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Research Article

Nearly Quadratic Mappings over p-Adic Fields

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We establish some stability results over p-adic fields for the generalized quadratic functional equation $\sum_{k=2}^{n}\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f(\sum_{i=1,i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}) + f(\sum_{i=1}^{n} x_i) = 2^{n-1}\sum_{i=1}^{n} f(x_i)$, where $n \in \mathbb{N}$ and $n \geq 2$.

1. Introduction and Preliminaries

In 1899, Hensel [1] discovered the p-adic numbers as a number of theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p. Then, p-adic absolute value $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$ are integers (see, e.g., [2, 3]). Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n.

During the last three decades, p-adic numbers have gained the interest of physicists for their research, in particular, in problems coming from quantum physics, p-adic strings, and superstrings [4, 5]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: For $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$.

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x+y| \leq \max\{|x|,|y|\} \leq |x|+|y|$ for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is nontrivial, that is, there is a $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then, $(X, \|\cdot\|)$ is called a non-Archimedean space. In any such a space, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if and only if $\{x_{n+1} - x_n\}_{n\in\mathbb{N}}$ converges to zero. By a complete non-Archimedean space, we mean one in which every Cauchy sequence is convergent.

The study of stability problems for functional equations is related to a question of Ulam [6] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [7]. Subsequently, the result of Hyers was generalized by Aoki [8] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influences in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Rassias [10] considered the Cauchy difference controlled by a product of different powers of norm. The above results have been generalized by Forti [11] and Găvruţa [12] who permitted the Cauchy difference to become arbitrary unbounded (see also [13–22]). Arriola and Beyer [23] investigated stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ε such that $|f(x+y) - f(x) - f(y)| \leq \varepsilon$ for all $x, y \in Q_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \varepsilon$ for all $x \in \mathbb{Q}_p$. For more details about the results concerning such problems, the reader is referred to [24–45].

Recently, Khodaei and Rassias [46] introduced the generalized additive functional equation

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r}\right) + f\left(\sum_{i=1}^{n} a_i x_i\right) = 2^{n-1} a_1 f(x_1) \tag{1.1}$$

and proved the generalized Hyers-Ulam stability of the above functional equation. The functional equation

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) \tag{1.2}$$

is related to symmetric biadditive function and is called a quadratic functional equation [47, 48]. Every solution of the quadratic equation (1.2) is said to be a quadratic function.

Now, we introduce the generalized quadratic functional equation in n-variables as follows:

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) = 2^{n-1}\sum_{i=1}^{n} f(x_i), \tag{1.3}$$

where $n \geq 2$. Moreover, we investigate the generalized Hyers-Ulam stability of functional equation (1.3) over the p-adic field $\mathbb{Q}_p$.

As a special case, if $n = 2$ in (1.3), then we have the functional equation (1.2). Also, if $n = 3$ in (1.3), we obtain

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3} f\left(\sum_{i=1, i\neq i_1,i_2}^{3} x_i - \sum_{r=1}^{2} x_{i_r}\right) + \sum_{i_1=2}^{3} f\left(\sum_{i=1, i\neq i_1}^{3} x_i - x_{i_1}\right) + f\left(\sum_{i=1}^{3} x_i\right) = 2^2 \sum_{i=1}^{3} f(x_i), \quad (1.4)$$

that is,

$$f(x_1 - x_2 - x_3) + f(x_1 - x_2 + x_3) + f(x_1 + x_2 - x_3) + f(x_1 + x_2 + x_3) = 4f(x_1) + 4f(x_2) + 4f(x_3). \quad (1.5)$$

2. Stability of Quadratic Functional Equation (1.3) over p-Adic Fields

We will use the following lemma.

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if the function f is quadratic.*

Proof. Let f satisfy the functional equation (1.3). Setting $x_i = 0$ ($i = 1, \ldots, n$) in (1.3), we have

$$\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f(0) + f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.1)$$

that is,

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{n} f(0) + \sum_{i_1=2}^{3}\sum_{i_2=i_1+1}^{4} \cdots \sum_{i_{n-2}=i_{n-3}+1}^{n} f(0) + \cdots + \sum_{i_1=2}^{n} f(0) + f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.2)$$

or

$$\left(\binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{1} + 1\right) f(0) = 2^{n-1}\sum_{i=1}^{n} f(0), \quad (2.3)$$

but $1 + \sum_{j=1}^{n-j}\binom{n-j}{j} = \sum_{j=0}^{n-j}\binom{n-j}{j} = 2^{n-j}$, and also $n > j \geq 1$ so $2^{n-1}(n-1)f(0) = 0$.

