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Research Article

Existence of Positive Solutions of Neutral Differential Equations

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The paper contains some suffcient conditions for the existence of positive solutions which are bounded below and above by positive functions for the nonlinear neutral differential equations of higher order. These equations can also support the existence of positive solutions approaching zero at infinity.

1. Introduction

This paper is concerned with the existence of a positive solution of the neutral differential equations of the form:

$$\frac{d^n}{dt^n}[x(t)-a(t)x(t-\tau)]=(-1)^{n+1}p(t)f(x(t-\sigma)),\quad t\geq t_0,\tag{1.1}$$

where $n>0$ is an integer, $\tau>0$, $\sigma\geq 0$, $a\in C([t_0,\infty),(0,\infty))$, $p\in C(R,(0,\infty))$, $f\in C(R,R)$, f is a nondecreasing function and $xf(x)>0$, $x\neq 0$.

By a solution of (1.1) we mean a function $x\in C([t_1-\tau,\infty),R)$ for some $t_1\geq t_0$, such that $x(t)-a(t)x(t-\tau)$ is n-times continuously differentiable on $[t_1,\infty)$ and such that (1.1) is satisfied for $t\geq t_1$.

The problem of the existence of solutions of neutral differential equations has been studied and discussed by several authors in the recent years. For related results we refer the reader to [1–17] and the references cited therein. However, there is no conception which guarantees the existence of positive solutions which are bounded below and above by positive functions. Maybe it is due to the technical difficulties arising in the analysis of the problem. In this paper we presented some conception. The method also supports the existence of positive solutions which approaching zero at infinity. Some examples illustrating the results.

The existence and asymptotic behavior of solutions of the nonlinear neutral differential equations and systems have been also solved in [1–7, 12, 15].

As much as we know for (1.1) in the literature, there is no result for the existence of solutions which are bounded by positive functions. Only the existence of solutions which are bounded by constants is treated and discussed, for example, in [10, 15, 17]. It seems that conditions of theorems are rather complicate, but cannot be simpler due to Corollaries 2.4, 2.8, and 3.3.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 (see [7, 10, 12] Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contractive and S_2 is completely continuous then the equation:*

$$S_1x + S_2x = x \tag{1.2}$$

has a solution in Ω.

2. The Existence of Positive Solution

In this section, we will consider the existence of a positive solution for (1.1) which is bounded by two positive functions. We will use the notation $m = \max\{\tau, \sigma\}$.

Theorem 2.1. *Suppose that there exist bounded functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$, and $t_1 \geq t_0 + m$ such that*

$$u(t) \leq v(t), \quad t \geq t_0, \tag{2.1}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1, \tag{2.2}$$

$$\begin{aligned} &\frac{1}{u(t-\tau)}\left(u(t) + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)f(v(s-\sigma))ds\right) \\ &\quad\leq a(t) \leq \frac{1}{v(t-\tau)}\left(v(t) + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)f(u(s-\sigma))ds\right) \\ &\quad\leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.3}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. Let $C([t_0, \infty), R)$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t \geq t_0}|x(t)|$. Then $C([t_0, \infty), R)$ is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), R) : u(t) \leq x(t) \leq v(t), \ t \geq t_0\}. \tag{2.4}$$

We now define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty), R)$ as follows:

$$(S_1x)(t) = \begin{cases} a(t)x(t-\tau), & t \geq t_1, \\ (S_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$
$$(S_2x)(t) = \begin{cases} -\dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}p(s)f(x(s-\sigma))ds, & t \geq t_1, \\ (S_2x)(t_1) + v(t) - v(t_1), & t_0 \leq t \leq t_1. \end{cases} \tag{2.5}$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$. For every $x, y \in \Omega$ and $t \geq t_1$ we obtain

$$(S_1x)(t) + (S_2y)(t) \leq a(t)v(t-\tau) - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)f(u(s-\sigma))ds \leq v(t). \tag{2.6}$$

For $t \in [t_0, t_1]$ we have

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\leq v(t_1) + v(t) - v(t_1) = v(t).\end{aligned} \tag{2.7}$$

