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Research Article

Coupled Fixed Point Theorems for Weak Contraction Mappings under F-Invariant Set

Wutiphol Sintunavarat,[1] Yeol Je Cho,[2] and Poom Kumam[1]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand*
[2] *Department of Mathematics Education and RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*

Correspondence should be addressed to Yeol Je Cho, yjcho@gnu.ac.kr
and Poom Kumam, poom.kum@kmutt.ac.th

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We extend the recent results of the coupled fixed point theorems of Cho et al. (2012) by weakening the concept of the mixed monotone property. We also give some examples of a nonlinear contraction mapping, which is not applied to the existence of the coupled fixed point by the results of Cho et al. but can be applied to our results. The main results extend and unify the results of Cho et al. and many results of the coupled fixed point theorems.

1. Introduction

Since Banach's fixed point theorem in 1922, because of its simplicity and usefullnes, it has become a very popular tool in solving the existence problems in many branches of nonlinear analysis. For some more results of the generalization of this principle, refer to [1–9] and references mentioned therein. Ran and Reurings [10] extended the Banach contraction principle to metric spaces endowed with a partial ordering, and they gave applications of their results to matrix equations. Afterward, Nieto and Rodriguez-López [11] extended Ran and Reurings's theorems in [10] for nondecreasing mappings and obtained a unique solution for a first order-ordinary differential equation with periodic boundary conditions.

In 2006, Gnana Bhaskar and Lakshmikantham [12] introduced the concept of the mixed monotone property and a coupled fixed point. They also established some coupled fixed point theorems for mappings that satisfy the mixed monotone property and gave some applications in the existence and uniqueness of a solution for a periodic boundary value problem. Because of their important role in the study of nonlinear differential equations, nonlinear integral equations and differential inclusions, a wide discussion on coupled fixed point theorems has been studied by many authors. A number of articles in this topic have been dedicated to the improvement and generalization; see [13–22] and references therein.

on the other hand, Huang and Zhang [23] introduced the notion of cone metric spaces and established the fixed point theorems for mappings on this space. In 2011, Cho et al. [24] introduced a new concept of the c-distance in cone metric spaces, which is a cone version of the w-distance of Kada et al. [25] and proved some fixed point theorems for some contractive type mappings in partially ordered cone metric spaces using the c-distance. Afterward, Sintunavarat et al. [26] studied and established the fixed point theorems for the generalized contraction mappings by using this concept. For some more results on the fixed point theory and applications in cone metric spaces by using the cone metric, w-cone-distance, and c-distance, we refer the readers to [27–35].

Recently, Cho et al. [36] established new coupled fixed point theorems under weak contraction mappings by using the concept of mixed monotone property and c-distance in partially cone metric spaces as follows.

Theorem 1.1 (see [36]). *Let $(X,\sqsubseteq)$ be a partially ordered set, and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X and $F : X\times X \to X$ a continuous function having the mixed monotone property such that*

$$q(F(x,y),F(x^*,y^*)) \preceq \frac{k}{2}(q(x,x^*)+q(y,y^*)) \tag{1.1}$$

for some $k\in[0,1)$ and all $x,y,x^*,y^*\in X$ with

$$(x\sqsubseteq x^*)\wedge(y\sqsupseteq y^*) \quad \text{or} \quad (x\sqsupseteq x^*)\wedge(y\sqsubseteq y^*). \tag{1.2}$$

If there exist $x_0,y_0\in X$ such that

$$x_0\sqsubseteq F(x_0,y_0), \qquad F(y_0,x_0)\sqsubseteq y_0, \tag{1.3}$$

then F has a coupled fixed point (u,v). Moreover, one has $q(v,v)=\theta$ and $q(u,u)=\theta$.

Theorem 1.2 (see [36]). *In addition to the hypotheses of Theorem 1.1, suppose that any two elements x and y in X are comparable. Then, the coupled fixed point has the form (u,u), where $u\in X$.*

Theorem 1.3 (see [36]). *Let $(X,\sqsubseteq)$ be a partially ordered set, and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X and $F : X\times X \to X$ a function having the mixed monotone property such that*

$$q(F(x,y),F(x^*,y^*)) \preceq \frac{k}{4}(q(x,x^*)+q(y,y^*)) \tag{1.4}$$

for some $k\in(0,1)$ and all $x,y,x^*,y^*\in X$ with

$$(x\sqsubseteq x^*)\wedge(y\sqsupseteq y^*) \quad \text{or} \quad (x\sqsupseteq x^*)\wedge(y\sqsubseteq y^*). \tag{1.5}$$