Putting $x_i = 0$ ($i = 2, \ldots, n-1$) in (1.3) and then using $f(0) = 0$, we get

$$f(x_1 - x_n) + \left(\binom{n-2}{1} f(x_1 - x_n) + \binom{n-2}{n-2} f(x_1 + x_n)\right) + \cdots$$
$$+ \left(\binom{n-2}{n-3} f(x_1 - x_n) + \binom{n-2}{2} f(x_1 + x_n)\right)$$

$$+\left(\binom{n-2}{n-2}f(x_1-x_n)+\binom{n-2}{1}f(x_1+x_n)\right)+f(x_1+x_n)$$
$$=2^{n-1}f(x_1)+2^{n-1}f(x_n), \tag{2.4}$$

that is,

$$\left(1+\sum_{j=1}^{n-2}\binom{n-2}{j}\right)\left(f(x_1+x_n)+f(x_1-x_n)\right)=2^{n-1}f(x_1)+2^{n-1}f(x_n), \tag{2.5}$$

for all $x_1, x_n \in X$, this shows that f satisfies the functional equation (1.2). So the function f is quadratic.

Conversely, suppose that f is quadratic, thus f satisfies the functional equation (1.2). Hence, we have $f(0)=0$ and f is even.

We are going to prove our assumption by induction on $n \geq 2$. It holds on $n=2$. Assume that it holds on the case where $n=t$; that is, we have

$$\sum_{k=2}^{t}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+1}=i_{t-k}+1}^{t}\right)f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+1}}^{t}x_i-\sum_{r=1}^{t-k+1}x_{i_r}\right)+f\left(\sum_{i=1}^{t}x_i\right)=2^{t-1}\sum_{i=1}^{t}f(x_i) \tag{2.6}$$

for all $x_1, \ldots, x_t \in X$. It follows from (1.2) that

$$f\left(\sum_{i=1}^{t}x_i+x_{t+1}\right)+f\left(\sum_{i=1}^{t}x_i-x_{t+1}\right)=2f\left(\sum_{i=1}^{t}x_i\right)+2f(x_{t+1}) \tag{2.7}$$

for all $x_1, \ldots, x_{t+1} \in X$. Replacing x_t by $-x_t$ in (2.7), we obtain

$$f\left(\sum_{i=1}^{t-1}x_i-x_t+x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i-x_t-x_{t+1}\right)=2f\left(\sum_{i=1}^{t-1}x_i-x_t\right)+2f(x_{t+1}) \tag{2.8}$$

for all $x_1, \ldots, x_{t+1} \in X$. Adding (2.7) to (2.8), we have

$$f\left(\sum_{i=1}^{t-1}x_i-x_t-x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i-x_t+x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t-x_{t+1}\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t+x_{t+1}\right)$$
$$=2\left[f\left(\sum_{i=1}^{t-1}x_i-x_t\right)+f\left(\sum_{i=1}^{t-1}x_i+x_t\right)\right]+4f(x_{t+1}) \tag{2.9}$$

for all $x_1, \ldots, x_{t+1} \in X$. Replacing x_{t-1} by $-x_{t-1}$ in (2.9), we get

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) = 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right)\right] + 4f(x_{t+1}) \quad (2.10)$$

for all $x_1, \ldots, x_{t+1} \in X$. Adding (2.9) to (2.10), one gets

$$f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t - x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t + x_{t+1}\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t + x_{t+1}\right)$$
$$+ f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t - x_{t+1}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} - x_t\right) + f\left(\sum_{i=1}^{t-2} x_i - x_{t-1} + x_t\right) + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} - x_t\right)\right.$$
$$\left. + f\left(\sum_{i=1}^{t-2} x_i + x_{t-1} + x_t\right)\right] + 8f(x_{t+1}) \quad (2.11)$$

for all $x_1, \ldots, x_{t+1} \in X$. By using the above method, for x_{t-2} until x_2, we infer that