Furthermore for $t \geq t_1$ we get

$$(S_1x)(t) + (S_2y)(t) \geq a(t)u(t-\tau) - \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}p(s)f(v(s-\sigma))ds \geq u(t). \tag{2.8}$$

Finally let $t \in [t_0, t_1]$ and with regard to (2.2) we get

$$v(t) - v(t_1) + u(t_1) \geq u(t), \quad t_0 \leq t \leq t_1. \tag{2.9}$$

Then for $t \in [t_0, t_1]$ and any $x, y \in \Omega$ we get

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\geq u(t_1) + v(t) - v(t_1) \geq u(t).\end{aligned} \tag{2.10}$$

Thus, we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq t_1$ we have

$$\left|(S_1x)(t) - (S_1y)(t)\right| = |a(t)|\left|x(t-\tau) - y(t-\tau)\right| \leq c\|x - y\|. \tag{2.11}$$

This implies that

$$\|S_1x - S_1y\| \leq c\|x - y\|. \tag{2.12}$$

Also for $t \in [t_0, t_1]$ the inequality above is valid. We conclude that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$\begin{aligned} &|(S_2x_k)(t) - (S_2x)(t)| \\ &\quad \leq \frac{1}{(n-1)!}\left|\int_t^\infty (s-t)^{n-1}p(s)\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right]ds\right| \\ &\quad \leq \frac{1}{(n-1)!}\int_{t_1}^\infty (s-t_1)^{n-1}p(s)\left|f(x_k(s-\sigma)) - f(x(s-\sigma))\right|ds. \end{aligned} \tag{2.13}$$

According to (2.8) we get

$$\int_{t_1}^\infty (s-t_1)^{n-1}p(s)f(v(s-\sigma))ds < \infty. \tag{2.14}$$

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k\to\infty} \|(S_2x_k)(t) - (S_2x)(t)\| = 0. \tag{2.15}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show, according to Levitan result [8], that for any given $\varepsilon > 0$ the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. With regard to the condition (2.14), for $x \in \Omega$ and any $\varepsilon > 0$ we take $t^* \geq t_1$ large enough so that

$$\frac{1}{(n-1)!}\int_{t^*}^\infty (s-t_1)^{n-1}p(s)f(x(s-\sigma))ds < \frac{\varepsilon}{2}. \tag{2.16}$$

Then for $x \in \Omega$, $T_2 > T_1 \geq t^*$ we have

$$\begin{aligned} &|(S_2x)(T_2) - (S_2x)(T_1)| \\ &\quad \leq \frac{1}{(n-1)!}\int_{T_2}^\infty (s-t_1)^{n-1}p(s)f(x(s-\sigma))ds + \frac{1}{(n-1)!}\int_{T_1}^\infty (s-t_1)^{n-1}p(s)f(x(s-\sigma))ds \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{2.17}$$

For $x \in \Omega$, $t_1 \le T_1 < T_2 \le t^*$ and $n \ge 2$ we get

$$\begin{aligned}
|(S_2x)(T_2) - (S_2x)(T_1)| &= \frac{1}{(n-1)!}\left|\int_{T_1}^{\infty} (s-T_1)^{n-1} p(s) f(x(s-\sigma))ds \right. \\
&\qquad \left. - \int_{T_2}^{\infty} (s-T_2)^{n-1} p(s) f(x(s-\sigma))ds \right| \\
&= \frac{1}{(n-1)!}\left|\int_{T_1}^{T_2} (s-T_1)^{n-1} p(s) f(x(s-\sigma))ds \right. \\
&\qquad + \int_{T_2}^{\infty} (s-T_1)^{n-1} p(s) f(x(s-\sigma))ds \\
&\qquad \left. - \int_{T_2}^{\infty} (s-T_2)^{n-1} p(s) f(x(s-\sigma))ds \right| \\
&\le \frac{1}{(n-1)!}\int_{T_1}^{T_2} s^{n-1} p(s) f(x(s-\sigma))ds \\
&\quad + \frac{1}{(n-1)!}\int_{T_2}^{\infty} \left[(s-T_1)^{n-1} - (s-T_2)^{n-1}\right] p(s) f(x(s-\sigma))ds \\
&\le \max_{t_1 \le s \le t^*}\left\{ \frac{1}{(n-1)!} s^{n-1} p(s) f(x(s-\sigma))\right\}(T_2 - T_1) \\
&\quad + \frac{1}{(n-1)!}\int_{T_2}^{\infty} [(s-T_1) - (s-T_2)] \\
&\quad \times \left[(s-T_1)^{n-2} + (s-T_1)^{n-3}(s-T_2) + \cdots + (s-T_1)(s-T_2)^{n-3} \right. \\
&\qquad \left. + (s-T_2)^{n-2}\right] p(s) f(x(s-\sigma))ds \\
&\le \max_{t_1 \le s \le t^*}\left\{ \frac{1}{(n-1)!} s^{n-1} p(s) f(x(s-\sigma))\right\}(T_2 - T_1) \\
&\quad + \frac{1}{(n-2)!}\int_{T_2}^{\infty} (T_2 - T_1)(s-T_1)^{n-2} p(s) f(x(s-\sigma))ds.
\end{aligned} \tag{2.18}$$

