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*Research Article*

# A Note on the Inverse Problem for a Fractional Parabolic Equation

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For a fractional inverse problem with an unknown time-dependent source term, stability estimates are obtained by using operator theory approach. For the approximate solutions of the problem, the stable difference schemes which have first and second orders of accuracy are presented. The algorithm is tested in a one-dimensional fractional inverse problem.

## 1. Introduction

Inverse problems arise in many fields of science and engineering such as ion transport problems, chromatography, and heat determination problems with an unknown internal energy source. Different typed of inverse problems have been investigated, and the main results obtained in this field of research were given by many researchers (see [1–10]). More than three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics. Fractional integrals and derivatives appear in the theory of control of dynamical systems, when the controlled system or/and the controller is described by a fractional differential equation (see [11]). Recently, many application areas such as bioengineering applications, image and signal processing are also related to fractional calculus. Methods of solutions of problems and theory of fractional calculus have been studied by many researchers [11–28]. Among them finite difference method is used for solving several fractional differential equations (see [20, 22, 23, 27] and the references therein).

### *1.1. Statement of the Problem*

Many scientists and researchers are trying to enhance mathematical models of real-life cases for investigating and understanding the behavior of them. Therefore, some phenomena have been modeled and investigated as fractional inverse problems (see [29–33] and the references therein). In this paper, we consider the fractional parabolic inverse problem with the Dirichlet condition

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} - a\frac{\partial^2 u(t,x)}{\partial x^2} - D_t^{1/2}u(t,x) + \sigma u(t,x) = p(t)q(x) + f(t,x), \quad 0 < x < \pi,\ 0 < t \le T,\\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le T,\\ u(0,x) = \varphi(x), \quad 0 \le x \le \pi,\\ u(t,x^*) = \rho(t), \quad 0 < x^* < \pi,\end{gathered} \tag{1.1}$$

where $u(t,x)$ and $p(t)$ are unknown functions, $a(x) \ge a > 0$, and $\sigma > 0$ is a sufficiently large number. Here, $D_t^{1/2} = D_{0+}^{1/2}$ is the standard Riemann-Liouville's derivative of order $1/2$.

Theorems on the stability of problem (1.1) are analyzed by assuming that $q(x)$ is a sufficiently smooth function, $q(0) = q(\pi) = 0$ and $q(x^*) \neq 0$.

## 2. Main Results

In this section, stability estimates for the solution of (1.1) are investigated. For the mathematical substantiation, we introduce the Banach space $\overset{\circ\,\alpha}{C}[0,\pi]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,\pi]$ with $\phi(0) = \phi(\pi) = 0$ satisfying a Hölder condition for which the following norm is finite

$$\|\phi\|_{\overset{\circ\,\alpha}{C}[0,\pi]} = \|\phi\|_{C[0,\pi]} + \sup_{0<x<x+h<\pi} \frac{|\phi(x+h) - \phi(x)|}{h^\alpha}, \tag{2.1}$$

where $C[0,\pi]$ is the space of all continuous function $\phi(x)$ defined on $[0,\pi]$ with the norm

$$\|\phi\|_{C[0,\pi]} = \max_{0\le x\le\pi} |\phi(x)|. \tag{2.2}$$

With the help of a positive operator $A$, we introduce the fractional spaces $E_\alpha$, $0 < \alpha < 1$, consisting of all $v$ in a Banach space $E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda>0} \lambda^{1-\alpha} \|A\exp\{-\lambda A\}v\|_E. \tag{2.3}$$

Throughout the paper, positive constants will be indicated by $M_i$ $(\alpha,\beta,\ldots)$. Here variables are used to focus on the fact that the constant depends only on $\alpha,\beta,\ldots$ and the subindex $i$ is used to indicate a different constant.

**Theorem 2.1.** *Let* $\varphi \in \overset{\circ}{C}{}^{2\alpha+2}[0,\pi]$, $F \in C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi])$, *and* $\rho' \in C[0,T]$. *Then for the solution of problem* (1.1), *the following coercive stability estimates*

$$\begin{aligned}\|u_t\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi])} + \|u\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha+2}[0,\pi])} &\le M(x^*,q)\|\rho'\|_{C[0,T]} + M(a,\delta,\sigma,\alpha,x^*,q,T) \\ &\quad \times \left(\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]} + \|F\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi])} + \|\rho\|_{C[0,T]}\right), \\ \|p\|_{C[0,T]} &\le M(x^*,q)\|\rho'\|_{C[0,T]} + M(a,\delta,\sigma,\alpha,x^*,q,T) \\ &\quad \times \left[\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]} + \|F\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi])} + \|\rho\|_{C[0,T]}\right]\end{aligned} \tag{2.4}$$

*hold.*

*Proof.* Let us search for the solution of inverse problem (1.1) in the following form (see [8]):

$$u(t,x) = \eta(t)q(x) + w(t,x), \tag{2.5}$$

where

$$\eta(t) = \int_0^t p(s)ds. \tag{2.6}$$

Using the overdetermined condition, we get

$$\eta(t) = \frac{\rho(t) - w(t,x^*)}{q(x^*)}, \tag{2.7}$$

$$p(t) = \frac{\rho'(t) - w_t(t,x^*)}{q(x^*)}. \tag{2.8}$$

Using identity (2.8) and the triangle inequality, it follows that

$$\begin{aligned}|p(t)| = \left|\frac{\rho'(t) - w_t(t,x^*)}{q(x^*)}\right| &\le M(x^*,q)\left(|\rho'(t)| + |w_t(t,x^*)|\right) \\ &\le M(x^*,q)\left(\max_{0\le t\le T}|\rho'(t)| + \max_{0\le t\le T}\max_{0\le x\le\pi}|w_t(t,x)|\right) \\ &\le M(x^*,q)\left(\max_{0\le t\le T}|\rho'(t)| + \max_{0\le t\le T}\|w_t(t,\cdot)\|_{\overset{\circ}{C}{}^{2\alpha}[0,\pi]}\right)\end{aligned} \tag{2.9}$$

for any $t, t \in [0,T]$. Here, $w(t,x)$ is the solution of the following problem:

$$\begin{aligned}
&\frac{\partial w(t,x)}{\partial t} - a\frac{\partial^2 w(t,x)}{\partial x^2} - a\frac{\rho(t) - w(t,x^*)}{q(x^*)}\frac{d^2 q(x)}{dx^2} - D_t^{1/2} w(t,x) \\
&\quad - \frac{D_t^{1/2}\rho(t) - D_t^{1/2} w(t,x^*)}{q(x^*)} q(x) + \sigma\frac{\rho(t) - w(t,x^*)}{q(x^*)} q(x) \\
&\quad + \sigma w(t,x) = f(t,x), \quad 0 < x < \pi,\ 0 < t \le T, \\
&w(t,0) = w(t,\pi) = 0, \quad 0 \le t \le T, \\
&w(0,x) = \varphi(x), \quad 0 \le x \le \pi.
\end{aligned} \tag{2.10}$$

