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Research Article

Strong Convergence to Solutions of Generalized Mixed Equilibrium Problems with Applications

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We introduce a Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. Strong convergence theorems are also established in this paper. As applications, we apply our main result to mixed equilibrium, generalized equilibrium, and mixed variational inequality problems in Banach spaces. Finally, examples and numerical results are also given.

1. Introduction

Let E be a real Banach space, C a nonempty, closed, and convex subset of E, and E^* the dual space of E. Let $T : C \to C$ be a nonlinear mapping. The fixed points set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

One classical way often used to approximate a fixed point of a nonlinear self-mapping T on C was firstly introduced by Halpern [1] which is defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \quad \forall n \geq 1, \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. He proved, in a real Hilbert space, a strong convergence theorem for a nonexpansive mapping T when $\alpha_n = n^{-a}$ for any $a \in (0, 1)$.

Subsequently, motivated by Halpern [1], many mathematicians devoted time to study algorithm (1.1) in different styles. Several strong convergence results for nonlinear mappings were also continuously established in some certain Banach spaces (see also [2–9]).

Let $f : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to E^*$ a mapping, and $\varphi : C \to \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \tag{1.2}$$

The solutions set of (1.2) is denoted by $\mathrm{GMEP}(f, A, \varphi)$ (see Peng and Yao [10]).

If $A \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \tag{1.3}$$

The solutions set of (1.3) is denoted by $\mathrm{MEP}(f, \varphi)$ (see Ceng and Yao [11]).

If $f \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following mixed variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \tag{1.4}$$

The solutions set of (1.4) is denoted by $\mathrm{VI}(C, A, \varphi)$ (see Noor [12]).

If $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.2) reduces to the following generalized equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The solutions set of (1.5) is denoted by $\mathrm{GEP}(f, A)$ (see Moudafi [13]).

If $\varphi \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The solutions set of (1.6) is denoted by $\mathrm{EP}(f)$ (see Combettes and Hirstoaga [14]).

If $f \equiv 0$, then the mixed equilibrium problem (1.3) reduces to the following convex minimization problem: finding $\hat{x} \in C$ such that

$$\varphi(y) \geq \varphi(\hat{x}), \quad \forall y \in C. \tag{1.7}$$

The solutions set of (1.7) is denoted by $\mathrm{CMP}(\varphi)$.

If $\varphi \equiv 0$, then the mixed variational inequality problem (1.4) reduces to the following variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.8}$$

The solutions set of (1.8) is denoted by $\mathrm{VI}(C, A)$ (see Stampacchia [7]).

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others. For more details on these topics, see, for instance, [14–34].

For solving the generalized mixed equilibrium problem, let us assume the following [25]:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The purpose of this paper is to investigate strong convergence of Halpern-type iteration for a generalized mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. As applications, our main result can be deduced to mixed equilibrium, generalized equilibrium, mixed variational inequality problems, and so on. Examples and numerical results are also given in the last section.

2. Preliminaries and Lemmas

In this section, we need the following preliminaries and lemmas which will be used in our main theorem.

Let E be a real Banach space and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if, for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function $\delta : [0, 2] \to [0, 1]$ called the *modulus of convexity* of E as follows:

$$\delta(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| : \ x, y \in E, \ \|x\| = \|y\| = 1, \ \|x - y\| \geq \varepsilon \right\}. \tag{2.3}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\} \tag{2.5}$$

for all $x \in E$. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E (see [35]).

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.6}$$

Remark 2.1. We know the following: for any $x, y, z \in E$,

(1) $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$;

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;

(3) $\phi(x, y) = \|x - y\|^2$ in a real Hilbert space.

Lemma 2.2 (see [36]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty}\phi(x_n, y_n) = 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty closed and convex subset of E. The *generalized projection mapping*, introduced by Alber [37], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem:

$$\phi(\overline{x}, x) = \min\{\phi(y, x) : y \in C\}. \tag{2.7}$$

In fact, we have the following result.