With regard to the condition (2.14) we have that

$$\frac{1}{(n-2)!}\int_{T_2}^{\infty} (s-T_1)^{n-2} p(s) f(x(s-\sigma))ds < B, \quad B > 0. \tag{2.19}$$

Then we obtain

$$|(S_2x)(T_2)-(S_2x)(T_1)| < \left(\max_{t_1\le s\le t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}+B\right)(T_2-T_1). \quad (2.20)$$

Thus there exists a $\delta_1 = \varepsilon/(M+B)$, where

$$M = \max_{t_1\le s\le t^*}\left\{\frac{1}{(n-1)!}s^{n-1}p(s)f(x(s-\sigma))\right\}, \quad (2.21)$$

such that

$$|(S_2x)(T_2)-(S_2x)(T_1)| < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_1. \quad (2.22)$$

For $n = 1$ we proceed by the similar way as above. Finally for any $x \in \Omega$, $t_0 \le T_1 < T_2 \le t_1$ there exists a $\delta_2 > 0$ such that

$$\begin{aligned} |(S_2x)(T_2)-(S_2x)(T_1)| = |v(T_1)-v(T_2)| &= \left|\int_{T_1}^{T_2} v'(s)ds\right| \\ &\le \max_{t_0\le s\le t_1}\{|v'(s)|\}(T_2-T_1) < \varepsilon \quad \text{if } 0 < T_2-T_1 < \delta_2. \end{aligned} \quad (2.23)$$

Then $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$ and hence $S_2\Omega$ is relatively compact subset of $C([t_0,\infty),R)$. By Lemma 1.1 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.2. *Suppose that all conditions of Theorem 2.1 are satisfied and*

$$\lim_{t\to\infty} v(t) = 0. \quad (2.24)$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 2.3. *Suppose that there exist bounded functions $u, v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \ge t_0 + m$ such that* (2.1), (2.3) *hold and*

$$v'(t) - u'(t) \le 0, \quad t_0 \le t \le t_1. \quad (2.25)$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. We only need to prove that condition (2.25) implies (2.2). Let $t \in [t_0, t_1]$ and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1). \quad (2.26)$$

Then with regard to (2.25), it follows that $H'(t) = v'(t) - u'(t) \leq 0$, $t_0 \leq t \leq t_1$. Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, this implies that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1. \tag{2.27}$$

Thus all conditions of Theorem 2.1 are satisfied. □

Corollary 2.4. *Suppose that there exists a bounded function $v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that*

$$a(t) = \frac{1}{v(t-\tau)} \left(v(t) + \frac{1}{(n-1)!} \int_t^{\infty} (s-t)^{n-1} p(s) f(v(s-\sigma)) ds \right) \leq c < 1, \quad t \geq t_1. \tag{2.28}$$

Then (1.1) *has a solution $x(t) = v(t)$, $t \geq t_1$.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.1. □