For simplicity, we assign

$$\begin{aligned}
F(t,x) &= \frac{a\rho(t)}{q(x^*)}\frac{d^2 q(x)}{dx^2} - \frac{\sigma\rho(t)}{q(x^*)} q(x) + \frac{D_t^{1/2}\rho(t)}{q(x^*)} q(x) + f(t,x), \\
G(t,x) &= Q_1(q,\rho,x,x^*,t) w(t,x^*) + Q_2(q,x,x^*) D_t^{1/2} w(t,x^*) + D_t^{1/2} w(t,x),
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
Q_1(q,\rho,x,x^*,t) &= \frac{1}{q(x^*)}\left(-a\frac{d^2 q(x)}{dx^2} + \sigma\rho(t)\right), \\
Q_2(q,x,x^*) &= -\frac{q(x)}{q(x^*)}.
\end{aligned} \tag{2.12}$$

Note that functions $F(t,x)$, $Q_1(q,\rho,x,x^*,t)$ and $Q_2(q,x,x^*)$ only contain given functions. Then, we can rewrite problem (2.10) as

$$\begin{aligned}
&\frac{\partial w(t,x)}{\partial t} - a\frac{\partial^2 w(t,x)}{\partial x^2} + \sigma w(t,x) = F(t,x) + G(t,x), \quad 0 < x < \pi,\ 0 < t \le T, \\
&w(t,0) = w(t,\pi) = 0, \quad 0 \le t \le T, \\
&w(0,x) = \varphi(x), \quad 0 \le x \le \pi.
\end{aligned} \tag{2.13}$$

So, the end of proof of Theorem 2.1 is based on estimate (2.9) and the following theorem. □

**Theorem 2.2.** *For the solution of problem* (2.10), *the following coercive stability estimate*

$$\begin{aligned}
\|w_t\|_{\overset{\circ}{C}{}^{2\alpha}[0,\pi]} &\le M(a,\delta,\sigma,\alpha,x^*,q,T) \\
&\quad \times \left(\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,\pi]} + \|F\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,\pi])} + \|\rho\|_{C[0,T]}\right)
\end{aligned} \tag{2.14}$$

*holds.*

*Proof.* In a Banach space $E = \overset{\circ}{C}[0,\pi]$, with the help of the positive operator $A$ defined by

$$Au = -a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma u, \tag{2.15}$$

with

$$D(A) = \{u(x) : u, u', u'' \in C[0,\pi],\ u(0) = u(\pi) = 0\}, \tag{2.16}$$

where $\sigma$ is a positive constant, problem (2.10) can be written in the abstract form as an initial-value problem

$$\begin{gathered} w_t + Aw = F(t) + G(t), \quad 0 < t \le T, \\ w(0) = \varphi. \end{gathered} \tag{2.17}$$

By the Cauchy formula, the solution can be written as

$$w(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A}(F(s) + G(s))ds. \tag{2.18}$$

Applying the formula

$$D_t^{1/2} u(t) = \int_0^t \frac{u'(\zeta)d\xi}{\sqrt{\pi}(t-\xi)^{1/2}}, \tag{2.19}$$

we get the following presentation of the solution of abstract problem (2.17):

$$\begin{aligned} D^{1/2}w(t) = & -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi - \int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi \\ & - \int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi + \int_0^t\int_0^{\xi} \frac{Ae^{-(\xi-s)A}F(s)}{\sqrt{\pi}(t-\xi)^{1/2}}\,ds\,d\xi \\ & + \int_0^t\int_0^{\xi} \frac{Ae^{-(\xi-s)A}G(s)}{\sqrt{\pi}(t-\xi)^{1/2}}\,ds\,d\xi. \end{aligned} \tag{2.20}$$

Changing the order of integration, we obtain that

$$\begin{aligned} D^{1/2}w(t) = & -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi - \int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi \\ & + \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi F(s)ds - \int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi \\ & + \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi G(s)ds = \sum_{k=1}^{5} J_k, \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} J_1(t) &= -\int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-p)^{1/2}}\,d\xi, \\ J_2(t) &= -\int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi, \\ J_3(t) &= \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi F(s)ds, \\ J_4(t) &= -\int_0^t \frac{G(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi, \\ J_5(t) &= \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi G(s)ds. \end{aligned} \tag{2.22}$$

Now, we estimate $J_k(t),\ k = 1,2,3,4,5$ separately. It is known that [13]

$$\left\| A^{\alpha} e^{-tA} \right\|_{E\to E} \le M, \quad 0 \le \alpha \le 1. \tag{2.23}$$

Since operators $A$ and $\exp(-tA)$ commute,

$$\left\| Ae^{-tA}\varphi \right\|_{E_\alpha} \le \left\| e^{-tA} \right\|_{E_\alpha \to E_\alpha} \|A\varphi\|_{E_\alpha} \le \left\| e^{-tA} \right\|_{E\to E} \|A\varphi\|_{E_\alpha}. \tag{2.24}$$

Applying the definition of norm of the spaces $E_\alpha$ and (2.23) and (2.24), we get

$$\|J_1(t)\|_{E_\alpha} = \left\| \int_0^t \frac{Ae^{-\xi A}\varphi}{\sqrt{\pi}(t-p)^{1/2}}\,d\xi \right\|_{E_\alpha} \le M_1 \|A\varphi\|_{E_\alpha} \tag{2.25}$$

for any $t,\ t \in [0,T]$. Estimation of $J_2(t)$ is as follows:

$$\begin{aligned} \|J_2(t)\|_{E_\alpha} &= \left\| \int_0^t \frac{F(\xi)}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi \right\|_{E_\alpha} \\ &\leq \|F(t)\|_{C(E_\alpha)} \int_0^t \frac{1}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi \leq M_2 \|F\|_{C(E_\alpha)}. \end{aligned} \tag{2.26}$$

Let us estimate $J_3(t)$:

$$\|J_3(t)\|_{E_\alpha} = \left\| \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi F(s) ds \right\|_{E_\alpha} \leq \left\| \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi ds \right\|_{E_\alpha \to E_\alpha} \|F\|_{C(E_\alpha)}. \tag{2.27}$$

It is proven that (see [28])

$$\left\| \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi \right\|_{E \to E} \leq \frac{M}{\sqrt{t-s}}. \tag{2.28}$$

