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Research Article

Algorithmic Approach to a Minimization Problem

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We first construct an implicit algorithm for solving the minimization problem $\min_{x\in\Omega}\|x\|$, where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a nonexpansive mapping, and the solution set of some variational inequality. Further, we suggest an explicit algorithm by discretizing this implicit algorithm. We prove that the proposed implicit and explicit algorithms converge strongly to a solution of the above minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of S by $F(S)$.

Let $B : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of (1.2) is denoted by $\mathrm{EP}(F, B)$. If $B = 0$, then (1.2) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The solution set of (1.3) is denoted by $\mathrm{EP}(F)$. If $F = 0$, then (1.2) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Bz, y - z \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The solution set of variational inequality (1.4) is denoted by $\mathrm{VI}(C, B)$.

Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. Equilibrium problems include variational inequalities, fixed point, Nash equilibrium, and game theory as special cases. The equilibrium problems and the variational inequality problems have been investigated by many authors. Please see [2–35] and the references therein. The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others.

On the other hand, we also notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset C of a Hilbert space H_1 and a bounded linear operator $R : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of R, $R_C^{\dagger}$, is then defined as the minimum-norm solution of the constrained minimization problem:

$$R_C^{\dagger}(b) := \arg\min_{x \in C} \|Rx - b\|, \tag{1.5}$$

which is equivalent to the fixed point problem:

$$x = P_C(x - \lambda R^*(Rx - b)), \tag{1.6}$$

where P_C is the metric projection from H_1 onto C, R^* is the adjoint of R, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{R(C)}}(b) \in R(C)$.

It is therefore an interesting problem to invent some algorithms that can generate schemes which converge strongly to the minimum-norm solution of a given problem.

In this paper, we focus on the following minimization problem: find $x^* \in \Omega$ such that

$$x^* = \arg\min_{x \in \Omega} \|x\|, \tag{1.7}$$

where Ω is the intersection set of the solution set of some equilibrium problem, the fixed points set of a nonexpansive mapping, and the solution set of some variational inequality. We will suggest and analyze two very simple algorithms for solving the above minimization problem.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x,x) = 0$ for all $x \in C$;

(H2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(H4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property:

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x,C). \tag{2.1}$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

We need the following well-known lemmas for proving our main results.

Lemma 2.1 (see [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)–(H4). Let $\mu > 0$ and $x \in C$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{\mu}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Further, if $T_\mu(x) = \{z \in C : F(z,y) + (1/\mu)\langle y - z, z - x\rangle \geq 0,$ for all $y \in C\}$, then the following hold:

(a) *T_μ is single-valued and T_μ is firmly nonexpansive, that is, for any $x, y \in C$, $\|T_\mu x - T_\mu y\|^2 \leq \langle T_\mu x - T_\mu y, x - y\rangle$;*

(b) *$\mathrm{EP}(F)$ is closed and convex and $\mathrm{EP}(F) = F(T_\mu)$.*

Lemma 2.2 (see [27]). *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)v_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|v_n - x_n\| = 0$.*

Lemma 2.3 (see [29]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.4 (see [29]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n, \tag{2.4}$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section we will introduce two algorithms (one implicit and one explicit) for finding the minimum norm element x^* of $\Omega := \mathrm{EP}(F,B) \cap \mathrm{VI}(C,A) \cap F(S)$. Namely, we want to find a point x^* which solves the following minimization problem:

$$x^* = \arg\min_{x\in\Omega} \|x\|. \tag{3.1}$$

Let $S : C \to C$ be a nonexpansive mapping and $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone mappings, respectively. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)–(H4). In order to solve the minimization problem (3.1), we first construct the following implicit algorithm by using the projection method:

$$x_t = SP_C\left[(1-t)(I-\lambda A)T_\mu(I-\mu B)x_t\right], \quad \forall t \in (0,1), \tag{3.2}$$

where T_μ is defined as Lemma 2.1 and λ, μ are two constants such that $\lambda \in (0, 2\alpha)$ and $\mu \in (0, 2\beta)$. We will show that the net $\{x_t\}$ defined by (3.2) converges to a solution of the minimization problem (3.1). First, we show that the net $\{x_t\}$ is well defined. As matter of fact, for each $t \in (0,1)$, we consider the following mapping W_t given by

$$W_t x = SP_C\left[(1-t)(I-\lambda A)T_\mu(I-\mu B)x\right], \quad \forall x \in C. \tag{3.3}$$

Since the mappings S, P_C, $I-\lambda A$, T_μ and $I-\mu B$ are nonexpansive, then we can check easily that $\|W_t x - W_t y\| \le (1-t)\|x-y\|$ which implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C, that is, $x_t = W_t x_t$ which is exactly (3.2).