Theorem 2.5. *Suppose that p is bounded and there exist bounded functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.2) *hold and*

$$\begin{aligned} &\frac{1}{u(t-\tau)} \left(u(t) - \frac{1}{(n-1)!} \int_{t_1}^{t} (t-s)^{n-1} p(s) f(u(s-\sigma)) ds \right) \\ &\quad \leq a(t) \leq \frac{1}{v(t-\tau)} \left(v(t) - \frac{1}{(n-1)!} \int_{t_1}^{t} (t-s)^{n-1} p(s) f(v(s-\sigma)) ds \right) \\ &\quad \leq c < 1, \quad t \geq t_1, \end{aligned} \tag{2.29}$$

if n is odd,

$$\begin{aligned} &\frac{1}{u(t-\tau)} \left(u(t) + \frac{1}{(n-1)!} \int_{t_1}^{t} (t-s)^{n-1} p(s) f(v(s-\sigma)) ds \right) \\ &\quad \leq a(t) \leq \frac{1}{v(t-\tau)} \left(v(t) + \frac{1}{(n-1)!} \int_{t_1}^{t} (t-s)^{n-1} p(s) f(u(s-\sigma)) ds \right) \\ &\quad \leq c < 1, \quad t \geq t_1, \end{aligned} \tag{2.30}$$

if n is even, and

$$\int_{t_1}^{t} (t-s)^{n-2} p(s) f(v(s-\sigma)) ds \leq K, \quad t \geq t_1, \ K > 0, \ n \geq 2. \tag{2.31}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. Let $C([t_0,\infty),R)$ be the set as in the proof of Theorem 2.1. We define a closed, bounded, and convex subset Ω of $C([t_0,\infty),R)$ as in the proof of Theorem 2.1. We define two maps S_1 and $S_2:\Omega\to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t)=\begin{cases}a(t)x(t-\tau), & t\geq t_1,\\ (S_1x)(t_1), & t_0\leq t\leq t_1,\end{cases}$$
$$(S_2x)(t)=\begin{cases}\dfrac{(-1)^{n+1}}{(n-1)!}\displaystyle\int_{t_1}^{t}(t-s)^{n-1}p(s)f(x(s-\sigma))ds, & t\geq t_1,\\ (S_2x)(t_1)+v(t)-v(t_1), & t_0\leq t\leq t_1.\end{cases} \tag{2.32}$$

We shall show that for any $x,y\in\Omega$ we have $S_1x+S_2y\in\Omega$. For n odd, every $x,y\in\Omega$ and $t\geq t_1$ we obtain

$$(S_1x)(t)+(S_2y)(t)\leq a(t)v(t-\tau)+\frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\leq v(t). \tag{2.33}$$

For $t\in[t_0,t_1]$, we have

$$\begin{aligned}(S_1x)(t)+(S_2y)(t)&=(S_1x)(t_1)+(S_2y)(t_1)+v(t)-v(t_1)\\ &\leq v(t_1)+v(t)-v(t_1)=v(t).\end{aligned} \tag{2.34}$$

Furthermore for $t\geq t_1$, we get

$$(S_1x)(t)+(S_2y)(t)\geq a(t)u(t-\tau)+\frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(u(s-\sigma))ds\geq u(t). \tag{2.35}$$

Let $t\in[t_0,t_1]$ and according to (2.2) we have

$$v(t)-v(t_1)+u(t_1)\geq u(t). \tag{2.36}$$

Then for $t\in[t_0,t_1]$ and any $x,y\in\Omega$ we get

$$\begin{aligned}(S_1x)(t)+(S_2y)(t)&=(S_1x)(t_1)+(S_2y)(t_1)+v(t)-v(t_1)\\ &\geq u(t_1)+v(t)-v(t_1)\geq u(t).\end{aligned} \tag{2.37}$$

Thus we have proved that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

For n even by the similar way as above we can prove that $S_1x+S_2y\in\Omega$ for any $x,y\in\Omega$.

As in the proof of Theorem 2.1, we can show that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First, we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$|(S_2x_k)(t) - (S_2x)(t)| \leq \frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)\left|f(x_k(s-\sigma)) - f(x(s-\sigma))\right|ds. \tag{2.38}$$

According to (2.33) there exists a positive constant M such that

$$\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds \leq M \quad \text{for } t \geq t_1. \tag{2.39}$$

The inequality above also holds for n even.