Using the definition of norm of the spaces $E_\alpha$, we can obtain that

$$\begin{aligned} \left\| \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi ds \right\|_{E_\alpha \to E_\alpha} &= \int_0^t \left\| \int_s^t \frac{Ae^{-(t-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}} d\xi \right\|_{E \to E} ds \\ &\quad + \sup_{\lambda > 0} \int_0^t \left\| \int_s^t \lambda^{1-\alpha} A e^{-\lambda A} \frac{Ae^{-(t-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi \right\|_{E \to E} ds. \end{aligned} \tag{2.29}$$

Using estimates (2.23) and (2.28), we get

$$\begin{aligned} \|J_3(t)\|_{E_\alpha} &\leq \left\| \int_0^t \int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\, d\xi ds \right\|_{E_\alpha \to E_\alpha} \|F\|_{C(E_\alpha)} \\ &\leq M_3 \|F\|_{C(E_\alpha)}. \end{aligned} \tag{2.30}$$

Expanding $G(s)$, estimation of $J_4(t)$ is as follows:

$$\begin{aligned} \|J_4(t)\|_{E_\alpha} &\leq \int_0^t \left\| \frac{Q_1(q,\rho,x,x^*,t) w(\xi,x^*)}{\sqrt{\pi}(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi \\ &\quad + \int_0^t \left\| \frac{Q_2(q,x,x^*) D_t^{1/2} w(\xi,x^*)}{\sqrt{\pi}(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi + \int_0^t \left\| \frac{D_t^{1/2} w(\xi,x)}{\sqrt{\pi}(t-\xi)^{1/2}} \right\|_{E_\alpha} d\xi. \end{aligned} \tag{2.31}$$

It is known that (see [34])

$$\|w\|_{E_\alpha} \leq M\left\|D_t^{1/2}w\right\|_{E_\alpha}. \tag{2.32}$$

Since $Q_1(q,\rho,x,x^*,t)$ and $Q_2(q,x,x^*)$ are known functions, it is easy to see that

$$\|J_4(t)\|_{E_\alpha} \leq M_4(q,\rho,x,x^*,T)\int_0^t \frac{1}{\sqrt{\pi}(t-\xi)^{1/2}}\left\|D_t^{1/2}w(\xi)\right\|_{E_\alpha}d\xi. \tag{2.33}$$

Estimation of $J_5(t)$ can be given similar to the estimation of $J_4(t)$. By (2.23) and (2.32),

$$\begin{aligned}\|J_5(t)\|_{E_\alpha} &\leq \int_0^t\int_s^t \frac{Ae^{-(\xi-s)A}}{\sqrt{\pi}(t-\xi)^{1/2}}\,d\xi G(s)ds \\ &\leq M_5(q,\rho,x,x^*,T)\int_0^t \left\|D_t^{1/2}w(s)\right\|_{E_\alpha}ds.\end{aligned} \tag{2.34}$$

Finally combining estimates (2.25), (2.26), (2.30), (2.33), and (2.34), we get

$$\begin{aligned}\left\|D_t^{1/2}w\right\|_{E_\alpha} &\leq M_1\|A\varphi\|_{E_\alpha} + (M_2+M_3)\|F\|_{C(E_\alpha)} \\ &\quad + \int_0^t\left(\frac{M_4}{\sqrt{\pi}(t-\xi)^{1/2}} + M_5\right)\left\|D_t^{1/2}w(s)\right\|_{E_\alpha}ds.\end{aligned} \tag{2.35}$$

Using the Gronwall's inequality, we can write

$$\left\|D_t^{1/2}w\right\|_{E_\alpha} \leq e^{M_6}\left(M_1\|A\varphi\|_{E_\alpha} + M_7\|F\|_{C(E_\alpha)}\right). \tag{2.36}$$

From the last estimate, we can obtain the estimate for $w_t(t)$ by using problem (2.17) and well-posedness of the Cauchy problem in $C(E_\alpha)$ (see [35]). So the following theorem finishes the proof of Theorem 2.2. □

**Theorem 2.3** (see, [36]). *For $0 < \alpha < 1/2$, the norms of the spaces $E_\alpha(C[0,\pi],A)$ and $C^{2\alpha}[0,\pi]$ are equivalent.*

## 3. Numerical Results

We have not been able to obtain a sharp estimate for the constants figuring in the stability inequalities. So we will provide the following results of numerical experiments of the following problem:

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t}=\frac{\partial^2 u(t,x)}{\partial x^2}-u(t,x)+D_t^{1/2}u(t,x)+p(t)\sin x+f(t,x),\\
f(t,x)=\left(-3t-\frac{1}{\sqrt{\pi}}t^{-1/2}+\frac{2}{\sqrt{\pi}}t^{1/2}\right)\sin x,\quad x\in(0,\pi),\ t\in(0,1],\\
u(0,x)=\sin x,\quad x\in[0,\pi],\\
u(t,0)=u(t,\pi)=0,\quad t\in[0,1],\\
u\left(t,\frac{\pi}{2}\right)=1-t.
\end{gathered}\tag{3.1}$$

The exact solution of the given problem is $u(t,x)=(1-t)\sin x$ and for the control parameter $p(t)$ is $1+t$.

### *3.1. The First Order of Accuracy Difference Scheme*

For the approximate solution of the problem (3.1), the Rothe difference scheme

$$\begin{gathered}
\frac{u_n^k-u_n^{k-1}}{\tau}=\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-u_n^k+D_\tau^{1/2}u_n^k+p^kq_n+f(t_k,x_n),\\
f(t_k,x_n)=\left(-3t_k-\frac{1}{\sqrt{\pi}}t_k^{-1/2}+\frac{2}{\sqrt{\pi}}t_k^{1/2}\right)\sin x_n,\\
p^k=p(t_k),\quad q_n=\sin x_n,\quad x_n=nh,\quad t_k=k\tau,\\
1\le k\le N,\quad 1\le n\le M-1,\quad Mh=\pi,\quad N\tau=1,\\
u_n^0=\sin x_n,\quad 0\le n\le M,\\
u_0^k=u_M^k=0,\quad 0\le k\le N,\\
u_s^k=\rho(t_k),\quad \rho(t_k)=1-t_k,\quad 0\le k\le N,\ s=\left\lfloor\frac{\pi}{2h}\right\rfloor,
\end{gathered}\tag{3.2}$$

where $\lfloor x \rfloor$ denotes greatest integer less than $x$ is constructed. Throughout the paper, let us denote

$$\begin{gathered}
\rho(t_k) = 1 - t_k, \qquad q_n = \sin x_n, \\
t_k = \{t_k = k\tau,\ 0 \le k \le N,\ N\tau = 1\}, \\
x_n = \{x_n = nh,\ 0 \le n \le M-1,\ Mh = \pi\}, \\
f(t_k, x_n) = \left(-3t_k - \frac{1}{\sqrt{\pi}} t_k^{-1/2} + \frac{2}{\sqrt{\pi}} t_k^{1/2}\right) \sin x_n, \\
F(t_k, x_n) = \frac{\rho(t_k)}{\sin(x_s)} \left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \sin(x_n)\right) \\
- \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+(1/2))}{(k-m)!} \frac{\tau^{1/2}}{\sin(x_s)} \sin(x_n) + f(t_k, x_n).
\end{gathered} \tag{3.3}$$