$$\sum_{k=2}^{t+1}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[\sum_{k=2}^{t}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+1}=i_{t-k}+1}^{t}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+1}}^{t} x_i - \sum_{r=1}^{t-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{t} x_i\right)\right] + 2^t f(x_{t+1}) \quad (2.12)$$

for all $x_1, \ldots, x_{t+1} \in X$. Now, by the case $n = t$, we lead to

$$\sum_{k=2}^{t+1}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{t-k+2}=i_{t-k+1}+1}^{t+1}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{t-k+2}}^{t+1} x_i - \sum_{r=1}^{t-k+2} x_{i_r}\right) + f\left(\sum_{i=1}^{t+1} x_i\right)$$
$$= 2\left[2^{t-1}\sum_{i=1}^{t} f(x_i)\right] + 2^t f(x_{t+1}) \quad (2.13)$$

for all $x_1, \ldots, x_{t+1} \in X$, so (1.3) holds for $n = t + 1$. This completes the proof of the lemma. □

Corollary 2.2. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if there exists a symmetric biadditive function $B_1 : X \times X \to Y$ such that $f(x) = B_1(x, x)$ for all $x \in X$.*

Now, we investigate the stability of the functional equation (1.3) from a Banach space B into p-adic field $\mathbb{Q}_p$. For convenience, we define the difference operator D_f for a given function f:

$$D_f(x_1, \ldots, x_n) := \sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f\left(\sum_{i=1,\, i\neq i_1,\ldots,i_{n-k+1}}^{n} x_i - \sum_{r=1}^{n-k+1} x_{i_r}\right) + f\left(\sum_{i=1}^{n} x_i\right) - 2^{n-1}\sum_{i=1}^{n} f(x_i). \tag{2.14}$$

Theorem 2.3. *Let B be a Banach space and let $\varepsilon > 0$, λ be real numbers. Suppose that a function $f : \mathbb{Q}_p \to B$ with $f(0) = 0$ satisfies the inequality*

$$\left\| D_f(x_1, \ldots, x_n) \right\| \leq \varepsilon \sum_{i=1}^{n} |x_i|_p^{\lambda} \tag{2.15}$$

for all $x_1, \ldots, x_n \in \mathbb{Q}_p$. Then there exists a unique quadratic function $Q : \mathbb{Q}_p \to B$ such that

$$\| f(x) - Q(x) \| \leq \begin{cases} \dfrac{\varepsilon}{2^{n-1} - 2^{n-\lambda-3}} |x|_p^{\lambda}, & p = 2,\ \lambda > -2; \\ \dfrac{\varepsilon}{3.2^{n-3}} |x|_p^{\lambda}, & p > 2; \end{cases} \tag{2.16}$$

for all nonzero $x \in \mathbb{Q}_p$.

Proof. Letting $x_1 = x_2 = x \neq 0$ and $x_i = 0$ $(i = 3, \ldots, n)$ in (2.15), we obtain

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{\varepsilon}{2^{n-1}} |x|_p^{\lambda} \tag{2.17}$$

for all $x \in \mathbb{Q}_p$. Hence,

$$\left\| \frac{1}{2^{2l}} f\left(2^l x\right) - \frac{1}{2^{2m}} f(2^m x) \right\| \leq \frac{\varepsilon}{2^{n-1}} \sum_{j=l}^{m-1} \frac{|2|_p^{\lambda j}}{2^{2j}} |x|_p^{\lambda} \tag{2.18}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in \mathbb{Q}_p$. It follows from (2.18) that the sequence $\{(1/2^{2m}) f(2^m x)\}$ is a Cauchy sequence for all $x \in \mathbb{Q}_p$. Since B is complete, the sequence $\{(1/2^{2m}) f(2^m x)\}$ converges. Therefore, one can define the function $Q : \mathbb{Q}_p \to B$ by

$$Q(x) := \lim_{m \to \infty} \frac{1}{2^{2m}} f(2^m x) \tag{2.19}$$

for all $x \in \mathbb{Q}_p$. It follows from (2.15) and (2.19) that

$$\left\| D_Q(x_1,\dots,x_n) \right\| = \lim_{m\to\infty} \frac{1}{2^{2m}} \left\| D_f(2^m x_1,\dots,2^m x_n) \right\| \le \lim_{m\to\infty} \frac{|2|_p^{\lambda m}}{2^{2m}} \sum_{i=1}^{n} \varepsilon |x_i|_p^{\lambda} = 0 \tag{2.20}$$

for all $x_1,\dots,x_n \in \mathbb{Q}_p$. So $D_Q(x_1,\dots,x_n) = 0$. By Lemma 2.1, the function $Q : \mathbb{Q}_p \to B$ is quadratic.