Next we show the first main result of the present paper.

Theorem 3.1. *Suppose that* $\Omega \ne \emptyset$. *Then the net* $\{x_t\}$ *generated by the implicit method* (3.2) *converges in norm, as* $t \to 0$, *to a solution of the minimization problem* (3.1).

Proof. Take $z \in \Omega$. First we need use the following facts:

(1) $z = Sz = T_\mu(z-\mu Bz) = P_C(I-\lambda A)z = SP_C[(I-\lambda A)T_\mu(I-\mu B)z]$ for all $\lambda > 0$, $\mu > 0$. In particular, for all $t \in (0,1)$,

$$\begin{aligned} z &= P_C[z-(1-t)\lambda Az] \\ &= P_C[tz+(1-t)(I-\lambda A)z] \\ &= SP_C\left[tz+(1-t)(I-\lambda A)T_\mu(I-\mu B)z\right]; \end{aligned} \tag{3.4}$$

(2) $I-\lambda A$ and $I-\mu B$ are nonexpansive and for all $x,y\in C$

$$\begin{aligned}&\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2\leq\|x-y\|^2+\lambda(\lambda-2\alpha)\|Ax-Ay\|^2;\\&\left\|(I-\mu B)x-(I-\mu B)y\right\|^2\leq\|x-y\|^2+\mu(\mu-2\beta)\|Bx-By\|^2.\end{aligned}\tag{3.5}$$

Set $u_t=T_\mu(I-\mu B)x_t$ and $y_t=(I-\lambda A)u_t$ for all $t\in(0,1)$. It follows that

$$\begin{aligned}\|u_t-z\|^2&=\left\|T_\mu(x_t-\mu Bx_t)-T_\mu(z-\mu Bz)\right\|^2\\&\leq\left\|(I-\mu B)x_t-(I-\mu B)z\right\|^2\\&\leq\|x_t-z\|^2+\mu(\mu-2\beta)\|Bx_t-Bz\|^2.\end{aligned}\tag{3.6}$$

From (3.2), we have

$$\begin{aligned}\|x_t-z\|&=\left\|SP_C\left[(1-t)(I-\lambda A)T_\mu(I-\mu B)x_t\right]-SP_C\left[tz+(1-t)(I-\lambda A)T_\mu(I-\mu B)z\right]\right\|\\&\leq\left\|\left[(1-t)(I-\lambda A)T_\mu(I-\mu B)x_t\right]-\left[tz+(1-t)(I-\lambda A)T_\mu(I-\mu B)z\right]\right\|\\&\leq(1-t)\left\|(I-\lambda A)T_\mu(I-\mu B)x_t-(I-\lambda A)T_\mu(I-\mu B)z\right\|+t\|z\|\\&\leq(1-t)\|x_t-z\|+t\|z\|,\end{aligned}\tag{3.7}$$

that is,

$$\|x_t-z\|\leq\|z\|.\tag{3.8}$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$, $\{Ax_t\}$, and $\{Bx_t\}$ are also bounded. Next we will use $M>0$ to denote some possible constant appearing in the following.

From (3.7), we have

$$\begin{aligned}\|x_t-z\|^2&\leq\left\|\left[(1-t)(I-\lambda A)T_\mu(I-\mu B)x_t\right]-\left[tz+(1-t)(I-\lambda A)T_\mu(I-\mu B)z\right]\right\|^2\\&=\left\|(1-t)\left[(I-\lambda A)T_\mu(I-\mu B)x_t-(I-\lambda A)T_\mu(I-\mu B)z\right]-tz\right\|^2\\&=(1-t)^2\|(I-\lambda A)u_t-(I-\lambda A)z\|^2\\&\quad-2t(1-t)\langle z,(I-\lambda A)u_t-(I-\lambda A)z\rangle+t^2\|z\|^2\\&\leq\|u_t-z\|^2+\lambda(\lambda-2\alpha)\|Au_t-Az\|^2+tM\\&\leq\|x_t-z\|^2+\mu(\mu-2\beta)\|Bx_t-Bz\|^2+\lambda(\lambda-2\alpha)\|Au_t-Az\|^2+tM,\end{aligned}\tag{3.9}$$

that is,

$$\lambda(2\alpha-\lambda)\|Au_t-Az\|^2+\mu(2\beta-\mu)\|Bx_t-Bz\|^2\leq tM\longrightarrow0.\tag{3.10}$$