Lemma 2.3 (see [37]). *Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Lemma 2.4 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E, $x \in E$, and $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle Jx - Jz, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.8}$$

Lemma 2.5 (see [36, 37]). *Let C be a nonempty closed and convex subset of a reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.9}$$

Lemma 2.6 (see [38]). *Let E be a uniformly convex and uniformly smooth Banach space and C a nonempty, closed, and convex subset of E. Then Π_C is uniformly norm-to-norm continuous on every bounded set.*

We make use of the following mapping V studied in Alber [37]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.10}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.7 (see [39]). *Let E be a reflexive, strictly convex, smooth Banach space. Then*

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, y^* \right\rangle \leq V(x, x^* + y^*) \tag{2.11}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.8 (see [25]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)–(A4), *and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq 0, \quad \forall y \in C. \tag{2.12}$$

Following [25, 40], we know the following lemma.

Lemma 2.9 (see [41]). *Let C be a nonempty closed and convex subset of a smooth, strictly convex, and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z), \quad \forall y \in C. \tag{2.13}$$

Define the mapping $T_r : E \to 2^C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \langle Az, y - z \rangle + \varphi(y) + \frac{1}{r}\langle Jz - Jx, y - z \rangle \geq \varphi(z),\ \forall y \in C \right\}. \tag{2.14}$$

Then, the followings hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive-type mapping [42], that is, for all* $x, y \in E$*,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.15}$$

(3) $F(T_r) = \mathrm{GMEP}(f, A, \varphi)$*;*

(4) $\mathrm{GMEP}(f, A, \varphi)$ *is closed and convex.*

Remark 2.10. It is known that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x) \tag{2.16}$$

for all $x, y \in \operatorname{dom} T$ (see [42]).

The following lemmas give us some nice properties of real sequences.

Lemma 2.11 (see [43]). *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n, \quad \forall n \geq 1, \tag{2.17}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{b_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$*;*

(b) $\limsup_{n\to\infty} b_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < +\infty$*.*

Then $\lim_{n\to\infty} a_n = 0$*.*

Lemma 2.12 (see [44]). *Let* $\{\gamma_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{\gamma_{n_j}\}$ *of* $\{\gamma_n\}$ *such that* $\gamma_{n_j} < \gamma_{n_j+1}$ *for all* $j \geq 1$*. Then there exists a nondecreasing sequence* $\{m_k\}$ *of* $\mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \geq 1$*:*

$$\gamma_{m_k} \leq \gamma_{m_k+1}, \quad \gamma_k \leq \gamma_{m_k+1}. \tag{2.18}$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, \ldots, k\}$ *such that the condition* $\gamma_n < \gamma_{n+1}$ *holds.*

3. Main Results

In this section, we prove our main theorem in this paper. To this end, we need the following proposition.

Proposition 3.1. *Let* C *be a nonempty closed and convex subset of a reflexive, strictly convex, and uniformly smooth Banach space* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)–(A4), $A : C \to E^*$ *a continuous and monotone mapping, and* φ *a lower semicontinuous and convex function* *from C to $\mathbb{R}$ such that* $\mathrm{GMEP}(f, A, \varphi) \neq \emptyset$. *Let* $\{r_n\} \subset (0, \infty)$ *be such that* $\liminf_{n\to\infty} r_n > 0$. *For each* $n \geq 1$, *let* T_{r_n} *be defined as in Lemma 2.9. Suppose that* $x \in C$ *and* $\{x_n\}$ *is a bounded sequence in* C *such that* $\lim_{n\to\infty}\|x_n - T_{r_n}x_n\| = 0$. *Then*

$$\limsup_{n\to\infty}\langle Jx - Jp, x_n - p\rangle \leq 0, \tag{3.1}$$

where $p = \Pi_{\mathrm{GMEP}(f,A,\varphi)}x$ *and* $\Pi_{\mathrm{GMEP}(f,A,\varphi)}$ *is the generalized projection of* C *onto* $\mathrm{GMEP}(f, A, \varphi)$.