Also, suppose that X has the following properties:

(a) *if (x_n) is a nondecreasing sequence in X with $x_n \to x$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$;*

(b) *if (x_n) is a nonincreasing sequence in X with $x_n \to x$, then $x \sqsubseteq x_n$ for all $n \in \mathbb{N}$.*

Assume there exist $x_0, y_0 \in X$ such that

$$x_0 \sqsubseteq F(x_0, y_0), \qquad F(y_0, x_0) \sqsubseteq y_0. \tag{1.6}$$

If $y_0 \sqsubseteq x_0$, then F has a coupled fixed point.

In this paper, we weaken the condition of the mixed monotone property in results of Cho et al. [36] by using the concept of the F-invariant set due to Samet and Vetro [37] in the cone version. We also give the example of a nonlinear contraction mapping, which is not applied by the main results of Cho et al. but can be applied to our results. The presented results extend and complement some recent results of Cho et al. [36] and some classical coupled fixed point theorems and several results in the literature.

2. Preliminaries

Throughout this paper $(X, \sqsubseteq)$ denotes a partially ordered set with the partial order $\sqsubseteq$. By $x \sqsubset y$, we mean $x \sqsubseteq y$ but $x \neq y$. A mapping $f : X \to X$ is said to be nondecreasing (nonincreasing) if, for all $x, y \in X$, $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ $(f(y) \sqsubseteq f(x)$, resp.).

Definition 2.1 (see [12]). Let $(X, \sqsubseteq)$ be a partial ordered set. A mapping $F : X \times X \to X$ is said to have a *mixed monotone property* if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$

$$\begin{aligned} &x_1, x_2 \in X,\ x_1 \sqsubseteq x_2 \Longrightarrow F(x_1, y) \sqsubseteq F(x_2, y), \\ &y_1, y_2 \in X,\ y_1 \sqsubseteq y_2 \Longrightarrow F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned} \tag{2.1}$$

Definition 2.2 (see [12]). Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of mapping $F : X \times X \to X$ if

$$x = F(x, y), \qquad y = F(y, x). \tag{2.2}$$

Next, we give some terminology of cone metric spaces and the concept of c-distance in cone metric spaces due to Cho et al. [24], which is a generalization of the w-distance of Kada et al. [25].

Let $(E, \|\cdot\|)$ be a real Banach space, θ a zero element in E, and P a subset of E with $\operatorname{int}(P) \neq \emptyset$. Then, P is called a *cone* if the following conditions are satisfied:

(1) P is closed and $P \neq \{\theta\}$;

(2) $a, b \in \mathbb{R}^+$, $x, y \in P$ implies $ax + by \in P$;

(3) $x \in P \cap -P$ implies $x = \theta$.

For a cone P, define the *partial ordering* $\preceq$ with respect to P by $x \preceq y$ if and only if $y-x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y-x \in \operatorname{int}(P)$.

It can be easily shown that $\lambda \operatorname{int}(P) \subseteq \operatorname{int}(P)$ for all positive scalars λ.

The cone P is called *normal* if there is a number $K > 0$ such that, for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|. \tag{2.3}$$

The least positive number satisfying the above is called the *normal constant* of P.

Definition 2.3 (see [23]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,y) \preceq d(x,z) + d(y,z)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X,d) is called a *cone metric space*.

Definition 2.4 (see [23]). Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If, for any $c \in X$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then $\{x_n\}$ is said to be *convergent* to a point $x \in X$ and x is the *limit* of $\{x_n\}$. One denotes this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) If, for any $c \in E$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) The space (X,d) is called a *complete cone metric space* if every Cauchy sequence is convergent.

Definition 2.5 (see [24]). Let (X,d) be a cone metric space. Then a function $q : X \times X \to E$ is called a *c-distance* on X if the following are satisfied:

(q1) $\theta \preceq q(x,y)$ for all $x, y \in X$;

(q2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x, y, z \in X$;

(q3) for any $x \in X$, if there exists $u = u_x \in P$ such that $q(x, y_n) \preceq u$ for each $n \in \mathbb{N}$, then $q(x,y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for any $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll c$ imply $d(x,y) \ll c$.

Remark 2.6. The c-distance q is a w-distance on X if we take (X,d) is a metric space, $E = \mathbb{R}^+$, $P = [0,\infty)$, and (q3) is replaced by the following condition:

for any $x \in X$, $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous.