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem we obtain that

$$\lim_{k\to\infty}\|(S_2x_k)(t) - (S_2x)(t)\| = 0. \tag{2.40}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For $n \geq 2$ and with regard to (2.31) we have

$$\begin{aligned}\left|\frac{d}{dt}(S_2x)(t)\right| &= \frac{1}{(n-2)!}\int_{t_1}^{t}(t-s)^{n-2}p(s)f(x(s-\sigma))ds\\ &\leq \frac{1}{(n-2)!}\int_{t_1}^{t}(t-s)^{n-2}p(s)f(v(s-\sigma)) \leq M_1,\end{aligned} \tag{2.41}$$

and for $n = 1$ we obtain

$$\left|\frac{d}{dt}(S_2x)(t)\right| = p(t)f(v(t-\sigma)) \leq M_2, \tag{2.42}$$

for $t \geq t_1$, $M_2 > 0$ and $|(d/dt)(S_2x)(t)| = |v'(t)| \leq M_3$ for $t_0 \leq t \leq t_1$, $M_3 > 0$, which shows the equicontinuity of the family $S_2\Omega$, (cf. [7, page 265]). Hence $S_2\Omega$ is relatively compact and therefore S_2 is completely continuous. By Lemma 1.1, there is $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. Thus $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.6. *Suppose that all conditions of Theorem 2.5 are satisfied and*

$$\lim_{t\to\infty} v(t) = 0. \tag{2.43}$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 2.7. *Suppose that p is bounded and there exist bounded functions $u, v \in C^1([t_0,\infty), (0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.29), (2.30), (2.31) *hold and*

$$v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1. \tag{2.44}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. The proof is similar to that of Corollary 2.3 and we omit it. □

Corollary 2.8. *Suppose that p is bounded and there exists a bounded function $v \in C^1([t_0,\infty), (0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.31) *holds and*

$$\begin{aligned} a(t) &= \frac{1}{v(t-\tau)}\left(v(t) + \frac{(-1)^n}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}p(s)f(v(s-\sigma))ds\right) \\ &\leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.45}$$

Then (1.1) *has a solution $x(t) = v(t)$, $t \geq t_1$.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.5. □

3. Applications and Examples

In this section, we give some applications of the theorems above.

Theorem 3.1. *Suppose that $0 < k_1 \leq k_2$ and there exist $\gamma \geq 0$, $c > 0$, $t_1 \geq t_0 + m$ such that*

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} p(t)dt\right) \geq 1, \tag{3.1}$$

$$\begin{aligned} &\exp\left(-k_2\int_{t-\tau}^{t} p(s)ds\right) + \frac{1}{(n-1)!}\exp\left(k_2\int_{t_0-\gamma}^{t-\tau} p(s)ds\right) \\ &\quad\times\int_t^{\infty}(s-t)^{n-1}p(s)f\left(\exp\left(-k_1\int_{t_0-\gamma}^{s-\sigma}p(\xi)d\xi\right)\right)ds \leq a(t) \\ &\leq \exp\left(-k_1\int_{t-\tau}^{t} p(s)ds\right) + \frac{1}{(n-1)!}\exp\left(k_1\int_{t_0-\gamma}^{t-\tau} p(s)ds\right) \\ &\quad\times\int_t^{\infty}(s-t)^{n-1}p(s)f\left(\exp\left(-k_2\int_{t_0-\gamma}^{s-\sigma}p(\xi)d\xi\right)\right)ds \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{3.2}$$

Then (1.1) *has a positive solution which is bounded by two exponential functions.*

Proof. We set

$$u(t) = \exp\left(-k_2 \int_{t_0-\gamma}^{t} p(s)ds\right), \quad v(t) = \exp\left(-k_1 \int_{t_0-\gamma}^{t} p(s)ds\right), \quad t \geq t_0. \tag{3.3}$$