Taking the limit $m \to \infty$ in (2.18) with $l = 0$, we find that the function Q is quadratic function satisfying the inequality (2.16) near the approximate function $f : \mathbb{Q}_p \to B$ of (1.3).

To prove the aforementioned uniqueness, we assume now that there is another additive function $Q' : \mathbb{Q}_p \to B$ which satisfies (1.3) and the inequality (2.16). So

$$\begin{aligned} \left\| Q(x) - Q'(x) \right\| &= \frac{1}{2^{2m}} \left\| Q(2^m x) - Q'(2^m x) \right\| \\ &\le \frac{1}{2^{2m}} \left(\left\| Q(2^m x) - f(2^m x) \right\| + \left\| f(2^m x) - Q'(2^m x) \right\| \right) \\ &\le \begin{cases} \dfrac{\varepsilon}{2^{2m+\lambda m}\left(2^{n-2} - 2^{n-\lambda-4}\right)} |x|_p^{\lambda}, & p = 2,\ \lambda > -2; \\ \dfrac{\varepsilon}{3.2^{2m+n-4}} |x|_p^{\lambda}, & p > 2; \end{cases} \end{aligned} \tag{2.21}$$

which tends to zero as $m \to \infty$ for all nonzero $x \in \mathbb{Q}_p$. This proves the uniqueness of Q, completing the proof of uniqueness. □

The following example shows that the above result is not valid over p-adic fields.

Example 2.4. Let $p > 2$ be a prime number and define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ by $f(x) = x^2 - 2x$. Since $|2^n|_p = 1$,

$$\left| D_f(x_1,\dots,x_n) \right|_p = \left| 2^n \sum_{i=2}^{n} x_i \right|_p = \left| \sum_{i=2}^{n} x_i \right|_p \le \sum_{i=1}^{n} |x_i|_p \tag{2.22}$$

for all $x_1,\dots,x_n \in \mathbb{Q}_p$. Hence, the conditions of Theorem 2.3 for $\varepsilon = 1$ and $\lambda = 1$ hold. However for each $n \in \mathbb{N}$, we have

$$\left| \frac{1}{2^{2(m+1)}} f(2^{m+1}x) - \frac{1}{2^{2m}} f(2^m x) \right|_p = \frac{|x|_p}{|2^m|_p} = |x|_p \tag{2.23}$$

for all $x \in \mathbb{Q}_p$. Hence $\{(1/2^{2m}) f(2^m x)\}$ is not convergent for all nonzero $x \in \mathbb{Q}_p$.

In the next result, which can be compared with Theorem 2.3, we will show that the stability of the functional equation (1.3) in non-Archimedean spaces over p-adic fields.

Theorem 2.5. *Let $\ell \in \{-1,1\}$ be fixed. Let $\mathcal{V}$ be a non-Archimedean space and $\mathcal{W}$ be a complete non-Archimedean space over $\mathbb{Q}_p$, where $p > 2$ is a prime number. Suppose that a function $f : \mathcal{V} \to \mathcal{W}$ satisfies the inequality*

$$\left\|D_f(x_1,\ldots,x_n)\right\|_{\mathcal{W}} \leq \begin{cases} \varepsilon\sum_{i=1}^{n}\|x_i\|_{\mathcal{V}}^{\lambda}, & \lambda\ell > 2\ell; \\ \varepsilon\sum_{i=2}^{n}\|x_1\|_{\mathcal{V}}^{\lambda_1}\|x_i\|_{\mathcal{V}}^{\lambda_i}, & (\lambda_1+\lambda_i)\ell > 2\ell; \\ \varepsilon\max\left\{\|x_i\|_{\mathcal{V}}^{\lambda}; 1 \leq i \leq n\right\}, & \lambda\ell > 2\ell; \end{cases} \tag{2.24}$$

for all $x_1,\ldots,x_n \in \mathcal{V}$, where $\varepsilon, \lambda_1,\ldots,\lambda_n$ and λ are nonnegative real numbers. Then, the limit

$$Q(x) := \lim_{m\to\infty}\frac{1}{p^{2\ell m}}f\left(p^{\ell m}x\right) \tag{2.25}$$