Moreover, (q3) holds whenever $q(x,\cdot)$ is lower semi-continuous. Thus, if (X,d) is a metric space, $E = \mathbb{R}^+$, and $P = [0,\infty)$, then every w-distance is a c-distance. But the converse is not true in the general case. Therefore, the c-distance is a generalization of the w-distance.

Example 2.7. Let (X,d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to E$ by $q(x,y) = d(x,y)$ for all $x, y \in X$. Then, q is c-distance.

Example 2.8. Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and

$$P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\} \tag{2.4}$$

(this cone is not normal). Let $X = [0,\infty)$, and define a mapping $d : X \times X \to E$ by

$$d(x,y) = |x-y|\varphi \tag{2.5}$$

for all $x, y \in X$, where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. Then, (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by

$$q(x,y) = (x+y)\varphi \tag{2.6}$$

for all $x, y \in X$. Then, q is a c-distance.

Example 2.9. Let (X,d) be a cone metric space and P a normal cone. Define a mapping $q : X \times X \to E$ by

$$q(x,y) = d(u,y) \tag{2.7}$$

for all $x, y \in X$, where u is a fixed point in X. Then, q is a c-distance.

Example 2.10. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$, and define a mapping $d : X \times X \to E$ by

$$d(x,y) = |x-y| \tag{2.8}$$

for all $x, y \in X$. Then, (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by

$$q(x,y) = y \tag{2.9}$$

for all $x, y \in X$. Then, q is a c-distance.

Remark 2.11. From Examples 2.9 and 2.10, we have two important results. For the c-distance, $q(x,y) = q(y,x)$ does not necessarily hold and $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

The following lemma is crucial in proving our results.

Lemma 2.12 (see [24]). *Let (X,d) be a cone metric space and q a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then, the following hold.*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to a point $z \in X$.*

(3) *If $q(x_n, x_m) \preceq u_n$ for each $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

3. Coupled Fixed Point under F-Invariant Set

In this section, we prove some coupled fixed point theorems by using the c-distance under the concept of the F-invariant in cone metric spaces and apply our results in partially ordered cone metric spaces. First of all, we give the concept of the F-invariant set in the cone version.

Definition 3.1. Let (X, d) be a cone metric space and $F : X \times X \to X$ a given mapping. Let M be a nonempty subset of X^4. One says that M is the *F-invariant* subset of X^4 if and only if, for all $x, y, z, w \in X$, one has

(a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$;

(b) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

We obtain that the set $M = X^4$ is trivially F-invariant.

The next example plays a key role in the proof of our main results in partially ordered set.

Example 3.2. Let (X, d) be a cone metric space endowed with a partial order $\sqsubseteq$. Let $F : X \times X \to X$ be a mapping satisfying the mixed monotone property, that is, for all $x, y \in X$, we have

$$\begin{aligned} &x_1, x_2 \in X,\ x_1 \sqsubseteq x_2 \Longrightarrow F(x_1, y) \sqsubseteq F(x_2, y), \\ &y_1, y_2 \in X,\ y_1 \sqsubseteq y_2 \Longrightarrow F(x, y_1) \sqsupseteq F(x, y_2). \end{aligned} \tag{3.1}$$

Define the subset $M \subseteq X^4$ by

$$M = \{(a, b, c, d) : c \sqsubseteq a,\ b \sqsubseteq d\}. \tag{3.2}$$

Then, M is the F-invariant of X^4.

Next, we prove the main results of this work.

Theorem 3.3. *Let (X, d) be a complete cone metric space. Let q be a c-distance on X, M a nonempty subset of X^4, and $F : X \times X \to X$ a continuous function such that*

$$q(F(x, y), F(x^*, y^*)) \preceq \frac{k}{2}(q(x, x^*) + q(y, y^*)) \tag{3.3}$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with

$$(x, y, x^*, y^*) \in M \quad \text{or} \quad (x^*, y^*, x, y) \in M. \tag{3.4}$$

If M is an F-invariant and there exist $x_0, y_0 \in X$ such that

$$(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M, \tag{3.5}$$

then F has a coupled fixed point (u, v). Moreover, if $(u, v, u, v) \in M$, then $q(v, v) = \theta$ and $q(u, u) = \theta$.

