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*Research Article*

# Computing the Fixed Points of Strictly Pseudocontractive Mappings by the Implicit and Explicit Iterations

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It is known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to study computing the fixed points of strictly pseudocontractive mappings by the iterations. Two iterative methods (one implicit and another explicit) for finding the fixed point of strictly pseudocontractive mappings have been constructed in Hilbert spaces. As special cases, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings.

## 1. Introduction

In this paper, we devote to study computing the fixed points of strictly pseudocontractive mappings by the iterations. Our motivations are mainly in two respects.

### *Motivation 1*

Iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1–35] and the references therein. It is known [36] that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. Therefore it is interesting to develop the algorithms for strictly pseudocontractive mappings.

*Motivation 2*

In many problems, it is needed to find a solution with minimum norm. In an abstract way, we may formulate such problems as finding a point $x^{\dagger}$ with the property

$$x^{\dagger} \in C, \qquad \left\| x^{\dagger} \right\| = \min_{x \in C} \|x\|, \tag{1.1}$$

where $C$ is a nonempty closed convex subset of a real Hilbert space $H$. A typical example is the least-squares solution to the constrained linear inverse problem [37]. Some related works for finding the minimum-norm solution (or fixed point of nonexpansive mappings) have been considered by some authors. The reader can refer to [38–41].

In the present paper, we present two iterative methods (one implicit and another explicit) for finding the fixed point of strictly pseudocontractive mappings in Hilbert spaces. As special cases, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $C$ be a nonempty closed convex subset of $H$.

### *2.1. Some Concepts*

Recall that a mapping $T : C \to C$ is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.1}$$

for all $x, y \in C$. And a mapping $T : C \to C$ is said to be strictly pseudocontractive if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2, \tag{2.2}$$

for all $x, y \in C$. For such a case, we also say that $T$ is a $\lambda$-strictly pseudocontractive mapping. It is clear that, in a real Hilbert space $H$, (2.2) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \lambda}{2} \|(I - T)x - (I - T)y\|^2, \tag{2.3}$$

for all $x, y \in C$. It is clear that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

Recall that the nearest point (or metric) projection from $H$ onto $C$ is defined as follows: for each point $x \in H$, $P_C[x]$ is the unique point in $C$ with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C. \tag{2.4}$$

Note that $P_C$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x \rangle \leq 0, \quad y \in C. \tag{2.5}$$

Consequently, $P_C$ is nonexpansive.

## 2.2. Several Useful Lemmas

**Lemma 2.1** (see [42]). *Let $H$ be a real Hilbert space. There holds the following identity:*

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \tag{2.6}$$

*for all $x, y \in H$.*

**Lemma 2.2** (see [43]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a $\lambda$-strict pseudocontraction. Then,*

(i) *$F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined;*

(ii) *$\kappa I + (1 - \kappa)T$ for $\kappa \in [\lambda, 1)$, is nonexpansive.*

**Lemma 2.3** (see [42]). *Let $C$ be a nonempty closed convex of a real Hilbert space $H$. Let $T : C \to C$ be a $\lambda$-strictly pseudocontractive mapping. Then $I - T$ is demiclosed at 0 that is if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

**Lemma 2.4** (see [44]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{2.7}$$

*for all $n \geq 0$ and*

$$\limsup_{n\to\infty} \left( \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{2.8}$$

*Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.5** (see [45]). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \gamma_n \sigma_n, \quad n \geq 0, \tag{2.9}$$

*where $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ satisfy*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.*

*Then $\{a_n\}_{n=0}^{\infty}$ converges to 0.*

We use the following notation:

(i) $\mathrm{Fix}(T)$ stands for the set of fixed points of $T$;

(ii) $x_n \rightharpoonup x$ stands for the weak convergence of $(x_n)$ to $x$;

(iii) $x_n \to x$ stands for the strong convergence of $(x_n)$ to $x$.

## 3. Iterations and Convergence Analysis

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a $\lambda$-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\kappa \in (0,1)$ be a constant. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1-\kappa) P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0. \tag{3.1}$$

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(u)$.*

*Proof. Step* 1. The sequence $\{x_n\}$ is well defined.