exists for all $x \in \mathcal{V}$ and $Q : \mathcal{V} \to \mathcal{W}$ is a unique quadratic function satisfying

$$\left\|f(x)-Q(x)\right\|_{\mathcal{W}} \leq \begin{cases} 2p^{1+\ell+(1-\ell)\lambda/2}\varepsilon\|x\|_{\mathcal{V}}^{\lambda}, \\ p^{1+\ell+((1-\ell)(\lambda_1+\lambda_2)/2)}\varepsilon\|x\|_{\mathcal{V}}^{\lambda_1+\lambda_2}, \\ p^{1+\ell+(1-\ell)\lambda/2}\varepsilon\|x\|_{\mathcal{V}}^{\lambda}, \end{cases} \tag{2.26}$$

for all $x \in \mathcal{V}$.

Proof. By (2.24),

$$\left\|D_f(x_1,\ldots,x_n)\right\|_{\mathcal{W}} \leq \varepsilon\sum_{i=1}^{n}\|x_i\|_{\mathcal{V}}^{\lambda} \tag{2.27}$$

for all $x_1,\ldots,x_n \in \mathcal{V}$, where $\lambda\ell > 2\ell$. Putting $x_i = 0$ $(i = 1,\ldots,n)$ in (2.27) to obtain $f(0) = 0$, setting $x_i = 0$ $(i = 3,\ldots,n)$ in (2.27), we obtain

$$\left\|2^{n-2}f(x_1+x_2)+2^{n-2}f(x_1-x_2)-2^{n-1}f(x_1)-2^{n-1}f(x_2)\right\|_{\mathcal{W}} \leq \varepsilon\left(\|x_1\|_{\mathcal{V}}^{\lambda}+\|x_2\|_{\mathcal{V}}^{\lambda}\right) \tag{2.28}$$

for all $x_1, x_2 \in \mathcal{V}$. So

$$\left\|f(x_1+x_2)+f(x_1-x_2)-2f(x_1)-2f(x_2)\right\|_{\mathcal{W}} \leq \varepsilon\left(\|x_1\|_{\mathcal{V}}^{\lambda}+\|x_2\|_{\mathcal{V}}^{\lambda}\right) \tag{2.29}$$

for all $x_1, x_2 \in \mathcal{V}$. Letting $x_1 = x_2 = x$ in (2.29), we have

$$\left\|f(2x)-4f(x)\right\|_{\mathcal{W}} \leq 2\varepsilon\|x\|_{\mathcal{V}}^{\lambda} \tag{2.30}$$

for all $x \in \mathcal{V}$. By induction on j, we will show that for each $j \geq 2$,

$$\left\| f(jx) - j^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.31}$$

for all $x \in \mathcal{V}$. It holds on $j = 2$; see (2.30). Let (2.31) hold for $j = 2, \ldots, k$. Replacing x_1 and x_2 by kx and x in (2.29), respectively, we get

$$\| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \|_{\mathcal{W}} \leq \varepsilon \left(1 + |k|_p^{\lambda}\right) \|x\|_{\mathcal{V}}^{\lambda} \tag{2.32}$$

for all $x \in \mathcal{V}$. It follows from (2.32) and our induction hypothesis that

$$\begin{aligned} \left\| f((k+1)x) - (k+1)^2 f(x) \right\|_{\mathcal{W}} &= \| f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) \\ &\quad - f((k-1)x) + (k-1)^2 f(x) - 2\left(f(kx) - k^2 f(x) \right) \Big\|_{\mathcal{W}} \\ &\leq \max\left\{ 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \varepsilon\left(1 + |k|_p^{\lambda}\right) \|x\|_{\mathcal{V}}^{\lambda} \right\} = 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{V}$. This proves (2.31) for each $j \geq 2$. In particular,

$$\left\| f(px) - p^2 f(x) \right\|_{\mathcal{W}} \leq 2\varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.34}$$

for all $x \in \mathcal{V}$. So

$$\begin{aligned} \left\| f(x) - \frac{1}{p^2} f(px) \right\|_{\mathcal{W}} &\leq 2p^2 \varepsilon \|x\|_{\mathcal{V}}^{\lambda}, \\ \left\| f(x) - p^2 f\left(\frac{x}{p}\right) \right\|_{\mathcal{W}} &\leq 2p^{\lambda} \varepsilon \|x\|_{\mathcal{V}}^{\lambda} \end{aligned} \tag{2.35}$$