Proof. As $F(X \times X) \subseteq X$, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_n = F(x_{n-1}, y_{n-1}), \quad y_n = F(y_{n-1}, x_{n-1}) \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Since $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (x_1, y_1, x_0, y_0) \in M$ and M is an F-invariant set, we get

$$(F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) = (x_2, y_2, x_1, y_1) \in M. \tag{3.7}$$

Again, using the fact that M is an F-invariant set, we have

$$(F(x_2, y_2), F(y_2, x_2), F(x_1, y_1), F(y_1, x_1)) = (x_3, y_3, x_2, y_2) \in M. \tag{3.8}$$

By repeating the argument to the above, we get

$$(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}), x_{n-1}, y_{n-1}) = (x_n, y_n, x_{n-1}, y_{n-1}) \in M \tag{3.9}$$

for all $n \in \mathbb{N}$. From (3.3), we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\preceq \frac{k}{2}(q(x_{n-1}, x_n) + q(y_{n-1}, y_n)), \end{aligned} \tag{3.10}$$

$$\begin{aligned} q(x_{n+1}, x_n) &= q(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\preceq \frac{k}{2}(q(x_n, x_{n-1}) + q(y_n, y_{n-1})). \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$q(x_n, x_{n+1}) + q(x_{n+1}, x_n) \preceq \frac{k}{2}(q(x_{n-1}, x_n) + q(y_{n-1}, y_n) + q(x_n, x_{n-1}) + q(y_n, y_{n-1})). \tag{3.12}$$

Since $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and M is an F-invariant set, we get

$$(y_{n-1}, x_{n-1}, y_n, x_n) \in M \tag{3.13}$$

for all $n \in \mathbb{N}$. From (3.3), we have

$$\begin{aligned} q(y_n, y_{n+1}) &= q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\preceq \frac{k}{2}(q(y_{n-1}, y_n) + q(x_{n-1}, x_n)), \\ q(y_{n+1}, y_n) &= q(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\preceq \frac{k}{2}(q(y_n, y_{n-1}) + q(x_n, x_{n-1})). \end{aligned} \tag{3.14}$$

From (3.14), we have

$$q(y_n, y_{n+1}) + q(y_{n+1}, y_n) \preceq \frac{k}{2}\left(q(x_{n-1}, x_n) + q(y_{n-1}, y_n) + q(x_n, x_{n-1}) + q(y_n, y_{n-1})\right). \tag{3.15}$$

Adding (3.12) and (3.15), we get

$$\begin{aligned} &q(x_n, x_{n+1}) + q(x_{n+1}, x_n) + q(y_n, y_{n+1}) + q(y_{n+1}, y_n) \\ &\quad \preceq k\left(q(x_{n-1}, x_n) + q(y_{n-1}, y_n) + q(x_n, x_{n-1}) + q(y_n, y_{n-1})\right). \end{aligned} \tag{3.16}$$

We repeat the above process for n-times, and we get

$$\begin{aligned} &q(x_n, x_{n+1}) + q(x_{n+1}, x_n) + q(y_n, y_{n+1}) + d(y_{n+1}, y_n) \\ &\quad \preceq k^n\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right). \end{aligned} \tag{3.17}$$

From (3.17), we can conclude that

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq k^n\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right), \\ q(y_n, y_{n+1}) &\preceq k^n\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right). \end{aligned} \tag{3.18}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Since

$$\begin{aligned} q(x_n, x_m) &\preceq \sum_{i=n}^{m-1} q(x_i, x_{i+1}), \\ q(y_n, y_m) &\preceq \sum_{i=n}^{m-1} q(y_i, y_{i+1}), \end{aligned} \tag{3.19}$$

and $0 \le k < 1$, we have

$$\begin{aligned} q(x_n, x_m) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right), \\ q(y_n, y_m) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right). \end{aligned} \tag{3.20}$$

Using Lemma 2.12(3), we have $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d). By the completeness of X, we get $x_n \to u$ and $y_n \to v$ for some $u, v \in X$.

Since F is continuous, taking $n \to \infty$ in (3.6), we get

$$\begin{aligned} \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} F(x_n, y_n) = F\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = F(u, v), \\ \lim_{n\to\infty} y_{n+1} &= \lim_{n\to\infty} F(y_n, x_n) = F\left(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\right) = F(v, u). \end{aligned} \tag{3.21}$$

By the uniqueness of the limits, we get $u = F(u,v)$ and $v = F(v,u)$. Therefore, (u,v) is a coupled fixed point of F.