Proof. Let $x \in C$ and put $p = \Pi_{\mathrm{GMEP}(f,A,\varphi)}x$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup v \in C$ and

$$\limsup_{n\to\infty}\langle Jx - Jp, x_n - p\rangle = \langle Jx - Jp, v - p\rangle. \tag{3.2}$$

Put $y_n = T_{r_n}x_n$. Since $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$, we have $y_{n_k} \rightharpoonup v$. On the other hand, since E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded subsets of E. So we have

$$\lim_{k\to\infty}\|Jx_{n_k} - Jy_{n_k}\| = 0. \tag{3.3}$$

Since $\liminf_{k\to\infty} r_{n_k} > 0$,

$$\lim_{k\to\infty}\frac{\|Jx_{n_k} - Jy_{n_k}\|}{r_{n_k}} = 0. \tag{3.4}$$

By the definition of $T_{r_{n_k}}$, for any $y \in C$, we see that

$$f(y_{n_k}, y) + \langle Ay_{n_k}, y - y_{n_k}\rangle + \varphi(y) + \frac{1}{r_{n_k}}\langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k}\rangle \geq \varphi(y_{n_k}). \tag{3.5}$$

By (A2), for each $y \in C$, we obtain

$$\begin{aligned} f(y, y_{n_k}) + \varphi(y_{n_k}) &\leq -f(y_{n_k}, y) + \varphi(y_{n_k}) \\ &\leq \langle Ay_{n_k}, y - y_{n_k}\rangle + \varphi(y) + \frac{1}{r_{n_k}}\langle Jy_{n_k} - Jx_{n_k}, y - y_{n_k}\rangle. \end{aligned} \tag{3.6}$$

For any $t \in (0, 1)$ and $y \in C$, we define $y_t = ty + (1 - t)v$. Then $y_t \in C$. It follows by the monotonicity of A that

$$\begin{aligned} f(y_t, y_{n_k}) + \varphi(y_{n_k}) &\leq \langle Ay_{n_k} - Ay_t, y_t - y_{n_k}\rangle + \langle Ay_t, y_t - y_{n_k}\rangle \\ &\quad + \varphi(y_t) + \frac{1}{r_{n_k}}\langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k}\rangle \\ &\leq \langle Ay_t, y_t - y_{n_k}\rangle + \varphi(y_t) + \frac{1}{r_{n_k}}\langle Jy_{n_k} - Jx_{n_k}, y_t - y_{n_k}\rangle. \end{aligned} \tag{3.7}$$

By (A4), (3.4), and the weakly lower semicontinuity of φ, letting $k \to \infty$, we obtain

$$f(y_t, v) + \varphi(v) \le \langle Ay_t, y_t - v\rangle + \varphi(y_t). \tag{3.8}$$

By (A1), (A4), and the convexity of φ, we have

$$\begin{aligned} 0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\le tf(y_t, y) + (1-t)f(y_t, v) + t\varphi(y) + (1-t)\varphi(v) - \varphi(y_t) \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)(f(y_t, v) + \varphi(v) - \varphi(y_t)) \\ &\le t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle Ay_t, y_t - v\rangle \\ &= t(f(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle Ay_t, y - v\rangle. \end{aligned} \tag{3.9}$$

It follows that

$$f(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle Ay_t, y - v\rangle \ge 0. \tag{3.10}$$

By (A3), the weakly lower semicontinuity of φ, and the continuity of A, letting $t \to 0$, we obtain

$$f(v, y) + \varphi(y) - \varphi(v) + \langle Av, y - v\rangle \ge 0, \quad \forall y \in C. \tag{3.11}$$

This shows that $v \in \mathrm{GMEP}(f, A, \varphi)$. By Lemma 2.4, we have

$$\limsup_{n\to\infty}\langle Jx - Jp, x_n - p\rangle = \langle Jx - Jp, v - p\rangle \le 0. \tag{3.12}$$

This completes the proof. □

Theorem 3.2. *Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *$A : C \to E^*$ a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that $\mathrm{GMEP}(f, A, \varphi) \ne \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$\begin{gathered} f(y_n, y) + \langle Ay_n, y - y_n\rangle + \varphi(y) + \frac{1}{r_n}\langle Jy_n - Jx_n, y - y_n\rangle \ge \varphi(y_n), \quad \forall y \in C, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n), \quad \forall n \ge 1, \end{gathered} \tag{3.13}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\lim\inf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_{\mathrm{GMEP}(f,A,\varphi)}x$, *where* $\Pi_{\mathrm{GMEP}(f,A,\varphi)}$ *is the generalized projection of* C *onto* $\mathrm{GMEP}(f,A,\varphi)$.