Set $\beta = \kappa/(1-(1-\kappa)\lambda)$. It is easily to check that $\beta \in (0,1)$. Then, we can rewrite (3.1) as

$$x_n = \frac{\beta(1-\lambda)}{1-\beta\lambda} T x_n + \frac{1-\beta}{1-\beta\lambda} P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0, \tag{3.2}$$

which is equivalent to the following:

$$x_n = \beta(\lambda x_n + (1-\lambda)T x_n) + (1-\beta) P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0. \tag{3.3}$$

Note that $\lambda I + (1-\lambda)T$ is nonexpansive (see Lemma 2.2). For fix $n$, we define a mapping $S_n : C \to C$ by

$$S_n x = \beta(\lambda x + (1-\lambda)T x) + (1-\beta) P_C[\alpha_n u + (1-\alpha_n)x], \quad x \in C. \tag{3.4}$$

For $x, y \in C$, we have

$$\begin{aligned}
\|S_n x - S_n y\| &= \big\|\beta(\lambda I + (1-\lambda)T)x + (1-\beta) P_C[\alpha_n u + (1-\alpha_n)x] \\
&\qquad -\beta(\lambda I + (1-\lambda)T)y - (1-\beta) P_C[\alpha_n u + (1-\alpha_n)y]\big\| \\
&\leq \beta\|x-y\| + (1-\beta)(1-\alpha_n)\|x-y\| \\
&= [1-(1-\beta)\alpha_n]\|x-y\|,
\end{aligned} \tag{3.5}$$

which implies that $S_n$ is a self-contraction of $C$ for every $n$. Hence $S_n$ has a unique fixed point $x_n \in C$ which is the unique solution of the fixed point equation (3.3).

*Step* 2. The sequence $\{x_n\}$ is bounded.

Pick up any $x^* \in \mathrm{Fix}(T)$. From (3.3), we have

$$\begin{aligned} \|x_n - x^*\| &= \left\|\beta(\lambda x_n + (1-\lambda)Tx_n) + (1-\beta)P_C[\alpha_n u + (1-\alpha_n)x_n] - x^*\right\| \\ &\le \beta\|\lambda x_n + (1-\lambda)Tx_n - x^*\| + (1-\beta)\|P_C[\alpha_n u + (1-\alpha_n)x_n] - x^*\| \\ &\le \beta\|x_n - x^*\| + (1-\beta)\|\alpha_n(u-x^*) + (1-\alpha_n)(x_n - x^*)\| \\ &\le \beta\|x_n - x^*\| + (1-\beta)[(1-\alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\|]. \end{aligned} \tag{3.6}$$

It follows that

$$\|x_n - x^*\| \le \|u - x^*\|. \tag{3.7}$$

Hence, $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

*Step* 3. $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

From (3.3), we have

$$\begin{aligned} \|x_n - Tx_n\| &\le \beta\lambda\|x_n - Tx_n\| + (1-\beta)\|P_C[\alpha_n u + (1-\alpha_n)x_n] - P_C[Tx_n]\| \\ &\le \beta\lambda\|x_n - Tx_n\| + (1-\beta)(\alpha_n\|u - Tx_n\| + (1-\alpha_n)\|x_n - Tx_n\|) \\ &= \left[1 - (1-\alpha_n-\lambda)\beta\right]\|x_n - Tx_n\| + (1-\beta)\alpha_n\|u - Tx_n\|. \end{aligned} \tag{3.8}$$

It follows that

$$\|x_n - Tx_n\| \le \frac{1-\beta}{(1-\alpha_n-\lambda)\beta}\alpha_n\|u - Tx_n\| \longrightarrow 0. \tag{3.9}$$

*Step* 4. $x_n \to \overline{x} \in P_{\mathrm{Fix}(T)}(u)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to a point $\overline{x} \in C$. Noticing (3.9) we can use Lemma 2.3 to get $\overline{x} \in \mathrm{Fix}(T)$.

By using the convexity of the norm and Lemma 2.1, for any $\widetilde{x} \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} \|x_n - \widetilde{x}\|^2 &= \left\|\beta(\lambda x_n + (1-\lambda)Tx_n - \widetilde{x}) + (1-\beta)(P_C[\alpha_n u + (1-\alpha_n)x_n] - \widetilde{x})\right\|^2 \\ &\le \beta\|\lambda x_n + (1-\lambda)Tx_n - \widetilde{x}\|^2 + (1-\beta)\|P_C[\alpha_n u + (1-\alpha_n)x_n] - \widetilde{x}\|^2 \\ &\le \beta\|x_n - \widetilde{x}\|^2 + (1-\beta)\|\alpha_n(u - \widetilde{x}) + (1-\alpha_n)(x_n - \widetilde{x})\|^2 \\ &= \beta\|x_n - \widetilde{x}\|^2 + (1-\beta)\left[(1-\alpha_n)^2\|x_n - \widetilde{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u - \widetilde{x}, x_n - \widetilde{x}\rangle + \alpha_n^2\|u - \widetilde{x}\|^2\right] \\ &= \beta\|x_n - \widetilde{x}\|^2 + (1-\beta)\Big[\|x_n - \widetilde{x}\|^2 - 2\alpha_n\|x_n - \widetilde{x}\|^2 + 2\alpha_n\langle u - \widetilde{x}, x_n - \widetilde{x}\rangle \\ &\qquad + \alpha_n^2\left(\|u - \widetilde{x}\|^2 + \|x_n - \widetilde{x}\|^2 - 2\langle u - \widetilde{x}, x_n - \widetilde{x}\rangle\right)\Big]. \end{aligned} \tag{3.10}$$