Finally, we assume that $(u,v,u,v) \in M$ and so $(v,u,v,u) \in M$. By (3.3), we have

$$\begin{aligned} q(u,u) &= q(F(u,v),F(u,v)) \preceq \frac{k}{2}\big(q(u,u)+q(v,v)\big), \\ q(v,v) &= q(F(v,u),F(v,u)) \preceq \frac{k}{2}\big(q(v,v)+q(u,u)\big). \end{aligned} \tag{3.22}$$

Therefore, we get

$$q(u,u)+q(v,v) \preceq k\big(q(v,v)+q(u,u)\big). \tag{3.23}$$

Since $0 \le k < 1$, we conclude that $q(u,u)+q(v,v) = \theta$ and hence $q(u,u) = \theta$ and $q(v,v) = \theta$. This completes the proof. □

Theorem 3.4. *In addition to the hypotheses of Theorem 3.3, suppose that any two elements x and y in X, we have*

$$(y,x,x,y) \in M \quad \text{or} \quad (x,y,y,x) \in M. \tag{3.24}$$

Then, the coupled fixed point has the form (u,u), where $u \in X$.

Proof. As in the proof of Theorem 3.3, there exists a coupled fixed point $(u,v) \in X \times X$. Hence,

$$u = F(u,v), \qquad v = F(v,u). \tag{3.25}$$

From the additional hypothesis and (3.3), we get

$$\begin{aligned} q(u,v) &= q(F(u,v),F(v,u)) \preceq \frac{k}{2}\big(q(u,v)+q(v,u)\big), \\ q(v,u) &= q(F(v,u),F(u,v)) \preceq \frac{k}{2}\big(q(v,u)+q(u,v)\big). \end{aligned} \tag{3.26}$$

Therefore, we have

$$q(u,v)+q(v,u) \preceq k\big(q(v,u)+q(u,v)\big). \tag{3.27}$$

Since $0 \le k < 1$, we get $q(u,v)+q(v,u) = \theta$. Therefore, $q(u,v) = \theta$ and $q(v,u) = \theta$.

Let $u_n = \theta$ and $x_n = u$. Then,

$$\begin{aligned} q(x_n,u) &\preceq u_n, \\ q(x_n,v) &\preceq u_n. \end{aligned} \tag{3.28}$$

From Lemma 2.12(1), we have $u = v$. Therefore, the coupled fixed point of F has the form (u, u). This completes the proof. □

Next, we apply Theorems 3.3 and 3.4 in partially ordered cone metric spaces. If we set M as Example 3.2, then we get to Theorems 3.1 and 3.2 of Cho et al. [36].

Corollary 3.5 (see [36, Theorem 3.1]). *Let $(X, \sqsubseteq)$ be a partially ordered set, and suppose that (X, d) is a complete cone metric space. Let q a c-distance on X and $F : X \times X \to X$ be a continuous function having the mixed monotone property such that*

$$q(F(x,y), F(x^*,y^*)) \preceq \frac{k}{2}(q(x,x^*) + q(y,y^*)) \tag{3.29}$$

for some $k \in [0,1)$ and all $x, y, x^*, y^* \in X$ with

$$(x \sqsubseteq x^*) \wedge (y \sqsupseteq y^*) \quad or \quad (x \sqsupseteq x^*) \wedge (y \sqsubseteq y^*). \tag{3.30}$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \sqsubseteq F(x_0, y_0), \qquad F(y_0, x_0) \sqsubseteq y_0, \tag{3.31}$$

then F has a coupled fixed point (u, v). Moreover, one has $q(v, v) = \theta$ and $q(u, u) = \theta$.

Corollary 3.6 (see [36, Theorem 3.2]). *In addition to the hypotheses of Corollary 3.5, suppose that any two elements x and y in X are comparable. Then the coupled fixed point has the form (u, u), where $u \in X$.*

Theorem 3.7. *Let (X, d) be a complete cone metric space. Let q be a c-distance on X, M a subset of X^4, and $F : X \times X \to X$ a function such that*

$$q(F(x,y), F(x^*,y^*)) \preceq \frac{k}{4}(q(x,x^*) + q(y,y^*)) \tag{3.32}$$

for some $k \in [0,1)$ and all $x, y, x^*, y^* \in X$ with

$$(x, y, x^*, y^*) \in M \quad or \quad (x^*, y^*, x, y) \in M. \tag{3.33}$$

Also, suppose that

(i) *there exist $x_0, y_0 \in X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$,*

(ii) *if two sequences $\{x_n\}$, $\{y_n\}$ are such that $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$, $\{y_n\} \to y$, then $(x, y, x_n, y_n) \in M$ for all $n \in \mathbb{N}$.*

If M is an F-invariant set, then F has a coupled fixed point.