Since $\lambda(2\alpha-\lambda)>0$ and $\mu(2\beta-\mu)>0$, we derive

$$\lim_{t\to 0}\|Au_t - Az\| = \lim_{t\to 0}\|Bx_t - Bz\| = 0. \tag{3.11}$$

From Lemma 2.1 and (2.2), we obtain

$$\begin{aligned}
\|u_t - z\|^2 &= \left\|T_\mu(x_t-\mu Bx_t) - T_\mu(z-\mu Bz)\right\|^2 \\
&\le \langle (x_t-\mu Bx_t) - (z-\mu Bz), u_t - z\rangle \\
&= \frac{1}{2}\Big(\left\|(x_t-\mu Bx_t) - (z-\mu Bz)\right\|^2 + \|u_t - z\|^2 \\
&\qquad - \left\|(x_t - z) - \mu(Bx_t - Bz) - (u_t - z)\right\|^2\Big) \\
&\le \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \left\|(x_t - u_t) - \mu(Bx_t - Bz)\right\|^2\Big) \\
&= \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 \\
&\qquad + 2\mu\langle x_t - u_t, Bx_t - Bz\rangle - \mu^2\|Bx_t - Bz\|^2\Big).
\end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned}
\|u_t - z\|^2 &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu\langle x_t - u_t, Bx_t - Bz\rangle - \mu^2\|Bx_t - Bz\|^2 \\
&\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2\mu\|x_t - u_t\|\|Bx_t - Bz\| \\
&\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Bx_t - Bz\|.
\end{aligned} \tag{3.13}$$

Set $z_t = P_C[(1-t)y_t]$ for all $t\in(0,1)$. By Lemma 2.1 and (2.2), we have

$$\begin{aligned}
\|z_t - z\|^2 &= \|P_C[(1-t)(I-\lambda A)u_t] - P_C[(I-\lambda A)z]\|^2 \\
&\le \langle (1-t)(I-\lambda A)u_t - (I-\lambda A)z, z_t - z\rangle \\
&= \frac{1}{2}\Big\{\left\|(I-\lambda A)u_t - (I-\lambda A)z - ty_t\right\|^2 + \|z_t - z\|^2 \\
&\qquad - \left\|(I-\lambda A)u_t - (I-\lambda A)z - (z_t - z) - ty_t\right\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|(I-\lambda A)u_t - (I-\lambda A)z\|^2 + tM + \|z_t - z\|^2 \\
&\qquad - \left\|u_t - z_t - \lambda(Au_t - Az) - ty_t\right\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\|u_t - z\|^2 + tM + \|z_t - z\|^2 - \|u_t - z_t\|^2 + 2\lambda\langle u_t - z_t, Au_t - Az\rangle \\
&\qquad + 2t\langle y_t, u_t - z_t\rangle - \left\|\lambda A(u_t - Az) + ty_t\right\|^2\Big\}
\end{aligned}$$

$$\leq \frac{1}{2}\Big\{\|u_t - z\|^2 + tM + \|z_t - z\|^2 - \|u_t - z_t\|^2 + 2\lambda\|u_t - z_t\|\|Au_t - Az\| + 2t\big\|y_t\big\|\|u_t - z_t\|\Big\}, \tag{3.14}$$

that is,

$$\begin{aligned}\|z_t - z\|^2 &\leq \|u_t - z\|^2 + tM - \|u_t - z_t\|^2 + 2\lambda\|u_t - z_t\|\|Au_t - Az\| + 2t\big\|y_t\big\|\|u_t - z_t\| \\ &\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Bx_t - Bz\| + tM - \|u_t - z_t\|^2 \\ &\quad + 2\lambda\|u_t - z_t\|\|Au_t - Az\| + 2t\big\|y_t\big\|\|u_t - z_t\|.\end{aligned} \tag{3.15}$$