Proof. From Lemma 2.9(4), we know that $\mathrm{GMEP}(f,A,\varphi)$ is closed and convex. Let $p = \Pi_{\mathrm{GMEP}(f,A,\varphi)}x$. Put $y_n = T_{r_n}x_n$ and $z_n = J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n)$ for all $n \in \mathbb{N}$. So, by Lemma 2.5, we have

$$\begin{aligned}\phi(p, x_{n+1}) &\leq \phi(p, z_n)\\ &\leq \alpha_n\phi(p, x) + (1-\alpha_n)\phi(p, y_n)\\ &\leq \alpha_n\phi(p, x) + (1-\alpha_n)\phi(p, x_n).\end{aligned} \tag{3.14}$$

By induction, we can show that $\phi(p, x_n) \leq \phi(p, x)$ for each $n \in \mathbb{N}$. Hence $\{\phi(p, x_n)\}$ is bounded and thus $\{x_n\}$ is also bounded.

We next show that if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\left(\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})\right) = 0, \tag{3.15}$$

then

$$\lim_{k\to\infty}\left(\phi(p, y_{n_k}) - \phi(p, x_{n_k})\right) = 0. \tag{3.16}$$

Since $\alpha_{n_k} \to 0$,

$$\lim_{k\to\infty}\|Jz_{n_k} - Jy_{n_k}\| = \lim_{k\to\infty}\alpha_{n_k}\|Jx - Jy_{n_k}\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, so is J^{-1}. It follows that

$$\lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0. \tag{3.18}$$

Since E is uniformly smooth and uniformly convex, by Lemma 2.6, Π_C is uniformly norm-to-norm continuous on bounded sets. So we obtain

$$\lim_{k\to\infty}\|x_{n_k+1} - y_{n_k}\| = \lim_{k\to\infty}\|\Pi_C z_{n_k} - \Pi_C y_{n_k}\| = 0, \tag{3.19}$$

and hence

$$\lim_{k\to\infty}\|Jx_{n_k+1} - Jy_{n_k}\| = 0. \tag{3.20}$$

Furthermore, $\lim_{k\to\infty}\phi(x_{n_k+1}, y_{n_k}) = 0$. Indeed, by the definition of ϕ, we observe that

$$\begin{aligned}\phi(x_{n_k+1}, y_{n_k}) &= \|x_{n_k+1}\|^2 - 2\langle x_{n_k+1}, Jy_{n_k}\rangle + \|y_{n_k}\|^2 \\ &= \langle x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k}\rangle + \langle y_{n_k} - x_{n_k+1}, Jy_{n_k}\rangle.\end{aligned} \tag{3.21}$$

It follows from (3.19) and (3.20) that $\lim_{k\to\infty}\phi(x_{n_k+1}, y_{n_k}) = 0$. On the other hand, from Remark 2.1(2), we have

$$\begin{aligned}\phi(p, y_{n_k}) - \phi(p, x_{n_k}) &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) + (\phi(p, y_{n_k}) - \phi(p, x_{n_k+1})) \\ &= (\phi(p, x_{n_k+1}) - \phi(p, x_{n_k})) \\ &\quad + \phi(x_{n_k+1}, y_{n_k}) + 2\langle p - x_{n_k+1}, Jx_{n_k+1} - Jy_{n_k}\rangle.\end{aligned} \tag{3.22}$$

It follows from (3.20) and (3.21) that $\lim_{k\to\infty}(\phi(p,y_{n_k})-\phi(p,x_{n_k}))=0.$

We next consider the following two cases.

Case 1. $\phi(p, x_{n+1}) \le \phi(p, x_n)$ for all sufficiently large n. Hence the sequence $\{\phi(p, x_n)\}$ is bounded and nonincreasing. So $\lim_{n\to\infty}\phi(p, x_n)$ exists. This shows that $\lim_{n\to\infty}(\phi(p, x_{n+1}) - \phi(p, x_n)) = 0$ and hence

$$\lim_{n\to\infty}(\phi(p, y_n) - \phi(p, x_n)) = 0. \tag{3.23}$$

Since T_{r_n} is of firmly nonexpansive type, by Remark 2.10, we have

$$\phi(y_n, p) + \phi(p, y_n) + \phi(y_n, x_n) + \phi(T_{r_n}p, p) \le \phi(y_n, p) + \phi(p, x_n), \tag{3.24}$$

