$

Then the problem (1.1) has at least one solution.

Proof. Let us define $\mathcal{B}_\tau = \{x \in \mathcal{X} \mid \|x\| < \tau\}$ and take $x \in \mathcal{X}$ such that $\|x\| = \tau$, that is, $x \in \partial\mathcal{B}_\tau$. As before, it can be shown that \mathcal{U} is completely continuous and

$$\|\mathcal{U}x\| \leq \max_{t \in [0, T]} \left\{ \frac{2|t^q| + T^q}{2\Gamma(q+1)} + \frac{|T - 2t|T^{q-1}}{4\Gamma(q)} + \frac{|t(T-t)|T^{q-2}}{4\Gamma(q-1)} + \frac{|6t^2T - 4t^3 - T^3|T^{q-3}}{48\Gamma(q-2)} + \frac{|2Tt^3 - t^4 - tT^3|T^{q-4}}{48\Gamma(q-3)} \right\} \delta \|x\|, \tag{3.11}$$

which in view of (3.10) yields $\|\mathcal{U}x\| \leq \|x\|$, $x \in \partial\mathcal{B}_\tau$. Therefore, by Theorem 3.2, the operator \mathcal{U} has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. \square

Our next existence result is based on Krasnoselskii’s fixed point theorem [29].

Theorem 3.5. *Let M be a closed convex and nonempty subset of a Banach space X . Let A and B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.6. *Let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a jointly continuous function. Further, we assume that*

$$(A_1) \quad |f(t, x) - f(t, y)| \leq L|x - y|, \text{ for all } t \in [0, T], \quad x, y \in \mathbb{R};$$

$$(A_2) \quad |f(t, x)| \leq \mu(t), \text{ for all } (t, x) \in [0, T] \times \mathbb{R}, \text{ and } \mu \in C([0, T], \mathbb{R}^+).$$

Then the problem (1.1) has at least one solution on $[0, T]$ if

$$\frac{LT^q}{2\Gamma(q+1)} \left(1 + \frac{q}{2} + \frac{q(q-1)}{8} + \frac{q(q-1)(q-2)}{24} + \frac{5q(q-1)(q-2)(q-3)}{384} \right) < 1. \tag{3.12}$$

Proof. Letting $\sup_{t \in [0,1]} |\mu(t)| = \|\mu\|$, we fix

$$\bar{r} \geq \frac{\|\mu\| T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2-9q+46)}{384} \right) \quad (3.13)$$

and consider $\mathcal{B}_{\bar{r}} = \{x \in \mathcal{E} : \|x\| \leq \bar{r}\}$. We define the operators \mathcal{U}_1 and \mathcal{U}_2 on $\mathcal{B}_{\bar{r}}$ as

$$\begin{aligned} (\mathcal{U}_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds, \\ (\mathcal{U}_2 x)(t) &= -\frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s)) ds + \frac{1}{4}(T-2t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s)) ds \\ &\quad + \frac{1}{4}(t(T-t)) \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s, x(s)) ds \\ &\quad + \frac{(6t^2T-4t^3-T^3)}{48} \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} f(s, x(s)) ds \\ &\quad + \frac{(2Tt^3-t^4-tT^3)}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} f(s, x(s)) ds. \end{aligned} \quad (3.14)$$

For $x, y \in \mathcal{B}_{\bar{r}}$, we find that

$$\|\mathcal{U}_1 x + \mathcal{U}_2 y\| \leq \frac{\|\mu\| T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2-9q+46)}{384} \right) \leq r. \quad (3.15)$$

Thus, $\mathcal{U}_1 x + \mathcal{U}_2 y \in \mathcal{B}_{\bar{r}}$. It follows from the assumption (A₁) that \mathcal{U}_2 is a contraction mapping for

$$\frac{LT^q}{2\Gamma(q+1)} \left(1 + \frac{q}{2} + \frac{q(q-1)}{8} + \frac{q(q-1)(q-2)}{24} + \frac{5q(q-1)(q-2)(q-3)}{384} \right) < 1. \quad (3.16)$$