Therefore, we have

$$\begin{aligned}\|x_t - z\|^2 &= \|Sz_t - z\|^2 \\ &\leq \|z_t - z\|^2 \\ &\leq \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Bx_t - Bz\| + tM - \|u_t - z_t\|^2 \\ &\quad + 2\lambda\|u_t - z_t\|\|Au_t - Az\| + 2t\big\|y_t\big\|\|u_t - z_t\|.\end{aligned} \tag{3.16}$$

Hence, we deduce

$$\begin{aligned}&\|x_t - u_t\|^2 + \|u_t - z_t\|^2 \\ &\leq M(\|Bx_t - Bz\| + t) + 2\lambda\|u_t - z_t\|\|Au_t - Az\| + 2t\big\|y_t\big\|\|u_t - z_t\| \\ &\longrightarrow 0.\end{aligned} \tag{3.17}$$

This denotes that

$$\lim_{t\to 0}\|x_t - u_t\| = \lim_{t\to 0}\|u_t - z_t\| = 0. \tag{3.18}$$

Note that

$$\big\|z_t - y_t\big\| = \big\|P_C\big[(1-t)y_t\big] - P_C y_t\big\| \leq t\big\|y_t\big\| \longrightarrow 0, \tag{3.19}$$

thus,

$$\begin{aligned}\big\|y_t - Sy_t\big\| &\leq \big\|y_t - z_t\big\| + \|z_t - u_t\| + \|u_t - x_t\| + \big\|x_t - Sy_t\big\| \\ &\leq 2\big\|y_t - z_t\big\| + \|z_t - u_t\| + \|u_t - x_t\| \\ &\longrightarrow 0.\end{aligned} \tag{3.20}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $u_n := u_{t_n}$ and $y_n := y_{t_n}$. From (3.20), we get

$$\|y_n - Sy_n\| \longrightarrow 0. \tag{3.21}$$

By (3.2), we have

$$\begin{aligned}\|x_t - z\|^2 &= \|SP_C[(1-t)y_t] - SP_Cz\|^2 \\ &\leq \|y_t - z - ty_t\|^2 \\ &= \|y_t - z\|^2 - 2t\langle y_t, y_t - z\rangle + t^2\|y_t\|^2 \\ &= \|y_t - z\|^2 - 2t\langle y_t - z, y_t - z\rangle - 2t\langle z, y_t - z\rangle + t^2\|y_t\|^2 \\ &\leq (1-2t)\|y_t - z\|^2 + 2t\langle z, z - y_t\rangle + t^2\|y_t\|^2 \\ &\leq (1-2t)\|x_t - z\|^2 + 2t\langle z, z - y_t\rangle + t^2\|y_t\|^2.\end{aligned} \tag{3.22}$$

It follows that

$$\|x_t - z\|^2 \leq \langle z, z - y_t\rangle + \frac{tM}{2}. \tag{3.23}$$

In particular,

$$\|x_n - z\|^2 \leq \langle z, z - y_n\rangle + \frac{t_nM}{2}, \quad z \in \Omega. \tag{3.24}$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $\{y_n\}$ converges weakly to a point $x^* \in C$. Hence, u_n and x_n also converge weakly to x^*. Noticing (3.21) we can use Lemma 2.3 to get $x^* \in F(S)$.

Now we show $x^* \in \mathrm{EP}(F, B)$. Since $u_n = T_\mu(x_n - \mu Bx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu Bx_n)\rangle \geq 0. \tag{3.25}$$

From the monotonicity of F, we have

$$\frac{1}{\mu}\langle y - u_n, u_n - (x_n - \mu Bx_n)\rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.26}$$

Hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.27}$$

Put $v_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. Then, we have $v_t \in C$. So, from (3.27) we have

$$\begin{aligned}\langle v_t - u_{n_i}, Bv_t\rangle &\geq \langle v_t - u_{n_i}, Bv_t\rangle - \left\langle v_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu} + Bx_{n_i}\right\rangle + F(v_t, u_{n_i}) \\ &= \langle v_t - u_{n_i}, Bv_t - Bu_{n_i}\rangle + \langle v_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ &\quad - \left\langle v_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\mu}\right\rangle + F(v_t, u_{n_i}).\end{aligned} \tag{3.28}$$