We search the solution of (3.2) in the following form:

$$u_n^k = \eta^k q_n + w_n^k, \tag{3.4}$$

where

$$\eta^k = \sum_{i=1}^{k} p^i \tau, \quad 1 \le k \le N,\ \eta^0 = 0. \tag{3.5}$$

Moreover for the interior grid point $u_s^k$, we have that

$$u_s^k = \eta^k q_s + w_s^k = \rho(t_k). \tag{3.6}$$

From (3.4), (3.5), and the condition $u_s^k = \rho(t_k)$, it follows that

$$\eta^k = \frac{\rho(t_k) - w_s^k}{q_s}, \tag{3.7}$$

$$p^k = \frac{\eta^k - \eta^{k-1}}{\tau}, \quad 1 \le k \le N, \tag{3.8}$$

$$u_n^k = \frac{\rho(t_k) - w_s^k}{q_s} q_n + w_n^k, \quad 0 \le k \le N,\ 0 \le n \le M, \tag{3.9}$$

where $w_n^k,\ 0 \le k \le N,\ 0 \le n \le M$ is the solution of the difference scheme

$$\begin{aligned}
\frac{w_n^k - w_n^{k-1}}{\tau} = {} & \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} - w_n^k \\
& - \frac{w_s^k}{\sin(x_s)}\left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \sin(x_n)\right) \\
& - \frac{1}{\sqrt{\pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+(1/2))}{(k-m)!}\left(\frac{w_s^m - w_s^{m-1}}{\sin(x_s)\tau^{1/2}}\sin(x_n) - \frac{w_n^m - w_n^{m-1}}{\tau^{1/2}}\right) \\
& + F(t_k, x_n), \quad 1 \le k \le N,\ 1 \le n \le M-1, \\
w_0^k = w_M^k = 0, & \quad 0 \le k \le N, \\
w_n^0 = \sin(x_n), & \quad 0 \le n \le M.
\end{aligned} \tag{3.10}$$

First, applying the first order of accuracy difference scheme (3.10), we obtain $(M+1)\times(M+1)$ system of linear equations and we write them in the matrix form

$$Aw^k + \sum_{j=0}^{k-1} B_j w^j = D\varphi^k, \quad 1 \le k \le N,\ w^0 = \{\sin(x_n)\}_{n=0}^{M}, \tag{3.11}$$

where

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & . & 0 & . & 0 & 0 & 0 \\
x & y & x & 0 & . & z_1 & . & 0 & 0 & 0 \\
0 & x & y & x & . & z_2 & . & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & z_{M-1} & . & x & y & x \\
0 & 0 & 0 & 0 & . & 0 & . & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_0 = \begin{bmatrix}
0 & 0 & 0 & . & 0 & . & 0 & 0 \\
0 & a & 0 & . & f_1 & . & 0 & 0 \\
0 & 0 & a & . & f_2 & . & 0 & 0 \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & . & f_s + a & . & 0 & 0 \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & . & f_{M-1} & . & 0 & a \\
0 & 0 & 0 & . & 0 & . & 0 & 0
\end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_j = \begin{bmatrix} 0 & 0 & 0 & . & 0 & . & 0 & 0 \\ 0 & c & 0 & . & d_1 & . & 0 & 0 \\ 0 & 0 & c & . & d_2 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & d_s + c & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & d_{M-1} & . & 0 & c \\ 0 & 0 & 0 & . & 0 & . & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}, \tag{3.12}$$

for any $j = 1, 2, \ldots, k-2$, and

$$B_{k-1} = \begin{bmatrix} 0 & 0 & 0 & . & 0 & . & 0 & 0 \\ 0 & v & 0 & . & c_1 & . & 0 & 0 \\ 0 & 0 & v & . & c_2 & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_s + v & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & c_{M-1} & . & 0 & v \\ 0 & 0 & 0 & . & 0 & . & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}. \tag{3.13}$$

Here, for any $n = 1, 2, \ldots, M-1$,

$$\begin{aligned} x &= -\frac{1}{h^2}, \qquad y = \frac{1}{\tau} + \frac{2}{h^2} + 1 + \frac{1}{\sqrt{\tau}}, \\ z_n &= \frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{\sin(x_s)h^2} - \frac{\sin(x_n)}{\sin(x_s)} - \frac{\sin(x_n)}{\sin(x_s)\sqrt{\tau}}, \quad \text{in } (s+1)\text{th column}, \\ a &= \frac{\Gamma(k-1/2)}{\sqrt{\tau\pi}(k-1)!}, \qquad f_n = -\frac{\sin(x_n)\Gamma(k-1/2)}{\sqrt{\tau\pi}\sin(x_s)(k-1)!}, \\ c &= \frac{1}{\sqrt{\tau\pi}}\left(\frac{\Gamma(k-j-1/2)}{(k-j-1)!} - \frac{\Gamma(k-j+1/2)}{(k-j)!}\right), \\ d_n &= \frac{\sin(x_n)}{\sqrt{\tau\pi}\sin(x_s)}\left(\frac{\Gamma(k-j-1/2)}{(k-j-1)!} - \frac{\Gamma(k-j+1/2)}{(k-j)!}\right), \\ v &= -\frac{1}{\sqrt{\tau}} - \frac{1}{\tau}, \qquad c_n = \frac{\sin(x_n)}{\sqrt{\tau}\sin(x_s)}, \end{aligned}$$

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \quad \text{for any } r = 0,1,\ldots,k,$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\phi_n^k = \left(\frac{\rho(t_k)}{\sin(x_s)}\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{h^2} - \frac{\rho(t_k)}{\sin(x_s)}\sin(x_n)\right)$$
$$- \frac{1}{\sqrt{\pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+1/2)}{(k-m)!}\frac{\tau^{1/2}}{\sin(x_s)}\sin(x_n) + f(t_k, x_n), \tag{3.14}$$

and $D$ is $(M+1)\times(M+1)$ identity matrix. Using (3.11), we can obtain that

$$w^k = A^{-1}\left(D\varphi^k - \sum_{j=0}^{k-1} B_j w^j\right), \quad k = 1,2,\ldots,N, \quad w^0 = \{\sin x_n\}_{n=0}^{M}. \tag{3.15}$$

To solve the resulting difference equations, we apply the method given in (3.15) step by step for $k = 1,2,\ldots,N$. For the evaluation of $w^r$, $r = 2,3,\ldots,N$, $w^{r-1}$ is needed. It is obtained in the previous step. Then, the solution pairs $(u,p)$ are obtained by using the last formulas (3.9) and (3.8).