Proof. As in the proof of Theorem 3.3, we can construct two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$(x_n, y_n, x_{n-1}, y_{n-1}) \in M \tag{3.34}$$

for all $n \in \mathbb{N}$. Moreover, we have that $\{x_n\}$ converges to a point $u \in X$ and $\{y_n\}$ converges to $v \in X$:

$$\begin{aligned} q(x_n, x_m) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right), \\ q(y_n, y_m) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right) \end{aligned} \tag{3.35}$$

for each $m > n \geq 1$. By (q3), we have

$$\begin{aligned} q(x_n, u) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right) \\ q(y_n, v) &\preceq \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right) \end{aligned} \tag{3.36}$$

and so

$$q(x_n, u) + q(y_n, v) \preceq \frac{2k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right). \tag{3.37}$$

By assumption (ii), we have $(u, v, x_{n-1}, y_{n-1}) \in M$ and $(y_{n-1}, x_{n-1}, v, u) \in M$. From (3.32), we have

$$\begin{aligned} q(x_n, F(u, v)) &= q\left(F(x_{n-1}, y_{n-1}), F(u, v)\right) \\ &\preceq \frac{k}{4}\left(q(x_{n-1}, u) + q(y_{n-1}, v)\right), \\ q(y_n, F(v, u)) &= q\left(F(y_{n-1}, x_{n-1}), F(v, u)\right) \\ &\preceq \frac{k}{4}\left(q(y_{n-1}, v) + q(x_{n-1}, u)\right). \end{aligned} \tag{3.38}$$

Thus, we have

$$q(x_n, F(u, v)) + q(y_n, F(v, u)) \preceq \frac{k}{2}\left(q(x_{n-1}, u) + q(y_{n-1}, v)\right). \tag{3.39}$$

By (3.37), we get

$$\begin{aligned} q(x_n, F(u, v)) + q(y_n, F(v, u)) &\preceq \frac{k}{2} \cdot \frac{2k^{n-1}}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right) \\ &= \frac{k^n}{1-k}\left(q(x_1, x_0) + q(y_1, y_0) + q(x_0, x_1) + q(y_0, y_1)\right). \end{aligned} \tag{3.40}$$

Therefore, we have

$$q(x_n, F(u,v)) \preceq \frac{k^n}{1-k}\left(q(x_1,x_0)+q(y_1,y_0)+q(x_0,x_1)+q(y_0,y_1)\right),$$
$$q(y_n, F(v,u)) \preceq \frac{k^n}{1-k}\left(q(x_1,x_0)+q(y_1,y_0)+q(x_0,x_1)+q(y_0,y_1)\right). \tag{3.41}$$

By using (3.36) and (3.41), Lemma 2.12(1) shows that $u = F(u,v)$ and $v = F(v,u)$. Therefore, (u,v) is a coupled fixed point of F. This completes the proof. □

Corollary 3.8 (see [36, Theorem 3.3]). *Let $(X,\sqsubseteq)$ be a partially ordered set, and suppose that (X,d) is a complete cone metric space. Let q be a c-distance on X and $F : X\times X \to X$ a function having the mixed monotone property such that*

$$q(F(x,y),F(x^*,y^*)) \preceq \frac{k}{4}\left(q(x,x^*)+q(y,y^*)\right) \tag{3.42}$$

for some $k\in(0,1)$ and all $x,y,x^*,y^*\in X$ with

$$(x\sqsubseteq x^*)\wedge(y\sqsupseteq y^*) \quad \text{or} \quad (x\sqsupseteq x^*)\wedge(y\sqsubseteq y^*). \tag{3.43}$$

Also, suppose that X has the following properties:

(a) *if $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$, then $x_n\sqsubseteq x$ for all $n\in\mathbb{N}$;*

(b) *if $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$, then $y\sqsubseteq y_n$ for all $n\in\mathbb{N}$.*

Assume there exist $x_0, y_0\in X$ such that

$$x_0\sqsubseteq F(x_0,y_0), \qquad F(y_0,x_0)\sqsubseteq y_0. \tag{3.44}$$

If $y_0\sqsubseteq x_0$, then F has a coupled fixed point.