Note that $\|Bu_{n_i} - Bx_{n_i}\| \leq (1/\beta)\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of B, we have $\langle v_t - u_{n_i}, Bv_t - Bu_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.28), we have

$$\langle v_t - x^*, Bv_t\rangle \geq F(v_t, x^*). \tag{3.29}$$

From (H1), (H4), and (3.29), we also have

$$\begin{aligned}0 = F(v_t, v_t) &\leq tF(v_t, y) + (1-t)F(v_t, x^*) \\ &\leq tF(v_t, y) + (1-t)\langle v_t - x^*, Bv_t\rangle \\ &= tF(v_t, y) + (1-t)t\langle y - x^*, Bv_t\rangle,\end{aligned} \tag{3.30}$$

and hence

$$0 \leq F(v_t, y) + (1-t)\langle Bv_t, y - x^*\rangle. \tag{3.31}$$

Letting $t \to 0$ in (3.31), we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Bx^*\rangle. \tag{3.32}$$

This implies that $x^* \in \mathrm{EP}(F, B)$. By the same argument as that of [13], we have $x^* \in \mathrm{VI}(C, A)$. Therefore, $x^* \in \Omega$.

We substitute x^* for z in (3.24) to get

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - y_n\rangle + \frac{t_n}{2}M. \tag{3.33}$$

Hence, the weak convergence of $\{y_n\}$ to x^* implies that $x_n \to x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Now we return to (3.24) and take the limit as $n \to \infty$ to get

$$\|x^* - z\|^2 \leq \langle z, z - x^*\rangle, \quad z \in \Omega. \tag{3.34}$$

To show that the entire net $\{x_t\}$ converges to x^*, assume $x_{s_n} \to \tilde{x} \in \Omega$, where $s_n \to 0$. In (3.34), we take $z = \tilde{x}$ to get

$$\|x^* - \tilde{x}\|^2 \leq \langle \tilde{x}, \tilde{x} - x^* \rangle. \tag{3.35}$$

Interchange x^* and $\tilde{x}$ to obtain

$$\|\tilde{x} - x^*\|^2 \leq \langle x^*, x^* - \tilde{x} \rangle. \tag{3.36}$$

Adding up (3.35) and (3.36) yields

$$2\|x^* - \tilde{x}\|^2 \leq \|x^* - \tilde{x}\|^2, \tag{3.37}$$

which implies that $\tilde{x} = x^*$.

We note that (3.34) is equivalent to

$$\|x^*\|^2 \leq \langle x^*, z \rangle, \quad z \in \Omega. \tag{3.38}$$

This clearly implies that

$$\|x^*\| \leq \|z\|, \quad z \in \Omega. \tag{3.39}$$

Therefore, x^* solves the minimization problem (3.1). This completes the proof. □

Next we introduce an explicit algorithm for finding a solution of the minimization problem (3.1). This scheme is obtained by discretizing the implicit scheme (3.2). We will show the strong convergence of this algorithm.

Theorem 3.2. *Suppose that $\Omega \neq \emptyset$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) SP_C\left[(1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)x_n\right], \quad n \geq 0, \tag{3.40}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to a solution of the minimization problem (3.1).

Proof. Take $z \in \Omega$. First we need use the following fact:

$z = Sz = T_\mu(I - \mu B)z = P_C(I - \lambda A)z = SP_C[(I - \lambda A)T_\mu(I - \mu B)z]$ for all $\lambda > 0, \mu > 0$. In particular,

$$\begin{aligned} z &= P_C[z - (1 - \alpha_n)\lambda A z] \\ &= P_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)z] \\ &= SP_C\left[\alpha_n z + (1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)z\right], \quad \forall n \geq 0. \end{aligned} \tag{3.41}$$