### *3.2. The Second Order of Accuracy Difference Scheme*

For the approximate solution of the problem (3.1), the Crank-Nicholson difference scheme

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{2h^2} + \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{2h^2} \\
- \frac{u_n^k + u_n^{k-1}}{2} + \frac{p^k + p^{k-1}}{2}q_n + D_\tau^{1/2}u\left(t_k - \frac{\tau}{2}, x_n\right) + f\left(t_k - \frac{\tau}{2}, x_n\right), \\
p^k = p(t_k), \quad 1 \le k \le N, \ 1 \le n \le M-1 \\
u_n^0 = \sin(x_n), \quad 0 \le n \le M, \\
u_0^k = u_M^k = 0, \quad 0 \le k \le N, \\
u_s^k + \frac{u_{s+1}^k - u_s^k}{h}(x^* - sh) = \rho(t_k), \quad 0 \le k \le N, \ s = \left\lfloor \frac{\pi}{2h} \right\rfloor
\end{gathered} \tag{3.16}$$

is constructed.

Here,

$$\Gamma\left(k-r+\frac{1}{2}\right)=\int_0^{\infty} t^{k-r+1/2}e^{-t}\,dt. \tag{3.17}$$

Moreover, applying the second order of approximation formula for

$$D_t^{1/2}u\left(t_k-\frac{\tau}{2}\right)=\frac{1}{\Gamma(1/2)}\int_0^{t_k-\tau/2}\left(t_k-\frac{\tau}{2}-s\right)^{-1/2}u'(s)ds, \tag{3.18}$$

it is obtained (see [27])

$$D_\tau^{1/2}u=\begin{cases}\dfrac{-d\sqrt{2}}{3}u_0+\dfrac{d\sqrt{2}}{3}u_1+\dfrac{d\tau}{3\sqrt{2}}\sin(x_n), & k=1,\\ -\dfrac{2d\sqrt{6}}{5}u_0+\dfrac{d\sqrt{6}}{5}u_1+\dfrac{d\sqrt{6}}{5}u_2-\dfrac{d\tau\sqrt{6}}{10}\sin(x_n), & k=2,\\ d\displaystyle\sum_{m=2}^{k-1}\{[(k-m)b_1+b_2]u_{m-2}+[(2m-2k-1)b_1-2b_2]u_{m-1} & \\ \qquad +[(k-m+1)b_1+b_2]u_m\}+\dfrac{d}{6\sqrt{2}}[-u_{k-2}-4u_{k-1}+5u_k], & 3\le k\le N.\end{cases} \tag{3.19}$$

Here and throughout the paper,

$$\begin{gathered}D_t^{1/2}u=D_\tau^{1/2}u\left(t_k-\frac{\tau}{2},x_n\right),\\ d=\frac{2}{\sqrt{\pi\tau}},\qquad b_1=\sqrt{k-m+\frac{1}{2}}-\sqrt{k-m-\frac{1}{2}},\\ b_2=-\frac{1}{3}\left(\left(k-m+\frac{1}{2}\right)^{3/2}-\left(k-m-\frac{1}{2}\right)^{3/2}\right).\end{gathered} \tag{3.20}$$

We search the solution of (3.16) in the following form:

$$u_n^k=\eta^k q_n+w_n^k, \tag{3.21}$$

where

$$\eta^k=\sum_{i=1}^{k}\frac{p^i+p^{i-1}}{2}\tau,\quad 1\le k\le N,\ \eta^0=0. \tag{3.22}$$

We have that

$$u_s^k + \frac{u_{s+1}^k - u_s^k}{h}(x^* - sh) = \eta^k\left(\left(1 - \frac{x^* - sh}{h}\right)q_s + \frac{x^* - sh}{h}q_{s+1}\right) + \left(1 - \frac{x^* - sh}{h}\right)w_s^k + \frac{x^* - sh}{h}w_{s+1}^k = \rho(t_k). \tag{3.23}$$

Let us denote

$$y = \frac{x^* - sh}{h} = \frac{x^*}{h} - \left\lfloor \frac{x^*}{h} \right\rfloor h, \tag{3.24}$$

where $0 \leq y < 1$. Then, one can write

$$\eta^k = \frac{\rho(t_k) - (1-y)w_s^k - yw_{s+1}^k}{(1-y)q_s + yq_{s+1}}. \tag{3.25}$$

So the values of $(p(t_k) + p(t_{k-1}))/2,\ 1 \leq k \leq N$ can be obtained by the following formula:

$$\frac{p^k + p^{k-1}}{2} = \frac{(\rho(t_k) - \rho(t_{k-1}))/\tau - (1-y)((w_s^k - w_s^{k-1})/\tau) - y\left(\left(w_{s+1}^k - w_{s+1}^{k-1}\right)/\tau\right)}{(1-y)q_s + yq_{s+1}}. \tag{3.26}$$

Let $w^r$ denote

$$w^r = \begin{bmatrix} w_0^r \\ \vdots \\ w_M^r \end{bmatrix}_{(M+1)\times 1} \quad \text{for } r = 0, 1, \ldots, N. \tag{3.27}$$

For $k = 1$, one can show that $w^1$ is the solution of the difference scheme

$$\frac{w_n^1 - w_n^0}{\tau} = \frac{w_{n+1}^1 - 2w_n^1 + w_{n-1}^1}{2h^2} + \frac{w_{n+1}^0 - 2w_n^0 + w_{n-1}^0}{2h^2} - \frac{w_n^1 + w_n^0}{2} + \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right)$$

$$\begin{aligned}
&\times\left(\frac{\rho(t_1)-(1-y)w_s^1-yw_{s+1}^1}{(1-y)q_s+yq_{s+1}}+\frac{\rho(t_0)-(1-y)w_s^0-yw_{s+1}^0}{(1-y)q_s+yq_{s+1}}\right)\\
&+\frac{d\sqrt{2}}{3}\left(\frac{\rho(t_1)-(1-y)w_s^1-yw_{s+1}^1}{(1-y)q_s+yq_{s+1}}q(n)+w_n^1\right)\\
&-\frac{d\sqrt{2}}{3}\left(\frac{\rho(t_0)-(1-y)w_s^0-yw_{s+1}^0}{(1-y)q_s+yq_{s+1}}q(n)+w_n^0\right)\\
&+\frac{d\tau}{3\sqrt{2}}q(n)+f\left(t_1-\frac{\tau}{2},x_n\right),\quad 1\le n\le M-1,\\
w_0^1 &= w_M^1 = 0,\\
w_n^0 &= \sin(x_n),\quad 0\le n\le M.
\end{aligned} \tag{3.28}$$