It turns out that

$$\begin{aligned}\|x_n-\tilde{x}\|^2 &\le \langle u-\tilde{x}, x_n-\tilde{x}\rangle + \frac{\alpha_n}{2}\left(\|u-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2+2\|u-\tilde{x}\|\|x_n-\tilde{x}\|\right)\\ &\le \langle u-\tilde{x}, x_n-\tilde{x}\rangle + \alpha_n M,\end{aligned} \tag{3.11}$$

where $M > 0$ is some constant such that

$$\sup \frac{1}{2}\left\{\|u-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2+2\|u-\tilde{x}\|\|x_n-\tilde{x}\|\right\} \le M. \tag{3.12}$$

Therefore we can substitute $\tilde{x}$ for $\overline{x}$ in (3.11) to get

$$\|x_n-\overline{x}\|^2 \le \langle u-\overline{x}, x_n-\overline{x}\rangle + \alpha_n M. \tag{3.13}$$

However, $x_n \rightharpoonup \overline{x}$. This together with (3.13) guarantees that $x_n \to \overline{x}$. It is clear that $\overline{x} = P_{Fix(T)}(u)$. As a matter of fact, in (3.11), if we let $n \to \infty$, then we get

$$\langle u-\tilde{x}, \overline{x}-\tilde{x}\rangle \ge 0, \quad \forall \tilde{x} \in \mathrm{Fix}(T). \tag{3.14}$$

This is equivalent to

$$\langle u-\overline{x}, \overline{x}-\tilde{x}\rangle \ge 0, \quad \forall \tilde{x} \in \mathrm{Fix}(T). \tag{3.15}$$

Hence, $\overline{x} = P_{\mathrm{Fix}(T)}(u)$. Therefore, $x_n \to \overline{x} = P_{\mathrm{Fix}(T)}(u)$. This completes the proof. □

**Corollary 3.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\kappa \in (0,1)$ be a constant. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1-\kappa) P_C[\alpha_n u + (1-\alpha_n) x_n], \quad n \ge 0. \tag{3.16}$$

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(u)$.*

**Corollary 3.3.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a $\lambda$-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\kappa \in (0,1)$ be a constant. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1-\kappa) P_C[(1-\alpha_n) x_n], \quad n \ge 0. \tag{3.17}$$

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(0)$ which is the minimum norm fixed point of $T$.*

**Corollary 3.4.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\kappa \in (0,1)$ be a constant. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following implicit manner:*

$$x_n = \kappa T x_n + (1-\kappa) P_C[(1-\alpha_n)x_n], \quad n \geq 0. \tag{3.18}$$

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(0)$ which is the minimum norm fixed point of $T$.*

Next, we introduce an explicit algorithm for finding the fixed point of $T$.

**Theorem 3.5.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a $\lambda$-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\beta$ and $\delta$ be two constants in $(0,1)$ satisfying $\beta+\delta < 1$. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1-\lambda)T x_n + (1-\beta-\delta) P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0, \tag{3.19}$$

*where $\alpha_n \in (0,1)$ satisfies the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(u)$.*

*Proof.* *Step* 1. The sequence $\{x_n\}$ is bounded.

First, we can rewrite (3.19) as

$$x_{n+1} = \beta(\lambda x_n + (1-\lambda)T x_n) + \delta x_n + (1-\beta-\delta) P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0. \tag{3.20}$$

Take $x^* \in \mathrm{Fix}(T)$. From (3.20), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \left\|\beta(\lambda x_n+(1-\lambda)T x_n-x^*)+\delta(x_n-x^*)+(1-\beta-\delta)(P_C[\alpha_n u+(1-\alpha_n)x_n]-x^*)\right\| \\
&\leq \beta\|\lambda x_n + (1-\lambda)T x_n - x^*\| + \delta\|x_n - x^*\| \\
&\quad + (1-\beta-\delta)\|P_C[\alpha_n u + (1-\alpha_n)x_n] - x^*\| \\
&\leq \beta\left\|x_n - p\right\| + \delta\|x_n - x^*\| + (1-\beta-\delta)(\alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|) \\
&= \left[1-(1-\beta-\delta)\alpha_n\right]\|x_n - x^*\| + (1-\beta-\delta)\alpha_n\|u - x^*\| \\
&\leq \max\{\|x_n - x^*\|, \|u - x^*\|\}.
\end{aligned} \tag{3.21}$$

By induction,

$$\|x_{n+1} - x^*\| \leq \max\{\|x_n - x^*\|, \|u - x^*\|\}. \tag{3.22}$$

Hence, the sequence $\{x_n\}$ is bounded and $\{T x_n\}$ is also bounded.