which implies

$$\phi(p, y_n) + \phi(y_n, x_n) \le \phi(p, x_n). \tag{3.25}$$

Hence

$$\phi(y_n, x_n) \le \phi(p, x_n) - \phi(p, y_n) \longrightarrow 0 \tag{3.26}$$

as $n \to \infty$. By Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.27}$$

Proposition 3.1 yields that

$$\limsup_{n\to\infty}\langle Jx - Jp, x_n - p\rangle \le 0. \tag{3.28}$$

It also follows that

$$\limsup_{n\to\infty}\langle Jx - Jp, y_n - p\rangle \le 0. \tag{3.29}$$

Finally, we show that $x_n \to p$. Using Lemma 2.7, we see that

$$\begin{aligned}
\phi(p, x_{n+1}) &\le \phi(p, z_n) \\
&= V(p, \alpha_n Jx + (1-\alpha_n)Jy_n) \\
&\le V(p, \alpha_n Jx + (1-\alpha_n)Jy_n - \alpha_n(Jx - Jp)) + \langle \alpha_n(Jx - Jp), z_n - p\rangle \\
&= V(p, \alpha_n Jp + (1-\alpha_n)Jy_n) + \alpha_n\langle Jx - Jp, z_n - p\rangle \\
&\le \alpha_n V(p, Jp) + (1-\alpha_n)V(p, Jy_n) + \alpha_n\langle Jx - Jp, z_n - p\rangle \\
&= (1-\alpha_n)\phi(p, y_n) + \alpha_n\langle Jx - Jp, z_n - p\rangle \\
&\le (1-\alpha_n)\phi(p, x_n) + \alpha_n\langle Jx - Jp, z_n - p\rangle \\
&= (1-\alpha_n)\phi(p, x_n) + \alpha_n(\langle Jx - Jp, z_n - y_n\rangle + \langle Jx - Jp, y_n - p\rangle).
\end{aligned} \tag{3.30}$$

Set $a_n = \phi(p, x_n)$ and $b_n = \alpha_n(\langle Jx - Jp, z_n - y_n\rangle + \langle Jx - Jp, y_n - p\rangle)$. We see that $\limsup_{n\to\infty} b_n/\alpha_n \le 0$. By Lemma 2.11, since $\sum_{n=1}^{\infty}\alpha_n = +\infty$, we conclude that $\lim_{n\to\infty}\phi(p, x_n) = 0$. Hence $x_n \to p$ as $n \to \infty$.

Case 2. There exists a subsequence $\{\phi(p, x_{n_j})\}$ of $\{\phi(p, x_n)\}$ such that $\phi(p, x_{n_j}) < \phi(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. By Lemma 2.12, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\phi(p, x_{m_k}) \le \phi(p, x_{m_k+1}), \quad \phi(p, x_k) \le \phi(p, x_{m_k+1}). \tag{3.31}$$

So we have

$$\begin{aligned}
0 &\le \lim_{k\to\infty}(\phi(p, x_{m_k+1}) - \phi(p, x_{m_k})) \\
&\le \limsup_{n\to\infty}(\phi(p, x_{n+1}) - \phi(p, x_n)) \\
&\le \limsup_{n\to\infty}(\phi(p, z_n) - \phi(p, x_n)) \\
&\le \limsup_{n\to\infty}(\alpha_n\phi(p, x) + (1-\alpha_n)\phi(p, y_n) - \phi(p, x_n)) \\
&= \limsup_{n\to\infty}(\alpha_n(\phi(p, x) - \phi(p, y_n)) + (\phi(p, y_n) - \phi(p, x_n))) \\
&\le \limsup_{n\to\infty}\alpha_n(\phi(p, x) - \phi(p, y_n)) = 0.
\end{aligned} \tag{3.32}$$

This shows that

$$\lim_{k\to\infty}\left(\phi(p,x_{m_k+1})-\phi(p,x_{m_k})\right)=0. \tag{3.33}$$

Following the proof line in Case 1, we can show that

$$\begin{gathered}\limsup_{k\to\infty}\langle Jx-Jp,y_{m_k}-p\rangle\le 0,\\ \phi(p,x_{m_k+1})\le(1-\alpha_{m_k})\phi(p,x_{m_k})+\alpha_{m_k}\left(\langle Jx-Jp,z_{m_k}-y_{m_k}\rangle+\langle Jx-Jp,y_{m_k}-p\rangle\right).\end{gathered} \tag{3.34}$$