We have the system of linear equations and we write them in the matrix form

$$A_1w^1+B_1w^0=D\varphi^1, \tag{3.29}$$

where

$$A_1=\begin{bmatrix}
1 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0\\
a & y_1 & a & 0 & \cdot & l_1 & c_1 & \cdot & 0 & 0 & 0\\
0 & a & y_1 & a & \cdot & l_2 & c_2 & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & a & \cdot & l_s+a & c_s & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cdot & l_{s+1}+y_1 & c_{s+1}+a & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & l_{M-1} & c_{M-1} & \cdot & a & y_1 & a\\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_1=\begin{bmatrix}
0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0\\
a & v_1 & a & 0 & \cdot & d_1 & e_1 & \cdot & 0 & 0 & 0\\
0 & a & v_1 & a & \cdot & d_2 & e_2 & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & a & \cdot & d_s+a & e_s & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cdot & d_{s+1}+v_1 & e_{s+1}+a & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & d_{M-1} & e_{M-1} & \cdot & a & v_1 & a\\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0
\end{bmatrix}_{(M+1)\times(M+1)}. \tag{3.30}$$

Here, for any $n = 1, 2, \ldots, M-1$,

$$
\begin{aligned}
a &= \left(-\frac{1}{2h^2}\right), \qquad y_1 = \left(\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - d\frac{\sqrt{2}}{3}\right), \\
v_1 &= \left(-\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + d\frac{\sqrt{2}}{3}\right), \\
l_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} + d\frac{\sqrt{2}}{3}q_n, \\
c_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} + d\frac{\sqrt{2}}{3}q_n, \\
d_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} - d\frac{\sqrt{2}}{3}q_n, \\
e_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} - d\frac{\sqrt{2}}{3}q_n, \\
\varphi^1 &= \begin{bmatrix} 0 \\ \phi_1^1 \\ \vdots \\ \phi_{M-1}^1 \\ 0 \end{bmatrix}_{(M+1)\times 1}, \\
\phi_n^1 &= \left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{2h^2} - \frac{\sin(x_n)}{2}\right)\frac{\rho(t_1) + \rho(t_0)}{(1-y)q_s + yq_{s+1}} \\
&\quad + \frac{d\sqrt{2}q_n}{3}\frac{\rho(t_1) - \rho(t_0)}{(1-y)q_s + yq_{s+1}} + \frac{d\tau}{3\sqrt{2}}q_n + f\left(t_1 - \frac{\tau}{2}, x_n\right)
\end{aligned}
\tag{3.31}
$$

and $D$ is $(M+1) \times (M+1)$ identity matrix. Using (3.29), we can obtain that

$$
w^1 = A_1^{-1}\left(D\varphi^1 - B_1 w^0\right), \qquad w^0 = \{\sin x_n\}_{n=0}^M. \tag{3.32}
$$

For $k = 2$, $w^2$ is the solution of the difference scheme

$$
\begin{aligned}
\frac{w_n^2 - w_n^1}{\tau} = \frac{w_{n+1}^2 - 2w_n^2 + w_{n-1}^2}{2h^2} - \frac{w_n^2 + w_n^1}{2} + \frac{w_{n+1}^1 - 2w_n^1 + w_{n-1}^1}{2h^2} \\
+ \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right)\left(\frac{\rho(t_2) - (1-y)w_s^2 - yw_{s+1}^2}{(1-y)q_s + yq_{s+1}}\right. \\
\left. + \frac{\rho(t_1) - (1-y)w_s^1 - yw_{s+1}^1}{(1-y)q_s + yq_{s+1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d\sqrt{6}}{5}\left(\frac{\rho(t_2)-(1-y)w_s^2-yw_{s+1}^2}{(1-y)q_s+yq_{s+1}}q(n)+w_n^2\right)\\
&+\frac{d\sqrt{6}}{5}\left(\frac{\rho(t_1)-(1-y)w_s^1-yw_{s+1}^1}{(1-y)q_s+yq_{s+1}}q(n)+w_n^1\right)\\
&-\frac{2d\sqrt{6}}{5}\left(\frac{\rho(t_0)-(1-y)w_s^0-yw_{s+1}^0}{(1-y)q_s+yq_{s+1}}q(n)+w_n^0\right)\\
&-\frac{d\tau\sqrt{6}}{10}q(n)+f\left(t_2-\frac{\tau}{2},x_n\right),\quad 1\le n\le M-1,\\
w_0^2 &= w_M^2=0,\\
w_n^0 &= \sin(x_n),\quad 0\le n\le M.
\end{aligned}
\tag{3.33}
$$

The system of linear equations given above can be written in the matrix form

$$
A_2w^2+B_2w^1+C_2w^0=D\varphi^2, \tag{3.34}
$$

where

$$
A_2=\begin{bmatrix}
1 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0\\
a & y_2 & a & 0 & \cdot & g_1 & h_1 & \cdot & 0 & 0 & 0\\
0 & a & y_2 & a & \cdot & g_2 & h_2 & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & g_s+a & h_s & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cdot & g_{s+1}+y_2 & h_{s+1}+a & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & g_{M-1} & h_{M-1} & \cdot & a & y_2 & a\\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 1
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$
B_2=\begin{bmatrix}
0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0\\
a & v_2 & a & 0 & \cdot & g_1 & h_1 & \cdot & 0 & 0 & 0\\
0 & a & v_2 & a & \cdot & g_2 & h_2 & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & g_s+a & h_s & \cdot & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cdot & g_{s+1}+v_2 & h_{s+1}+a & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & 0 & 0 & \cdot & g_{M-1} & h_{M-1} & \cdot & a & v_2 & a\\
0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0
\end{bmatrix}_{(M+1)\times(M+1)},
$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & z & 0 & 0 & \cdot & i_1 & j_1 & \cdot & 0 & 0 & 0 \\ 0 & 0 & z & 0 & \cdot & i_2 & j_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & i_s + z & j_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & i_{s+1} & j_{s+1} + z & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & i_{M-1} & j_{M-1} & \cdot & 0 & z & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)}. \tag{3.35}$$