Set $u_n = T_\mu(x_n - \mu B x_n)$, $y_n = (I - \lambda A)u_n$ and $z_n = P_C[(1 - \alpha_n)y_n]$ for all $n \geq 0$. From (3.40), we get

$$\begin{aligned} \|x_{n+1} - z\| &= \big\|\beta_n(x_n - z) + (1 - \beta_n)\big(SP_C[(1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)x_n] \\ &\qquad - SP_C[\alpha_n z + (1 - \alpha_n)(I - \lambda A)T_\mu(I - \mu B)z]\big)\big\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)\big\|(1 - \alpha_n)\big[(I - \lambda A)T_\mu(I - \mu B)x_n \\ &\qquad - (I - \lambda A)T_\mu(I - \mu B)z\big] - \alpha_n z\big\| \\ &\leq \beta_n\|x_n - z\| + (1 - \beta_n)[(1 - \alpha_n)\|x_n - z\| + \alpha_n\|z\|] \\ &= [1 - (1 - \beta_n)\alpha_n]\|x_n - z\| + \alpha_n(1 - \beta_n)\|z\| \\ &\leq \max\{\|x_n - z\|, \|z\|\}. \end{aligned} \tag{3.42}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\{\|x_0 - z\|, \|z\|\}. \tag{3.43}$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. We will use $M > 0$ to denote some possible constant appearing in the following.

Define $x_{n+1} = \beta_n x_n + (1 - \beta_n)v_n$ for all $n \geq 0$. It follows that

$$\begin{aligned} \|v_{n+1} - v_n\| &= \big\|SP_C[(1 - \alpha_{n+1})y_{n+1}] - SP_C[(1 - \alpha_n)y_n]\big\| \\ &\leq \big\|(1 - \alpha_{n+1})y_{n+1} - (1 - \alpha_n)y_n\big\| \\ &\leq \|y_{n+1} - y_n\| + \alpha_{n+1}\|y_{n+1}\| + \alpha_n\|y_n\| \\ &\leq \|(I - \lambda A)u_{n+1} - (I - \lambda A)u_n\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|u_{n+1} - u_n\| + M(\alpha_{n+1} + \alpha_n) \\ &= \|T_\mu(I - \mu B)x_{n+1} - T_\mu(I - \mu B)x_n\| + M(\alpha_{n+1} + \alpha_n) \\ &\leq \|x_{n+1} - x_n\| + M(\alpha_{n+1} + \alpha_n). \end{aligned} \tag{3.44}$$

This together with (a) implies that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.45}$$

Hence by Lemma 2.2, we get

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{3.46}$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|v_n-x_n\|=0. \tag{3.47}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \left\|\beta_n(x_n-z)+(1-\beta_n)(v_n-z)\right\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\big\|P_C\left[(1-\alpha_n)(I-\lambda A)T_\mu(I-\mu B)x_n\right]\\
&\qquad -P_C\left[\alpha_n z+(1-\alpha_n)(I-\lambda A)T_\mu(I-\mu B)z\right]\big\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\big\|(1-\alpha_n)\left[(I-\lambda A)T_\mu(I-\mu B)x_n\right.\\
&\qquad \left.-(I-\lambda A)T_\mu(I-\mu B)z\right]-\alpha_n z\big\|^2\\
&= \beta_n\|x_n-z\|^2+(1-\beta_n)\Big[(1-\alpha_n)^2\|(I-\lambda A)u_n-(I-\lambda A)z\|^2\\
&\qquad -2\alpha_n(1-\alpha_n)\langle (I-\lambda A)u_n-(I-\lambda A)z,z\rangle+\alpha_n^2\|z\|^2\Big]\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\Big[\|u_n-z\|^2+\lambda(\lambda-2\alpha)\|Au_n-Az\|^2+\alpha_n M\Big]\\
&= \beta_n\|x_n-z\|^2+(1-\beta_n)\Big[\left\|T_\mu(I-\mu B)x_n-T_\mu(I-\mu B)z\right\|^2\\
&\qquad +\lambda(\lambda-2\alpha)\|Au_n-Az\|^2+\alpha_n M\Big]\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)\Big[\|x_n-z\|^2+\mu(\mu-2\beta)\|Bx_n-Bz\|^2\\
&\qquad +\lambda(\lambda-2\alpha)\|Au_n-Az\|^2+\alpha_n M\Big].
\end{aligned} \tag{3.48}$$

It follows that

$$\begin{aligned}
&(1-\beta_n)\lambda(2\alpha-\lambda)\|Au_n-Az\|^2+(1-\beta_n)\mu(2\beta-\mu)\|Bx_n-Bz\|^2\\
&\quad\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n M\\
&\quad\le (\|x_n-z\|+\|x_{n+1}-z\|)\|x_n-x_{n+1}\|+\alpha_n M.
\end{aligned} \tag{3.49}$$