This implies

$$\begin{aligned}\alpha_{m_k}\phi(p,x_{m_k})&\le\phi(p,x_{m_k})-\phi(p,x_{m_k+1})\\ &\quad+\ \alpha_{m_k}\left(\langle Jx-Jp,z_{m_k}-y_{m_k}\rangle+\langle Jx-Jp,y_{m_k}-p\rangle\right)\\ &\le\alpha_{m_k}\left(\langle Jx-Jp,z_{m_k}-y_{m_k}\rangle+\langle Jx-Jp,y_{m_k}-p\rangle\right).\end{aligned} \tag{3.35}$$

Hence $\lim_{k\to\infty}\phi(p,x_{m_k})=0$. Using this and (3.33) together, we conclude that

$$\limsup_{k\to\infty}\phi(p,x_k)\le\lim_{k\to\infty}\phi(p,x_{m_k+1})=0. \tag{3.36}$$

This completes the proof. □

As a direct consequence of Theorem 3.2, we obtain the following results.

Corollary 3.3. *Let C be nonempty closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that* $\mathrm{MEP}(f,\varphi)\neq\emptyset$*. Define the sequence $\{x_n\}$ as follows: $x_1=x\in C$ and*

$$\begin{gathered}f(y_n,y)+\varphi(y)+\frac{1}{r_n}\langle Jy_n-Jx_n,y-y_n\rangle\ge\varphi(y_n),\quad\forall y\in C,\\ x_{n+1}=\Pi_C J^{-1}(\alpha_n Jx+(1-\alpha_n)Jy_n),\quad\forall n\ge 1,\end{gathered} \tag{3.37}$$

where $\{\alpha_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfy the following conditions:

(a) $\lim_{n\to\infty}\alpha_n=0$;

(b) $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(c) $\liminf_{n\to\infty}r_n>0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{MEP}(f,\varphi)}x$, where $\Pi_{\mathrm{MEP}(f,\varphi)}$ is the generalized projection of C onto $\mathrm{MEP}(f,\varphi)$.

Corollary 3.4. *Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *and $A : C \to E^*$ a continuous and monotone mapping such that* $\mathrm{GEP}(f, A) \neq \emptyset$. *Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$\begin{aligned} &f(y_n, y) + \langle Ay_n, y - y_n\rangle + \frac{1}{r_n}\langle Jy_n - Jx_n, y - y_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \Pi_C J^{-1}\big(\alpha_n Jx + (1-\alpha_n)Jy_n\big), \quad \forall n \geq 1, \end{aligned} \tag{3.38}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{GEP}(f,A)}x$, where $\Pi_{\mathrm{GEP}(f,A)}$ is the generalized projection of C onto $\mathrm{GEP}(f, A)$.

Corollary 3.5. *Let C be nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, and φ a lower semicontinuous and convex function from C to $\mathbb{R}$ such that* $\mathrm{VI}(C, A, \varphi) \neq \emptyset$. *Define the sequence $\{x_n\}$ as follows: $x_1 = x \in C$ and*

$$\begin{aligned} &\langle Ay_n, y - y_n\rangle + \varphi(y) + \frac{1}{r_n}\langle Jy_n - Jx_n, y - y_n\rangle \geq \varphi(y_n), \quad \forall y \in C, \\ &x_{n+1} = \Pi_C J^{-1}\big(\alpha_n Jx + (1-\alpha_n)Jy_n\big), \quad \forall n \geq 1, \end{aligned} \tag{3.39}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{VI}(C,A,\varphi)}x$, where $\Pi_{\mathrm{VI}(C,A,\varphi)}$ is the generalized projection of C onto $\mathrm{VI}(C, A, \varphi)$.

4. Examples and Numerical Results

In this section, we give examples and numerical results for our main theorem.