Here, for any $n = 1, 2, \dots, M-1$,

$$\begin{aligned}
a &= -\frac{1}{2h^2}, \qquad y_2 = \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{d\sqrt{6}}{5}, \\
v_2 &= -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{d\sqrt{6}}{5}, \\
g_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} \\
&\quad + \frac{d\sqrt{6}q_n(1-y)}{5((1-y)q_s + yq_{s+1})}, \quad \text{in } (s+1)\text{th column}, \\
h_n &= \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} \\
&\quad + \frac{d\sqrt{6}q_n y}{5((1-y)q_s + yq_{s+1})}, \quad \text{in } (s+2)\text{th column}, \\
i_n &= -\frac{2d\sqrt{6}q_n(1-y)}{5((1-y)q_s + yq_{s+1})}, \qquad j_n = -\frac{2\sqrt{6}q_n y}{5((1-y)q_s + yq_{s+1})}, \qquad z = \frac{2d\sqrt{6}}{5}, \\
\varphi^2 &= \begin{bmatrix} 0 \\ \phi_1^2 \\ \vdots \\ \phi_{M-1}^2 \\ 0 \end{bmatrix}_{(M+1)\times 1}, \\
\phi_n^2 &= \left(\frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{2h^2} - \frac{\sin(x_n)}{2} + \frac{d\sqrt{6}q_n}{5}\right) \\
&\quad \times \frac{\rho(t_2) + \rho(t_1)}{(1-y)q_s + yq_{s+1}} - \frac{2d\sqrt{6}q_n}{5}\frac{\rho(t_0)}{(1-y)q_s + yq_{s+1}} - \frac{d\tau\sqrt{6}}{10}q_n + f\left(t_2 - \frac{\tau}{2}, x_n\right).
\end{aligned} \tag{3.36}$$

Using (3.34), we can obtain that

$$w^2 = A_2^{-1}\left(D\varphi^2 - B_2 w^1 - C_2 w^0\right), \quad w^0 = \{\sin x_n\}_{n=0}^{M}. \tag{3.37}$$

For $3 \le k \le N$, we can obtain the following difference scheme:

$$\begin{aligned}
\frac{w_n^k - w_n^{k-1}}{\tau} = {} & \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{2h^2} + \frac{w_{n+1}^{k-1} - 2w_n^{k-1} + w_{n-1}^{k-1}}{2h^2} \\
& - \frac{w_n^k + w_n^{k-1}}{2} + \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right) \\
& \times \left(\frac{\rho(t_k) - (1-y)w_s^k - yw_{s+1}^k}{(1-y)q_s + yq_{s+1}}\right) + \left(\frac{q_{n+1} - 2q_n + q_{n-1}}{2h^2} - \frac{q_n}{2}\right) \\
& \times \left(\frac{\rho(t_{k-1}) - (1-y)w_s^{k-1} - yw_{s+1}^{k-1}}{(1-y)q_s + yq_{s+1}}\right) \\
& + d\sum_{m=2}^{k-1}\Bigg\{((k-m)b_1 + b_2) \\
& \qquad \times \left(\frac{\rho(t_{m-2}) - (1-y)w_s^{m-2} - yw_{s+1}^{m-2}}{(1-y)q_s + yq_{s+1}} q(n)\right) + ((2m-2k-1)b_1 - 2b_2) \\
& \qquad \times \left(\frac{\rho(t_{m-1}) - (1-y)w_s^{m-1} - yw_{s+1}^{m-1}}{(1-y)q_s + yq_{s+1}} q(n)\right) \\
& \qquad + ((2m-2k-1)b_1 - 2b_2)w_n^{m-1} \\
& \qquad \times \left(\frac{\rho(t_m) - (1-y)w_s^m - yw_{s+1}^m}{(1-y)q_s + yq_{s+1}} q(n)\right) + ((k-m)b_1 + b_2)w_n^{m-2} \\
& \qquad + ((k-m-1)b_1 + b_2)w_n^m + ((2m-2k-1)b_1 - 2b_2)w_n^{m-1}\Bigg\} \\
& - \frac{d}{6\sqrt{2}}\left(\frac{\rho(t_{k-2}) - (1-y)w_s^{k-2} - yw_{s+1}^{k-2}}{(1-y)q_s + yq_{s+1}} q(n) + w_n^{k-2}\right) \\
& - \frac{4d}{6\sqrt{2}}\left(\frac{\rho(t_{k-1}) - (1-y)w_s^{k-1} - yw_{s+1}^{k-1}}{(1-y)q_s + yq_{s+1}} q(n) + w_n^{k-1}\right) \\
& + \frac{5d}{6\sqrt{2}}\left(\frac{\rho(t_k) - (1-y)w_s^k - yw_{s+1}^k}{(1-y)q_s + yq_{s+1}} q(n) + w_n^k\right) + f\left(t_k - \frac{\tau}{2}, x_n\right), \\
& \qquad\qquad 1 \le n \le M-1, \\
& w_0^k = w_M^k = 0, \quad 3 \le k \le N, \\
& w_n^0 = \sin(x_n), \quad 0 \le n \le M.
\end{aligned} \tag{3.38}$$

This system can be written in matrix form as

$$A_3 w^k + B_3 w^{k-1} + C_3 w^{k-2} + \sum_{j=0}^{k-3} E_j w^j = D\varphi^k, \tag{3.39}$$

where

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ a & y_3 & a & 0 & \cdot & r_1 & m_1 & \cdot & 0 & 0 & 0 \\ 0 & a & y_3 & a & \cdot & r_2 & m_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & r_s + a & m_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & r_{s+1} + y_3 & m_{s+1} + a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & r_{M-1} & s_{M-1} & \cdot & a & y_3 & a \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 1 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ a & X & a & 0 & \cdot & Y_1 & Z_1 & \cdot & 0 & 0 & 0 \\ 0 & a & X & a & \cdot & Y_2 & Z_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & Y_s + a & Z_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & Y_{s+1} + X & Z_{s+1} + a & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & Y_{M-1} & Z_{M-1} & \cdot & a & X & a \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & S & 0 & 0 & \cdot & R_1 & T_1 & \cdot & 0 & 0 & 0 \\ 0 & 0 & S & 0 & \cdot & R_2 & T_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & R_s + S & T_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & R_{s+1} & T_{s+1} + S & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & R_{M-1} & T_{M-1} & \cdot & 0 & S & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & G & 0 & 0 & \cdot & H_1 & I_1 & \cdot & 0 & 0 & 0 \\ 0 & 0 & G & 0 & \cdot & H_2 & I_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & H_s + G & I_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & H_{s+1} & I_{s+1} + G & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & H_{M-1} & I_{M-1} & \cdot & 0 & G & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & G1 & 0 & 0 & \cdot & J_1 & K_1 & \cdot & 0 & 0 & 0 \\ 0 & 0 & G1 & 0 & \cdot & J_2 & K_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & J_s + G1 & K_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & J_{s+1} & K_{s+1} + G1 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & J_{M-1} & K_{M-1} & \cdot & 0 & G1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

$$E_j = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \\ 0 & G2 & 0 & 0 & \cdot & P_1 & L_1 & \cdot & 0 & 0 & 0 \\ 0 & 0 & G2 & 0 & \cdot & P_2 & L_2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & P_s + G2 & L_s & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & P_{s+1} & L_{s+1} + G2 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & P_{M-1} & L_{M-1} & \cdot & 0 & G2 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}_{(M+1)\times(M+1)},$$

(3.40)

for $j = 2, 3, \ldots k - 3$.