We will show that the conditions of Corollary 2.3 are satisfied. With regard to (3.1) for $t \in [t_0, t_1]$ we get

$$\begin{aligned} v'(t) - u'(t) &= -k_1 p(t)v(t) + k_2 p(t)u(t) \\ &= p(t)v(t)\left[-k_1 + k_2 u(t) \exp\left(k_1 \int_{t_0-\gamma}^{t} p(s)ds\right)\right] \\ &= p(t)v(t)\left[-k_1 + k_2 \exp\left((k_1 - k_2) \int_{t_0-\gamma}^{t} p(s)ds\right)\right] \\ &\leq p(t)v(t)\left[-k_1 + k_2 \exp\left((k_1 - k_2) \int_{t_0-\gamma}^{t_0} p(s)ds\right)\right] \leq 0. \end{aligned} \tag{3.4}$$

Other conditions of Corollary 2.3 are also satisfied. The proof is complete. □

Corollary 3.2. *Suppose that all conditions of Theorem 3.1 are satisfied and*

$$\int_{t_0}^{\infty} p(t)dt = \infty. \tag{3.5}$$

Then (1.1) *has a positive solution which tends to zero.*

Corollary 3.3. *Suppose that* $k > 0,\ c > 0,\ t_1 \geq t_0 + m$ *and*

$$\begin{aligned} a(t) = {}& \exp\left(-k \int_{t-\tau}^{t} p(s)ds\right) + \frac{1}{(n-1)!} \exp\left(k \int_{t_0}^{t-\tau} p(s)ds\right) \\ & \times \int_{t}^{\infty} (s-t)^{n-1} p(s) f\left(\exp\left(-k \int_{t_0}^{s-\sigma} p(\xi)d\xi\right)\right) ds \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{3.6}$$

Then (1.1) *has a solution:*

$$x(t) = \exp\left(-k \int_{t_0}^{t} p(s)ds\right), \quad t \geq t_1. \tag{3.7}$$

Proof. We put $k_1 = k_2 = k,\ \gamma = 0$ and apply Theorem 3.1. □

Example 3.4. Consider the nonlinear neutral differential equation:

$$[x(t) - a(t)x(t-2)]' = px^3(t-1), \quad t \geq t_0, \tag{3.8}$$

where $p \in (0, \infty)$. We will show that the conditions of Theorem 3.1 are satisfied. The condition (3.1) has a form:

$$\frac{k_1}{k_2} \exp\left((k_2 - k_1)p\gamma\right) \geq 1, \tag{3.9}$$

$0 < k_1 \leq k_2$, $\gamma \geq 0$. For function $a(t)$, we obtain

$$\begin{aligned}
&\exp(-2pk_2) + \frac{1}{3k_1} \exp\left(p\left[k_2(\gamma - t_0 - 2) - 3k_1(\gamma - t_0 - 1) + (k_2 - 3k_1)t\right]\right) \\
&\quad \leq a(t) \leq \exp(-2pk_1) \\
&\qquad + \frac{1}{3k_2} \exp\left(p\left[k_1(\gamma - t_0 - 2) - 3k_2(\gamma - t_0 - 1) + (k_1 - 3k_2)t\right]\right), \quad t \geq t_0.
\end{aligned} \tag{3.10}$$

For $p = 1,\ k_1 = 1,\ k_2 = 2,\ \gamma = 1,\ t_0 = 1$, the condition (3.9) is satisfied and

$$e^{-4} + \frac{1}{3e} e^{-t} \leq a(t) \leq e^{-2} + \frac{e^4}{6} e^{-5t}, \quad t \geq t_1 \geq 3. \tag{3.11}$$

If the function $a(t)$ satisfies (3.11), then (3.8) has a solution which is bounded by the functions $u(t) = \exp(-2t),\ v(t) = \exp(-t),\ t \geq 3$.