for all $x \in \mathcal{V}$. Hence,

$$\left\| \frac{1}{p^{2\ell j}} f\left(p^{\ell j} x\right) - \frac{1}{p^{2\ell(j+1)}} f\left(p^{\ell(j+1)} x\right) \right\|_{\mathcal{W}} \leq \frac{2p^{2\ell j + (1-\ell)\lambda/2 + 1 + \ell}}{p^{\lambda \ell j}} \varepsilon \|x\|_{\mathcal{V}}^{\lambda} \tag{2.36}$$

for all $x \in \mathcal{V}$. Since the right side of the above inequality tends to zero as $j \to \infty$, $\{(1/p^{2\ell m}) f(p^{\ell m} x)\}$ is a Cauchy sequence in complete non-Archimedean space $\mathcal{W}$, thus it converges to some function $Q(x) = \lim_{m\to\infty}(1/p^{2\ell m})f(p^{\ell m}x)$ for all $x \in \mathcal{V}$. Using (2.35) and induction, one can show that for any $m \in \mathbb{N}$, we have

$$\begin{aligned}\left\|f(x) - \frac{1}{p^{2\ell m}}f(p^{\ell m}x)\right\|_{\mathcal{W}} &= \left\|\sum_{j=0}^{m-1}\frac{1}{p^{2\ell j}}f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}}f(p^{\ell(j+1)}x)\right\|_{\mathcal{W}} \\ &\le \max\left\{\left\|\frac{1}{p^{2\ell j}}f(p^{\ell j}x) - \frac{1}{p^{2\ell(j+1)}}f\left(p^{\ell(j+1)}x\right)\right\|_{\mathcal{W}}; 0 \le j < m\right\} \\ &\le \max\left\{2p^{1+\ell+(1-\ell)\lambda/2+\ell j(2-\lambda)}\varepsilon\|x\|_{\mathcal{V}}^{\lambda}; 0 \le j < m\right\}\end{aligned} \tag{2.37}$$

for all $x \in \mathcal{V}$. Letting $m \to \infty$ in this inequality, we see that

$$\|f(x) - Q(x)\|_{\mathcal{W}} \le 2p^{1+\ell+(1-\ell)\lambda/2}\varepsilon\|x\|_{\mathcal{V}}^{\lambda} \tag{2.38}$$

for all $x \in \mathcal{V}$. Moreover,

$$\|D_Q(x_1,\ldots,x_n)\|_{\mathcal{W}} = \lim_{m\to\infty}\left\|\frac{1}{p^{2\ell m}}D_f(p^{\ell m}x_1,\ldots,p^{\ell m}x_n)\right\|_{\mathcal{W}} \le \lim_{m\to\infty}\frac{p^{2\ell m}}{p^{\lambda\ell m}}\sum_{i=1}^{n}\varepsilon\|x_i\|_{\mathcal{V}}^{\lambda} = 0 \tag{2.39}$$

for all $x_1,\ldots,x_n \in \mathcal{V}$. So $D_Q(x_1,\ldots,x_n) = 0$. By Lemma 2.1, the function $Q : \mathcal{V} \to \mathcal{W}$ is quadratic.

Now, let $Q' : \mathcal{V} \to \mathcal{W}$ be another quadratic function satisfying (1.3) and (2.38). So

$$\begin{aligned}\|Q(x) - Q'(x)\|_{\mathcal{W}} &\le p^{2\ell m}\max\left\{\left\|Q\left(p^{\ell m}x\right) - f\left(p^{\ell m}x\right)\right\|_{\mathcal{W}'}\left\|f\left(p^{\ell m}x\right) - Q'\left(p^{\ell m}x\right)\right\|_{\mathcal{W}}\right\} \\ &\le \frac{2p^{2\ell m+(1-\ell)\lambda/2+1+\ell}}{p^{\lambda\ell m}}\varepsilon\|x\|_{\mathcal{V}}^{\lambda},\end{aligned} \tag{2.40}$$

which tends to zero as $m \to \infty$ for all $x \in \mathcal{V}$. This proves the uniqueness of Q.

The rest of the proof is similar to the above proof, hence it is omitted. □

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