*Step* 2. $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

We can rewrite (3.20) as

$$x_{n+1} = \delta x_n + (1-\delta)y_n, \tag{3.23}$$

where

$$y_n = \frac{\beta}{1-\delta}(\lambda x_n + (1-\lambda)Tx_n) + \frac{1-\beta-\delta}{1-\delta}P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \geq 0. \tag{3.24}$$

It follows that

$$\begin{aligned}
\left\|y_{n+1} - y_n\right\| &\leq \frac{\beta}{1-\delta}\|(\lambda x_{n+1} + (1-\lambda)Tx_{n+1}) - (\lambda x_n + (1-\lambda)Tx_n)\| \\
&\quad + \frac{1-\beta-\delta}{1-\delta}\|P_C[\alpha_{n+1}u + (1-\alpha_{n+1})x_{n+1}] - P_C[\alpha_n u + (1-\alpha_n)x_n]\| \\
&\leq \frac{\beta}{1-\delta}\|x_{n+1} - x_n\| + \frac{1-\beta-\delta}{1-\delta}|\alpha_{n+1} - \alpha_n|(\|u\| + \|x_n\|) \\
&\quad + \frac{1-\beta-\delta}{(1-\delta)}(1-\alpha_{n+1})\|x_{n+1} - x_n\| \\
&\leq \|x_{n+1} - x_n\| + \frac{1-\beta-\delta}{1-\delta}|\alpha_{n+1} - \alpha_n|(\|u\| + \|x_n\|).
\end{aligned} \tag{3.25}$$

Thus,

$$\limsup_{n\to\infty}\left(\left\|y_{n+1} - y_n\right\| - \|x_{n+1} - x_n\|\right) \leq \limsup_{n\to\infty}\frac{1-\beta-\delta}{1-\delta}|\alpha_{n+1} - \alpha_n|(\|u\| + \|x_n\|) = 0. \tag{3.26}$$

This together with Lemma 2.4 implies that

$$\lim_{n\to\infty}\left\|y_n - x_n\right\| = 0. \tag{3.27}$$

Note that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \left\|x_n - y_n\right\| + \left\|y_n - Tx_n\right\| \\
&= \left\|x_n - y_n\right\| + \left\|\frac{\beta}{1-\delta}(\lambda x_n + (1-\lambda)Tx_n) + \frac{1-\beta-\delta}{1-\delta}P_C[\alpha_n u + (1-\alpha_n)x_n] - Tx_n\right\| \\
&\leq \left\|x_n - y_n\right\| + \frac{\beta}{1-\delta}\lambda\|x_n - Tx_n\| + \frac{1-\beta-\delta}{1-\delta}\|P_C[\alpha_n u + (1-\alpha_n)x_n] - Tx_n\|
\end{aligned}$$

$$\leq \left\|x_n - y_n\right\| + \frac{\beta}{1-\delta}\lambda\|x_n - Tx_n\| + \frac{1-\beta-\delta}{1-\delta}\alpha_n\|u - Tx_n\| + \frac{1-\beta-\delta}{1-\delta}(1-\alpha_n)\|x_n - Tx_n\|. \tag{3.28}$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{1-\delta}{\beta(1-\lambda)}\left\|x_n - y_n\right\| + \frac{1-\beta-\delta}{\beta(1-\lambda)}\alpha_n\|u - Tx_n\|. \tag{3.29}$$

Thus,

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.30}$$

*Step* 3. $\limsup_{n\to\infty}\langle u-\tilde{x}, x_n-\tilde{x}\rangle \leq 0$, where $\tilde{x} = P_{\mathrm{Fix}(T)}(u)$.

To see this, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying the properties

$$\limsup_{n\to\infty}\langle u-\tilde{x}, x_n-\tilde{x}\rangle = \lim_{k\to\infty}\langle u-\tilde{x}, x_{n_k}-\tilde{x}\rangle, \tag{3.31}$$

$$x_{n_k} \rightharpoonup x^* \quad \text{as } k \longrightarrow \infty. \tag{3.32}$$

By the demiclosed principle (see Lemma 2.3) and (3.30), we have that $x^* \in \mathrm{Fix}(T)$. So,

$$\limsup_{n\to\infty}\langle u-\tilde{x}, x_n-\tilde{x}\rangle = \langle u-\tilde{x}, x^*-\tilde{x}\rangle \leq 0. \tag{3.33}$$

*Step* 4. $x_n \longrightarrow \tilde{x}$.