Here, for any $n = 1, 2, \ldots, M - 1$,

$$a = -\frac{1}{2h^2}, \qquad d = \frac{2}{\sqrt{\pi\tau}}, \qquad y_3 = \frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} - \frac{5d}{6\sqrt{2}},$$

$$v_3 = -\frac{1}{\tau} + \frac{1}{h^2} + \frac{1}{2} + \frac{4d}{6\sqrt{2}},$$

$$\alpha_n = \frac{dq(n)(1-y)}{6\sqrt{2}((1-y)q_s + yq_{s+1})}, \qquad \beta_n = \frac{dq(n)y}{6\sqrt{2}((1-y)q_s + yq_{s+1})},$$

$$r_n = \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} + 5\frac{dq(n)(1-y)}{6\sqrt{2}((1-y)q_s + yq_{s+1})},$$

$$s_n = \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} + 5\frac{dq(n)y}{6\sqrt{2}((1-y)q_s + yq_{s+1})},$$

$$X = v_3 - pp \times d, \qquad Y_{n-1} = \gamma_{n-1} + 6\sqrt{2}pp \times \alpha_{n-1},$$

$$Z_{n-1} = \delta_{n-1} + 6\sqrt{2}pp \times \beta_{n-1}, \qquad S = d\left(\frac{1}{6\sqrt{2}} - pp - nn\right),$$

$$R_{n-1} = -\alpha_{n-1} + 6\sqrt{2}(nn + pp)\alpha_{n-1},$$

$$T_{n-1} = -\beta_{n-1} + 6\sqrt{2}(nn + pp)\beta_{n-1},$$

$$\gamma_n = \frac{(q_{n+1} - 2q_n + q_{n-1})(1-y)}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n(1-y)}{2((1-y)q_s + yq_{s+1})} - 4\frac{dq(n)(1-y)}{6\sqrt{2}((1-y)q_s + yq_{s+1})},$$

$$\delta_n = \frac{(q_{n+1} - 2q_n + q_{n-1})y}{2h^2((1-y)q_s + yq_{s+1})} + \frac{q_n y}{2((1-y)q_s + yq_{s+1})} - 4\frac{dq(n)y}{6\sqrt{2}((1-y)q_s + yq_{s+1})},$$

$$mm = (k-m)b_1 + b_2, \qquad nn = (2m - 2k - 1)b_1 - 2b_2,$$

$$pp = (k - m + 1)b_1 - 2b_2,$$

$$nn1 = (2(m+1) - 2k - 1)b_1 - 2b_2,$$

$$pp1 = (k - (m+2) + 1)b_1 + b_2,$$

$$G = -d \times mm, \qquad G1 = -d \times (mm + nn1),$$

$$G2 = -d \times (mm + nn1 + pp1),$$

$$H_{n-1} = 6\sqrt{2} \times \alpha_1 \times mm, \qquad I_{n-1} = 6\sqrt{2} \times \beta_1 \times mm,$$

$$J_{n-1} = 6\sqrt{2} \times \alpha_{n-1} \times (mm + nn1),$$

$$K_{n-1} = 6\sqrt{2} \times \beta_{n-1} \times (mm + nn1),$$

$$P_{n-1} = 6\sqrt{2} \times \alpha_{n-1} \times (mm + nn1 + pp1),$$

$$L_{n-1} = 6\sqrt{2} \times \beta_{n-1} \times (mm + nn1 + pp1),$$

$$\varphi^k = \begin{bmatrix} 0 \\ \phi_1^k \\ \vdots \\ \phi_{M-1}^k \\ 0 \end{bmatrix}_{(M+1)\times 1},$$

$$\begin{aligned}
\phi_n^k = {} & \left( \frac{\sin(x_{n+1}) - 2\sin(x_n) + \sin(x_{n-1})}{2h^2} - \frac{\sin(x_n)}{2} \right) \frac{\rho(t_k) + \rho(t_{k-1})}{(1-y)q_s + yq_{s+1}} \\
& + \frac{5dq_n\rho(t_k)}{6\sqrt{2}((1-y)q_s + yq_{s+1})} - \frac{4dq_n\rho(t_{k-1})}{6\sqrt{2}((1-y)q_s + yq_{s+1})} \\
& - \frac{dq_n\rho(t_{k-2})}{6\sqrt{2}((1-y)q_s + yq_{s+1})} + f\left(t_k - \frac{\tau}{2}, x_n\right) + \frac{dq_n}{(1-y)q_s + yq_{s+1}} \\
& \times \sum_{m=2}^{k-1} \{((k-m)b_1 + b_2)\rho(t_{m-2}) \\
& \qquad + ((2m - 2k - 1)b_1 - 2b_2)\rho(t_{m-1}) + ((k - m + 1)b_1 + b_2)\rho(t_m)\}.
\end{aligned} \tag{3.41}$$

**Table 1:** Comparison of exact solution and approximate solutions.

| Method | $N = M = 15$ | $N = M = 45$ | $N = M = 75$ |
|---|---|---|---|
| 1st order of accuracy | 0.1190 | 0.0126 | 0.0045 |
| 2nd order of accuracy | 0.0055 | $6.0917 \times 10^{-4}$ | $2.1932 \times 10^{-4}$ |

Finally, from (3.39), it follows that

$$w^k = (A_3)^{-1}\left(D\varphi^k - B_3 w^{k-1} - C_3 w^{k-2} - \sum_{j=0}^{k-3} E_j w^j\right), \quad 3 \leq k \leq N. \tag{3.42}$$

Applying the last formula step by step, we can reach $w^k$. Then, using (3.21), (3.25), and (3.26), we reach the approximate solutions of $u(t,x)$ and $(p(t_k) + p(t_{k-1}))/2$.

### 3.3. *Error Analysis*

In this part, the results of the numerical analysis is given. The numerical solutions are recorded for different values of $N$ and $M$ and $u_n^k$ represents the approximate solution of $u(t,x)$ at grid points $(t_k, x_n)$. Table 1 gives the error analysis between the exact solution and the solutions derived by difference schemes. Table 1 is constructed for $N = M = 15,\ 45$, and 75, respectively. For their comparison, the errors are computed by

$$E = \max_{1\leq k\leq N_{1\leq n\leq M}} \left| u(t_k, x_n) - u_n^k \right|. \tag{3.43}$$

Thus, the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

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