Proof. Let $M = \{(a,b,c,d) : c\sqsubseteq a,\ b\sqsubseteq d\}$. We obtain that M is an F-invariant set. By (3.42), we have

$$q(F(x,y),F(x^*,y^*)) \preceq \frac{k}{4}\left(q(x,x^*)+q(y,y^*)\right) \tag{3.45}$$

for some $k\in[0,1)$ and all $x,y,x^*,y^*\in X$ with $(x,y,x^*,y^*)\in M$ or $(x^*,y^*,x,y)\in M$. From assumptions (a) and (b), we know that, for two sequences $\{x_n\}$, $\{y_n\}$ such that $\{x_n\}$ is a non-decreasing sequence in X with $x_n \to x$ and $\{y_n\}$ is a non-increasing sequence in X with $y_n \to y$,

$$x_0\sqsubseteq x_1\sqsubseteq\cdots\sqsubseteq x_n\sqsubseteq\cdots\sqsubseteq x,$$
$$y_0\sqsupseteq y_1\sqsupseteq\cdots y_n\sqsupseteq\cdots\sqsupseteq y \tag{3.46}$$

for all $n \in \mathbb{N}$. Therefore, we have $(x, y, x_n, y_n) \in M$ for all $n \in \mathbb{N}$. Since $y_0 \sqsubseteq x_0$, we have

$$y \sqsubseteq y_n \sqsubseteq y_0 \sqsubseteq x_0 \sqsubseteq x_n \sqsubseteq x \tag{3.47}$$

for all $n \in \mathbb{N}$. Therefore, we can conclude that $(y_n, x_n, y, x) \in M$ for all $n \in \mathbb{N}$. Now, all the hypotheses of Theorem 3.7 hold. Thus, F has a coupled fixed point. □

Example 3.9. Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and

$$P = \{x \in E : x(t) \geq 0,\ t \in [0,1]\}. \tag{3.48}$$

Let $X = [0, \infty)$ (with usual order), and let $d : X \times X \to E$ be defined by

$$d(x,y)(t) = |x - y| e^t. \tag{3.49}$$

Then, (X, d) is an ordered cone metric space (see [24, Example 2.9]). Further, let $q : X \times X \to E$ be defined by

$$q(x,y)(t) = y e^t. \tag{3.50}$$

It is easy to check that q is a c-distance. Consider now the function $F : X \times X \to X$ defined by

$$F(x,y) = \begin{cases} \frac{1}{7}(x+y), & x \geq y, \\ 0, & x < y. \end{cases} \tag{3.51}$$

For $y_1 = 2$ and $y_2 = 3$, we have $y_1 \sqsubseteq y_2$ but $F(x, y_1) \sqsubseteq F(x, y_2)$ for all $x > 3$. So the mapping F does not satisfy the mixed monotone property. Hence, the main results of Cho et al. [36] cannot be applied to this example. But it is easy to see that

$$q(F(x,y), F(u,v)) \preceq \frac{1}{5}(q(x,u) + q(y,v)) \tag{3.52}$$

for all $x, y, u, v \in X^4$. Note that there exists $0, 1 \in X$ such that

$$(F(0,1), F(1,0), 0, 1) \in X^4. \tag{3.53}$$

Now, we can apply Theorem 3.3 with $M = X^4$. Therefore, F has a unique coupled fixed point, that is, a point $(0,0)$.

Remark 3.10. Although the main results of Cho et al. [36] are an essential tool in the partially ordered cone metric spaces to claim the existence of the coupled fixed points the mappings do not have the mixed monotone property in general case such as the mapping in the above example. Therefore, it is very interesting to consider our theorems as another auxiliary tool to claim the existence of a coupled fixed point.

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References

[1] M. Abbas, S. H. Khan, and T. Nazir, "Common fixed points of R-weakly commuting maps in generalized metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 41, 2011.

[2] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.

[3] C. Mongkolkeha, W. Sintunavarat, and P. Kumam, "Fixed point theorems for contraction mappings in modular metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 93, 2011.

[4] W. Sintunavarat and P. Kumam, "Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1877–1881, 2009.

[5] W. Sintunavarat and P. Kumam, "Weak condition for generalized multi-valued (f, α, β)-weak contraction mappings," *Applied Mathematics Letters*, vol. 24, no. 4, pp. 460–465, 2011.

[6] W. Sintunavarat and P. Kumam, "Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 923458, 12 pages, 2011.

[7] W. Sintunavarat and P. Kumam, "Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type," *Journal of Inequalities and Applications*, vol. 2011, article 3, 2011.