Continuity of f implies that the operator \mathcal{U}_1 is continuous. Also, \mathcal{U}_1 is uniformly bounded on $\mathcal{B}_{\bar{r}}$ as

$$\|\mathcal{U}_1 x\| \leq \frac{\|\mu\| T^q}{\Gamma(q+1)}. \quad (3.17)$$

Now we prove the compactness of the operator \mathcal{U}_1 . In view of (A₁), we define

$$\sup_{(t,x) \in [0,T] \times \mathcal{B}_{\bar{r}}} |f(t, x)| = f_m < \infty, \quad (3.18)$$

and consequently, for $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
 |(\mathcal{U}_1 x)(t_2) - (\mathcal{U}_1 x)(t_1)| &\leq \frac{f_m}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds \right. \\
 &\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right|,
 \end{aligned} \tag{3.19}$$

which is independent of x and tends to zero as $t_2 - t_1 \rightarrow 0$. So \mathcal{U}_1 is relatively compact on $\mathcal{B}_{\bar{r}}$. Hence, By the Arzela-Ascoli theorem, \mathcal{U}_1 is compact on $\mathcal{B}_{\bar{r}}$. Thus all the assumptions of Theorem 3.5 are satisfied. Therefore, the conclusion of Theorem 3.5 applies and the antiperiodic fractional boundary value problem (1.1) has at least one solution on $[0, T]$. This completes the proof. \square

Theorem 3.7. Assume that $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a jointly continuous function satisfying the condition

$$|f(t, x) - f(t, y)| \leq L|x - y|, \quad \forall t \in [0, T], \quad x, y \in \mathbb{R} \tag{3.20}$$

with

$$L\Delta < 1, \tag{3.21}$$

$$\Delta = \frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right). \tag{3.22}$$

Then the antiperiodic boundary value problem (1.1) has a unique solution.

Proof. Let us define $\sup_{t \in [0, T]} |f(t, 0)| = M$ and select $r_\kappa \geq M\Delta / (1 - \kappa)$ where $L\Delta \leq \kappa < 1$. Then we show that $\mathcal{U}B_{r_\kappa} \subset B_{r_\kappa}$, where $B_{r_\kappa} = \{x \in \mathcal{X} : \|x\| \leq r_\kappa\}$. For $x \in B_{r_\kappa}$, we have

$$\begin{aligned}
 \|(\mathcal{U}x)\| &\leq \max_{t \in [0, T]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \right. \\
 &\quad + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
 &\quad \left. + \frac{1}{4} |T - 2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
& + \frac{1}{48} |6t^2T - 4t^3 - T^3| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
& + \left. \frac{|2Tt^3 - t^4 - tT^3|}{48} \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \right\} \\
\leq & (Lr_\kappa + M) \max_{t \in [0, T]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\
& + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \\
& + \frac{1}{48\Gamma(q-3)} |6t^2T - 4t^3 - T^3| \int_0^T (T-s)^{q-4} ds \\
& \left. + \frac{|2Tt^3 - t^4 - tT^3|}{48\Gamma(q-4)} \int_0^T (T-s)^{q-5} ds \right\} \\
\leq & (Lr_\kappa + M) \left[\frac{T^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) \right] \\
= & (Lr_\kappa + M)\Delta \leq \kappa r_\kappa + M\Delta \leq r_\kappa,
\end{aligned} \tag{3.23}$$

where (3.22) is used. Now, for $x, y \in \mathcal{X}$, we obtain

$$\begin{aligned}
& \|(\mathcal{U}x) - (\mathcal{U}y)\| \\
\leq & \max_{t \in [0, T]} \left\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, x(s)) - f(s, y(s))\| ds \right. \\
& + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \|f(s, x(s)) - f(s, y(s))\| ds \\
& + \frac{1}{4} |T-2t| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \|f(s, x(s)) - f(s, y(s))\| ds \\
& + \frac{1}{4} |t(T-t)| \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} \|f(s, x(s)) - f(s, y(s))\| ds \\
& + \frac{1}{48} |6t^2T - 4t^3 - T^3| \int_0^T \frac{(T-s)^{q-4}}{\Gamma(q-3)} \|f(s, x(s)) - f(s, y(s))\| ds \\
& \left. + \frac{1}{48} |2Tt^3 - t^4 - tT^3| \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} \|f(s, x(s)) - f(s, y(s))\| ds \right\}
\end{aligned}$$