Since $\liminf_{n\to\infty}(1-\beta_n)\lambda(2\alpha-\lambda)>0$, $\liminf_{n\to\infty}(1-\beta_n)\mu(2\beta-\mu)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we derive

$$\lim_{n\to\infty}\|Au_n-Az\|=\lim_{n\to\infty}\|Bx_n-Bz\|=0. \tag{3.50}$$

From Lemma 2.1 and (2.2), we obtain

$$\begin{aligned}
&\|u_n-z\|^2\\
&\quad=\left\|T_\mu(x_n-\mu Bx_n)-T_\mu(z-\mu Bz)\right\|^2\\
&\quad\le\langle(x_n-\mu Bx_n)-(z-\mu Bz),u_n-z\rangle\\
&\quad=\frac{1}{2}\left(\left\|(x_n-\mu Bx_n)-(z-\mu Bz)\right\|^2+\|u_n-z\|^2-\left\|(x_n-u_n)-\mu(Bx_n-Bz)\right\|^2\right)\\
&\quad\le\frac{1}{2}\left(\|x_n-z\|^2+\|u_n-z\|^2-\left\|(x_n-u_n)-\mu(Bx_n-Bz)\right\|^2\right)\\
&\quad=\frac{1}{2}\left(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2+2\mu\langle x_n-u_n,Bx_n-Bz\rangle-\mu^2\|Bx_n-Bz\|^2\right).
\end{aligned} \tag{3.51}$$

It follows that

$$\begin{aligned}
\|u_n-z\|^2&\le\|x_n-z\|^2-\|x_n-u_n\|^2+2\mu\langle x_n-u_n,Bx_n-Bz\rangle\\
&\quad-\mu^2\|Bx_n-Bz\|^2\\
&\le\|x_n-z\|^2-\|x_n-u_n\|^2+2\mu\|x_n-u_n\|\|Bx_n-Bz\|\\
&\le\|x_n-z\|^2-\|x_n-u_n\|^2+M\|Bx_n-Bz\|.
\end{aligned} \tag{3.52}$$

Again, by Lemma 2.1 and (2.2), we have

$$\begin{aligned}
\|z_n-z\|^2&=\|P_C[(1-\alpha_n)(I-\lambda A)u_n]-P_C[(I-\lambda A)z]\|^2\\
&\le\langle(1-\alpha_n)(I-\lambda A)u_n-(I-\lambda A)z,z_n-z\rangle\\
&=\frac{1}{2}\Big\{\left\|(I-\lambda A)u_n-(I-\lambda A)z-\alpha_ny_n\right\|^2+\|z_n-z\|^2\\
&\qquad-\left\|(I-\lambda A)u_n-(I-\lambda A)z-(z_n-z)-\alpha_ny_n\right\|^2\Big\}\\
&\le\frac{1}{2}\Big\{\|(I-\lambda A)u_n-(I-\lambda A)z\|^2+\alpha_nM+\|z_n-z\|^2\\
&\qquad-\left\|u_n-z_n-\lambda(Au_n-Az)-\alpha_ny_n\right\|^2\Big\}\\
&\le\frac{1}{2}\Big\{\|u_n-z\|^2+\alpha_nM+\|z_n-z\|^2-\|u_n-z_n\|^2+2\lambda\langle u_n-z_n,Au_n-Az\rangle\\
&\qquad+2\alpha_n\langle y_n,u_n-z_n\rangle-\left\|\lambda(Au_n-Az)+\alpha_ny_n\right\|^2\Big\}
\end{aligned}$$

$$\leq \frac{1}{2}\Big\{\|u_n - z\|^2 + \alpha_n M + \|z_n - z\|^2 - \|u_n - z_n\|^2 + 2\lambda\|u_n - z_n\|\|Au_n - Az\| + 2\alpha_n \big\|y_n\big\|\|u_n - z_n\|\Big\}, \tag{3.53}$$

that is,

$$\begin{aligned}\|z_n - z\|^2 &\leq \|u_n - z\|^2 + \alpha_n M - \|u_n - z_n\|^2 + 2\lambda\|u_n - z_n\|\|Au_n - Az\| \\ &\quad + 2\alpha_n\big\|y_n\big\|\|u_n - z_n\| \\ &\leq \|x_n - z\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bz\| + \alpha_n M - \|u_n - z_n\|^2 \\ &\quad + 2\lambda\|u_n - z_n\|\|Au_n - Az\| + 2\alpha_n\big\|y_n\big\|\|u_n - z_n\|.\end{aligned} \tag{3.54}$$