Example 3.5. Consider the nonlinear differential equation:

$$[x(t) - a(t)x(t-\pi)]' = p(t)f(x(t-\pi)), \quad t \geq 0, \tag{3.12}$$

where $f(x) = \sqrt{x},\ x > 0,\ p(t) = 0.8\exp(\pi - t + 0.05\cos t),\ t \geq 0$, and

$$e^{0.1\cos t}\left(e^{0.1\cos t} + 0.8\left(e^{\pi - t} - 1\right)\right) \leq a(t) \leq e^{0.1\cos t}\left(e^{0.1\cos t} + \frac{0.8}{\sqrt{b}}\left(e^{\pi - t} - 1\right)\right) < 1, \tag{3.13}$$

for $t \geq \pi,\ b \in [1,2]$. Set

$$u(t) = e^{0.1\cos t}, \quad v(t) = be^{0.1\cos t}, \quad t \geq 0. \tag{3.14}$$

Then we have

$$\begin{aligned}
v'(t) - u'(t) &= -0.1be^{0.1\cos t}\sin t + 0.1e^{0.1\cos t}\sin t \\
&= -0.1(b-1)e^{0.1\cos t}\sin t \leq 0 \quad \text{for } t \in [0, \pi].
\end{aligned} \tag{3.15}$$

By Corollary 2.7, (3.12) has a solution which is bounded by the functions $e^{0.1\cos t}$ and $be^{0.1\cos t}$, $t \geq \pi$. If

$$a(t) = e^{0.1\cos t}\left(e^{0.1\cos t} + 0.8\left(e^{\pi - t} - 1\right)\right) \quad \text{for } t \geq \pi, \tag{3.16}$$

then (3.12) has the positive periodic solution $x(t) = u(t) = e^{0.1\cos t}$, $t \geq \pi$.

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References

[1] J. Diblík, "Asymptotic Representation of Solutions of Equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$," *Journal of Mathematical Analysis and Applications*, vol. 217, no. 1, pp. 200–215, 1998.
[2] J. Diblík, "A criterion for existence of positive solutions of systems of retarded functional-differential equations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 38, no. 3, pp. 327–339, 1999.
[3] J. Diblík and M. Kúdelčíková, "Existence and asymptotic behavior of positive solutions of functional differential equations of delayed type," *Abstract and Applied Analysis*, vol. 2011, Article ID 754701, 16 pages, 2011.
[4] J. Diblík and M. Kúdelčíková, "Two classes of asymptotically different positive solutions of the equation over $\dot{y}(t) = -f(t, y_t)$," *Nonlinear Analysis*, vol. 70, no. 10, pp. 3702–3714, 2009.
[5] J. Diblík and M. Růžičková, "Convergence of the solutions of the equation over $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$ in the critical case," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1361–1370, 2007.
[6] J. Diblík and M. Růžičková, "Existence of positive solutions of a singular initial problem for a nonlinear system of differential equations," *Rocky Mountain Journal of Mathematics*, vol. 34, no. 3, pp. 923–944, 2004.
[7] J. Diblík and M. Růžičková, "Exponential solutions of equation $\dot{y}(t) = \beta(t)[y(t - \delta) - y(t - \tau(t))]$," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 273–287, 2004.
[8] J. Diblík, Z. Svoboda, and Z. Šmarda, "Retract principle for neutral functional differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 12, pp. e1393–e1400, 2009.
[9] B. Dorociaková, A. Najmanová, and R. Olach, "Existence of nonoscillatory solutions of first-order neutral differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 346745, 9 pages, 2011.
[10] L. H. Erbe, Q. K. Kong, and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, NY, USA, 1995.
[11] B. M. Levitan, "Some questions of the theory of almost periodic functions. I," *Uspekhi Matematicheskikh Nauk*, vol. 2, no. 5, pp. 133–192, 1947.
[12] E. Špániková and H. Šamajová, "Asymptotic properties of solutions to n-dimensional neutral differential systems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2877–2885, 2009.
[13] X. Lin, "Oscillation of second-order nonlinear neutral differential equations," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 442–452, 2005.
[14] Z. Lin, L. Chen, S. M. Kang, and S. Y. Cho, "Existence of nonoscillatory solutions for a third order nonlinear neutral delay differential equation," *Abstract and Applied Analysis*, vol. 2011, Article ID 93890, 23 pages, 2011.
[15] X. Wang and L. Liao, "Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 279, no. 1, pp. 326–338, 2003.

[16] Y. Zhou, "Existence for nonoscillatory solutions of second-order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 91–96, 2007.

[17] Y. Zhou and B. G. Zhang, "Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients," *Applied Mathematics Letters*, vol. 15, no. 7, pp. 867–874, 2002.