$$\begin{aligned}
 &\leq L\|x - y\| \max_{t \in [0, T]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\
 &\quad + \frac{|T-2t|}{4\Gamma(q-1)} \int_0^T (T-s)^{q-2} ds + \frac{|t(T-t)|}{4\Gamma(q-2)} \int_0^T (T-s)^{q-3} ds \\
 &\quad + \frac{1}{48\Gamma(q-3)} |6t^2T - 4t^3 - T^3| \int_0^T (T-s)^{q-4} \Gamma(q-3) ds \\
 &\quad \left. + \frac{1}{48} |2Tt^3 - t^4 - tT^3| \int_0^T \frac{(T-s)^{q-5}}{\Gamma(q-4)} ds \right\} \\
 &\leq \frac{LT^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) \|x - y\| = \Delta L \|x - y\|,
 \end{aligned} \tag{3.24}$$

where we have used (3.22). It follows by the condition (3.21) that \mathcal{U} is a contraction. So, by Banach’s contraction mapping principle, problem (1.1) has a unique solution. \square

Example 3.8. Consider the following antiperiodic fractional boundary value problem:

$$\begin{aligned}
 {}^C D^q x(t) &= \frac{e^{(1-\cos^2 x(t))^2} [4 \sin 2t + 8 \ln(17 + 5 \cos^2 x(t))]}{\sqrt{(17 + \sin x(t))}}, \quad 0 < t < 1, \\
 x(0) &= -x(1), \quad x'(0) = -x'(1), \quad x''(0) = -x''(1), \\
 x'''(0) &= -x'''(1), \quad x^{iv}(0) = -x^{iv}(1),
 \end{aligned} \tag{3.25}$$

where $4 < q \leq 5$ and $T = 1$.

Clearly, $|f(t, x)| \leq L_1 = e(1 + 2 \ln 22)$, and the hypothesis of Theorem 3.3 holds. Therefore, the conclusion of Theorem 3.3 applies to problem (3.25).

Example 3.9. Consider the following problem:

$${}^C D^q x(t) = x \left(a^2 + x^3(t) \right)^{1/2} + 2 \left(1 + t^4 \right)^3 (1 - \cos x(t)), \quad x \neq 0, \quad a > 0, \quad 0 < t < 1, \tag{3.26}$$

$$\begin{aligned}
 x(0) &= -x(1), \quad x'(0) = -x'(1), \quad x''(0) = -x''(1), \\
 x'''(0) &= -x'''(1), \quad x^{iv}(0) = -x^{iv}(1),
 \end{aligned} \tag{3.27}$$

where $4 < q \leq 5$, and $T = 1$.

For sufficiently small x (ignoring x^2 and higher powers of x), we have

$$\left| x \left(a^2 + x^3(t) \right)^{1/2} + 2 \left(1 + t^4 \right)^3 (1 - \cos x(t)) \right| \leq a|x|, \tag{3.28}$$

where $a \leq \delta$, and (3.10) takes the form

$$\left(\frac{3}{2\Gamma(q+1)} + \frac{1}{4\Gamma(q)} + \frac{1}{16\Gamma(q-1)} + \frac{1}{48\Gamma(q-2)} + \frac{5}{768\Gamma(q-3)} \right) \delta \leq 1 \quad (3.29)$$

(in particular, for $q = 9/2$, $\delta \leq 1920\sqrt{\pi}/313$). Thus all the assumptions of Theorem 3.4 hold. Consequently, the conclusion of Theorem 3.4 implies that the problem (3.26) has at least one solution

Example 3.10. Consider the following antiperiodic fractional boundary value problem:

$$\begin{aligned} {}^c D^{9/2} x(t) &= \frac{1}{\sqrt{(t+2025)}} \left(\frac{|x|}{1+|x|} + \tan^{-1} x \right) + \sin t, \quad t \in [0, \pi], \\ x(0) &= -x(\pi), \quad x'(0) = -x'(\pi), \quad x''(0) = -x''(\pi), \\ x'''(0) &= -x'''(\pi), \quad x^{(iv)}(0) = -x^{(iv)}(\pi), \end{aligned} \quad (3.30)$$

where $q = 9/2$, and $T = \pi$. Clearly, $L = 2/45$ as $|f(t, x) - f(t, y)| \leq 2/45|x - y|$. Further,

$$L\Delta = \frac{LT^q}{2\Gamma(q+1)} \left(3 + \frac{q}{2} + \frac{q(q-1)(5q^2 - 9q + 46)}{384} \right) = \frac{313\pi^4}{43200} < 1. \quad (3.31)$$

Thus, all the assumptions of by Theorem 3.7 are satisfied. Hence, the fractional boundary value problem (3.30) has a unique solution on $[0, \pi]$.

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