From (3.20), we get

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\leq \beta\|\lambda x_n + (1-\lambda)Tx_n - \tilde{x}\|^2 + \delta\|x_n-\tilde{x}\|^2 \\
&\quad + (1-\beta-\delta)\|P_C[\alpha_n u + (1-\alpha_n)x_n] - \tilde{x}\|^2 \\
&\leq (\beta+\delta)\|x_n-\tilde{x}\|^2 + (1-\beta-\delta)\|\alpha_n(u-\tilde{x}) + (1-\alpha_n)(x_n-\tilde{x})\|^2 \\
&= (\beta+\delta)\|x_n-\tilde{x}\|^2 + (1-\beta-\delta)\Big[(1-2\alpha_n)\|x_n-\tilde{x}\|^2 + 2\alpha_n\langle u-\tilde{x}, x_n-\tilde{x}\rangle \\
&\qquad + \alpha_n^2\Big(\|u-\tilde{x}\|^2 + \|x_n-\tilde{x}\|^2 - 2\langle u-\tilde{x}, x_n-\tilde{x}\rangle\Big)\Big] \\
&= \left[1 - 2(1-\beta-\delta)\alpha_n\right]\|x_n-\tilde{x}\|^2 \\
&\quad + 2(1-\beta-\delta)\alpha_n\Big(\langle u-\tilde{x}, x_n-\tilde{x}\rangle + \frac{\alpha_n}{2}\Big(\|u-\tilde{x}\|^2 + \|x_n-\tilde{x}\|^2 - 2\langle u-\tilde{x}, x_n-\tilde{x}\rangle\Big)\Big) \\
&= (1-\delta_n)\|x_n-\tilde{x}\|^2 + \delta_n\theta_n,
\end{aligned} \tag{3.34}$$

where $\delta_n = 2(1-\beta-\delta)\alpha_n$ and

$$\theta_n = \langle u - \tilde{x}, x_n - \tilde{x}\rangle + \frac{\alpha_n}{2}\left(\|u-\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2 - 2\langle u - \tilde{x}, x_n - \tilde{x}\rangle\right). \quad (3.35)$$

It is easy to see that $\lim_{n\to\infty}\delta_n = 0$ and $\limsup_{n\to\infty}\theta_n \le 0$. We can therefore apply Lemma 2.5 to (3.34) and conclude that $x_{n+1} \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

**Corollary 3.6.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\beta$ and $\delta$ be two constants in $(0,1)$ satisfying $\beta+\delta<1$. For $u \in H$ and any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1-\lambda)Tx_n + (1-\beta-\delta)P_C[\alpha_n u + (1-\alpha_n)x_n], \quad n \ge 0, \quad (3.36)$$

*where $\alpha_n \in (0,1)$ satisfies the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(u)$.*

**Corollary 3.7.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a $\lambda$-strictly pseudocontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\beta$ and $\delta$ be two constants in $(0,1)$ satisfying $\beta+\delta<1$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1-\lambda)Tx_n + (1-\beta-\delta)P_C[(1-\alpha_n)x_n], \quad n \ge 0, \quad (3.37)$$

*where $\alpha_n \in (0,1)$ satisfies the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(0)$ which is the minimum norm fixed point of $T$.*

**Corollary 3.8.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\beta$ and $\delta$ be two constants in $(0,1)$ satisfying $\beta+\delta<1$. For any $x_0 \in C$, let $\{x_n\}$ be the sequence defined by the following explicit manner:*

$$x_{n+1} = (\beta\lambda + \delta)x_n + \beta(1-\lambda)Tx_n + (1-\beta-\delta)P_C[(1-\alpha_n)x_n], \quad n \ge 0, \quad (3.38)$$

*where $\alpha_n \in (0,1)$ satisfies the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

*Then the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(T)}(0)$ which is the minimum norm fixed point of $T$.*

## 4. Conclusion

Finding fixed points of nonlinear mappings (especially, nonexpansive mappings) has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is wellknown that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to construct the methods for computing the fixed points of strictly pseudocontractive mappings. Two iterative methods have been presented. Especially, we can use these two methods to find the minimum norm fixed point of strictly pseudocontractive mappings. The ideas contained in the present paper can help us to solve the minimum norm problems in the applied science.

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