Hence,

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\|z_n - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1 - \beta_n)\Big[\|x_n - z\|^2 - \|x_n - u_n\|^2 + M\|Bx_n - Bz\| \\ &\qquad + \alpha_n M - \|u_n - z_n\|^2 + 2\lambda\|u_n - z_n\|\|Au_n - Az\| \\ &\qquad + 2\alpha_n\big\|y_n\big\|\|u_n - z_n\|\Big] \\ &\leq \|x_n - z\|^2 - (1 - \beta_n)\|x_n - u_n\|^2 + (\|Bx_n - Bz\| + \alpha_n)M \\ &\quad - (1 - \beta_n)\|u_n - z_n\|^2 + 2\lambda\|u_n - z_n\|\|Au_n - Az\| + 2\alpha_n\big\|y_n\big\|\|u_n - z_n\|.\end{aligned} \tag{3.55}$$

It follows that

$$\begin{aligned}&(1 - \beta_n)\Big(\|x_n - u_n\|^2 + \|u_n - z_n\|^2\Big) \\ &\quad\leq \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|) + (\|Bx_n - Bz\| + \alpha_n)M \\ &\quad\quad + 2\lambda\|u_n - z_n\|\|Au_n - Az\| + 2\alpha_n\big\|y_n\big\|\|u_n - z_n\|.\end{aligned} \tag{3.56}$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, $\|Au_n - Az\| \to 0$ and $\|Bx_n - Bz\| \to 0$, we derive that

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.57}$$

Note that

$$\big\|z_n - y_n\big\| = \big\|P_C[(1 - \alpha_n)y_n] - P_C y_n\big\| \leq \alpha_n\big\|y_n\big\| \longrightarrow 0, \tag{3.58}$$

therefore,

$$\begin{aligned}\|Sy_n - y_n\| &\leq \|Sy_n - Sz_n\| + \|Sz_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - y_n\| \\ &\leq 2\|y_n - z_n\| + \|Sz_n - x_n\| + \|x_n - u_n\| + \|u_n - z_n\| \\ &\longrightarrow 0.\end{aligned} \tag{3.59}$$

Next we prove

$$\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle \leq 0, \tag{3.60}$$

where $x^* = P_{\Omega}(0)$ is a solution of the minimization problem (3.1).

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle = \lim_{i\to\infty}\langle x^*, x^* - y_{n_i}\rangle. \tag{3.61}$$

Without loss of generality, we may further assume that $y_{n_i} \to \tilde{x}$ weakly. By the same argument as that of Theorem 3.1, we can deduce that $\tilde{x} \in \Omega$. Therefore,

$$\limsup_{n\to\infty}\langle x^*, x^* - y_n\rangle = \langle x^*, x^* - \tilde{x}\rangle \leq 0. \tag{3.62}$$

Finally, we prove $x_n \to x^*$. As a matter of fact, we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad= \left\|\beta_n(x_n - x^*) + (1-\beta_n)\left(SP_C[(1-\alpha_n)y_n] - SP_C[x^*]\right)\right\|^2 \\ &\quad\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|(1-\alpha_n)(y_n - x^*) - \alpha_n x^*\right\|^2 \\ &\quad= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)^2\left\|y_n - x^*\right\|^2 - 2\alpha_n(1-\alpha_n)\langle x^*, y_n - x^*\rangle + \alpha_n^2\|x^*\|^2\right] \\ &\quad\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n(1-\alpha_n)\langle x^*, y_n - x^*\rangle + \alpha_n^2\|x^*\|^2\right] \\ &\quad\leq \left[1 - 2(1-\beta_n)\alpha_n\right]\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle x^*, x^* - y_n\rangle + (1-\beta_n)\alpha_n^2 M \\ &\quad= (1-\gamma_n)\|x_n - x^*\|^2 + \delta_n\gamma_n,\end{aligned} \tag{3.63}$$

where $\gamma_n = 2(1-\beta_n)\alpha_n$ and $\delta_n = (1-\alpha_n)\langle x^*, x^* - y_n\rangle + (\alpha_n M/2)$. It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta \leq 0$. Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$ strongly. This completes the proof. □

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