Example 4.1. Let $E = \mathbb{R}$ *and* $C = [-1, 1]$. Let $f(x, y) = -9x^2 + xy + 8y^2$, $\varphi(x) = 3x^2$, and $Ax = 2x$. Find $\widehat{x} \in [-1, 1]$ such that

$$f(\widehat{x}, y) + \langle A\widehat{x}, y - \widehat{x}\rangle + \varphi(y) \geq \varphi(\widehat{x}), \quad \forall y \in [-1, 1]. \tag{4.1}$$

Solution. It is easy to check that f, φ, and A satisfy all conditions in Theorem 3.2. For each $r > 0$ and $x \in [-1,1]$, Lemma 2.9 ensures that there exists $z \in [-1,1]$ such that, for any $y \in [-1,1]$,

$$\begin{aligned}
&f(z,y) + \langle Az, y-z\rangle + \varphi(y) + \frac{1}{r}\langle z-x, y-z\rangle \geq \varphi(z) \\
&\Longleftrightarrow -9z^2 + yz + 8y^2 + 2z(y-z) + 3y^2 + \frac{1}{r}(z-x)(y-z) \geq 3z^2 \\
&\Longleftrightarrow 11ry^2 + (3rz + z - x)y - \left(14rz^2 + z^2 - xz\right) \geq 0.
\end{aligned} \tag{4.2}$$

Put $G(y) = 11ry^2 + (3rz + z - x)y - (14rz^2 + z^2 - xz)$. Then G is a quadratic function of y with coefficient $a = 11r$, $b = (3rz + z - x)$, and $c = -(14rz^2 + z^2 - xz)$. We next compute the discriminant Δ of G as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= [(3r+1)z - x]^2 + 44r\left(14rz^2 + z^2 - xz\right) \\
&= x^2 - 2(3r+1)xz + (3r+1)^2z^2 + 616r^2z^2 + 44rz^2 - 44rxz \\
&= x^2 - 50rxz - 2xz + 625r^2z^2 + 50rz^2 + z^2 \\
&= x^2 - 2(25rz + z)x + \left(625r^2z^2 + 50rz^2 + z^2\right) \\
&= [x - (25rz + z)]^2.
\end{aligned} \tag{4.3}$$

We know that $G(y) \geq 0$ for all $y \in [-1,1]$ if it has at most one solution in $[-1,1]$. So $\Delta \leq 0$ and hence $x = 25rz + z$. Now we have $z = T_r x = x/(25r+1)$.

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by $x_1 = x \in [-1,1]$ and

$$\begin{aligned}
&f(y_n, y) + \langle Ay_n, y - y_n\rangle + \varphi(y) + \frac{1}{r_n}\langle y_n - x_n, y - y_n\rangle \geq \varphi(y_n), \quad \forall y \in [-1,1], \\
&x_{n+1} = \alpha_n x + (1-\alpha_n)y_n, \quad \forall n \geq 1,
\end{aligned} \tag{4.4}$$

and, equivalently,

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T_{r_n}x_n, \quad \forall n \geq 1. \tag{4.5}$$

We next give two numerical results for algorithm (4.5).

Algorithm 4.2. Let $\alpha_n = 1/80n$ and $r_n = n/(n+1)$. Choose $x_1 = x = 1$. Then algorithm (4.5) becomes

$$x_{n+1} = \frac{1}{80n} + \left(1 - \frac{1}{80n}\right)\left(\frac{n+1}{26n+1}\right)x_n, \quad \forall n \geq 1. \tag{4.6}$$

Table 1

n	x_n
1	1.0000
2	0.0856
3	0.0111
4	0.0047
5	0.0033
$\vdots$	$\vdots$
261	0.0001
262	0.0000

Table 2

n	x_n
1	−1.0000
2	−0.0481
3	−0.0074
4	−0.0038
5	−0.0027
$\vdots$	$\vdots$
217	−0.0001
218	0.0000

Numerical Result I

See Table 1.

Algorithm 4.3. Let $\alpha_n = 1/100n$ and $r_n = (n+1)/2n$. Choose $x_1 = x = -1$. Then algorithm (4.5) becomes

$$x_{n+1} = -\frac{1}{100n} + \left(1 - \frac{1}{100n}\right)\left(\frac{2n}{27n+25}\right)x_n, \quad \forall n \geq 1. \tag{4.7}$$

Numerical Result II

See Table 2.

5. Conclusion

Tables 1 and 2 show that the sequence $\{x_n\}$ converges to 0 which solves the generalized mixed equilibrium problem. On the other hand, using Lemma 2.9(3), we can check that $\mathrm{GMEP}(f, A, \varphi) = F(T_r) = \{0\}$.

Remark 5.1. In the view of computation, our algorithm is simple in order to get strong convergence for generalized mixed equilibrium problems.

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