[8] W. Sintunavarat and P. Kumam, "Common fixed point theorem for hybrid generalized multi-valued contraction mappings," *Applied Mathematics Letters*, vol. 25, no. 1, pp. 52–57, 2012.

[9] W. Sintunavarat and P. Kumam, "Common fixed point theorems for generalized $\mathcal{JH}$-operator classes and invariant approximations," *Journal of Inequalities and Applications*, vol. 2011, article 67, 2011.

[10] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[11] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[12] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis A*, vol. 65, no. 7, pp. 1379–1393, 2006.

[13] M. Abbas, A. R. Khan, and T. Nazir, "Coupled common fixed point results in two generalized metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6328–6336, 2011.

[14] M. Abbas, W. Sintunavarat, and P. Kumam, "Coupled fixed point in partially ordered G-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 31, 2012.

[15] Y. J. Cho, G. He, and N.-j. Huang, "The existence results of coupled quasi-solutions for a class of operator equations," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 455–465, 2010.

[16] Y. J. Cho, M. H. Shah, and N. Hussain, "Coupled fixed points of weakly F-contractive mappings in topological spaces," *Applied Mathematics Letters*, vol. 24, no. 7, pp. 1185–1190, 2011.

[17] D. Dorića, Z. Kadelburgb, and S. Radenović, "Coupled fixed point results for mappings without mixed monotone property," *Applied Mathematics Letters*. In press.

[18] Z. Golubović, Z. Kadelburg, and S. Radenović, "Coupled coincidence points of mappings in ordered partial metric spaces," *Abstract and Applied Analysis*. In press.

[19] M. Eshaghi Gordji, H. Baghani, and Y. J. Cho, "Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 1897–1906, 2011.

[20] Z. Kadelburg and S. Radenović, "Coupled fixed point results under tvs-cone metric and w-cone distance," *Advanced Fixed Point Theory*. In press.
[21] W. Shatanawi, M. Abbas, and T. Nazir, "Common coupled fixed points results in two generalized metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 80, 2011.
[22] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces," *Fixed Point Theory and Applications*, vol. 2011, article 81, 2011.
[23] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.
[24] Y. J. Cho, R. Saadati, and S. Wang, "Common fixed point theorems on generalized distance in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1254–1260, 2011.
[25] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.
[26] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Common fixed point theorems for c-distance in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1969–1978, 2011.
[27] M. Abbas, Y. J. Cho, and T. Nazir, "Common fixed point theorems for four mappings in TVS-valued cone metric spaces," *Journal of Mathematical Inequalities*, vol. 5, no. 2, pp. 287–299, 2011.
[28] I. Altun and G. Durmaz, "Some fixed point theorems on ordered cone metric spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 58, no. 2, pp. 319–325, 2009.
[29] L. Ćirić, H. Lakzian, and V. Rakočević, "Fixed point theorems for w-cone distance contraction mappings in tvs-cone metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 3, 2012.
[30] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović, and D. Spasić, "Fixed point results under c-distance in tvs-cone metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 29, 2011.
[31] S. Janković, Z. Kadelburg, and S. Radenović, "On cone metric spaces: a survey," *Nonlinear Analysis A*, vol. 74, no. 7, pp. 2591–2601, 2011.
[32] Z. Kadelburg, M. Pavlović, and S. Radenović, "Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces," *Computers & Mathematics with Applications*, vol. 59, no. 9, pp. 3148–3159, 2010.
[33] Z. Kadelburg, S. Radenović, and V. Rakočević, "Topological vector space-valued cone metric spaces and fixed point theorems," *Fixed Point Theory and Applications*, Article ID 170253, 17 pages, 2010.
[34] S. Radenović and B. E. Rhoades, "Fixed point theorem for two non-self mappings in cone metric spaces," *Computers & Mathematics with Applications*, vol. 57, no. 10, pp. 1701–1707, 2009.
[35] W. Sintunavarat and P. Kumam, "Common fixed points of f-weak contractions in cone metric spaces," *Bulletin of the Iranian Mathematical Society*. In press.
[36] Y.J. Cho, Z. Kadelburg, R. Saadati, and W. Shatanawi, "Coupled fixed point theorems under weak contractions," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 184534, 9 pages, 2012.
[37] B. Samet and C. Vetro, "Coupled fixed point, F-invariant set and fixed point of N-order," *Annals of Functional Analysis*, vol. 1, no. 2, pp. 46–56, 2010.