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Research Article

Variational Approach to Quasi-Periodic Solution of Nonautonomous Second-Order Hamiltonian Systems

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We deal with the quasi-periodic solutions of the following second-order Hamiltonian systems $\ddot{x}(t) = \nabla F(t, x(t))$, where $x(t) = (x_1(t), \ldots, x_N(t))$, and we present a new approach via variational methods and Minmax method to obtain the existence of quasi-periodic solutions to the above equation.

1. Introduction

In this paper, we consider the quasi-periodic solutions of the following second-order Hamiltonian system:

$$\ddot{x}(t) = \nabla F(t, x(t)), \tag{1.1}$$

where $x(t) = (x_1(t), \ldots, x_N(t))$, $\nabla F(t, x) = (\partial F/\partial x_1, \ldots, \partial F/\partial x_N)$ and $\partial F/\partial x_k \in C(R \times R^N, R), k = 1, \ldots, N$.

A special class of the system (1.1) is the following autonomous second-order Hamilton system with convex potential Φ:

$$\ddot{x}(t) = \nabla\Phi(x(t)) + e(t), \tag{1.2}$$

For the scalar case [1] and for the vectorial case [2], Berger and Chen have established the existence and uniqueness of almost periodic solution of (1.2). In [2], Berger and Chen assume that e is almost periodic, and the potential Φ is of the form

$$\Phi(x) = \frac{1}{2}\langle Ax \mid x\rangle + U(x), \tag{1.3}$$

where A is a symmetric positive-definite matrix and $U \in C^2(R^N, R)$ is a convex function. They also need the growth condition.

In [3], Carminati states a local version of the results of Berger and Chen, assuming that Φ is convex only near the minimum of Φ. The above growth condition is not used by Carminati. To prove the existence and uniqueness of bounded or almost periodic solution of (1.2), Carminati assumes that e is bounded or almost periodic and the potential Φ is of form (1.3), where A is a symmetric positive-definite matrix and U is a convex function of class C^1 on the ball $\overline{B}(x_0,\rho)$ $(\rho > 0)$, where Φ reaches its minimum in this ball at x_0.

When F is autonomous in the system (1.1), Padilla [4] states the existence of the quasi periodic solution by using critical point theory, but it assumes that the Diophantine condition is satisfied.

As to the system (1.1), using a variational method, Zakharin and Parasyuk [5] have studied the existence of almost (quasi)periodic solutions for the system

$$x''(t) = \nabla_x\Phi(t, x(t)), \tag{1.4}$$

where $\Phi \in C^0(R \times K, R)$, K is a compact convex subset of R^N and $\Phi(t,\cdot)$ is convex and differentiable on K, for each $t \in R$. The authors use a variational method on a Hilbert space of Besicovitch almost periodic functions which looks like a Sobolev space. By this method, the authors establish the existence of generalized solutions and in the quasi-periodic case, they prove that these solutions are classical. To prove the existence of quasi-periodic solutions, Zakharin and Parasyuk [5] assume that $\nabla_x\Phi$ is quasi-periodic in t, and $\nabla_x\Phi(t,\cdot)$ is strongly monotone on K with positive modulus c. They also assume that the boundary ∂K of K is a differentiable manifold of class C^1 such that, for each $x \in \partial K$, the gradient $\nabla_x\Phi(t, x)$ makes an acute angle with an external normal unit vector to ∂K at the point x in the Theorem 4.3 of [5] or a similar condition using the projection operator on the closed convex in the Theorem 4.2 of [5].

More recently in [6], Ayachi and Blot provided new variational settings to study the almost periodic solutions of a class of nonlinear neutral delay equation

$$\begin{aligned}
D_1L\big(x(t-r), x(t-2r), x'(t-r), x'(t-2r), t-r\big) &+ D_2L\big(x(t), x(t-r), x'(t), x'(t-r), t\big) \\
= \frac{d}{dt}\Big[D_3L\big(x(t-r), x(t-2r), &x'(t-r), x'(t-2r), t-r\big) \\
&+D_4L\big(x(t), x(t-r), x'(t), x'(t-r), t\big)\Big],
\end{aligned} \tag{1.5}$$

where $L : (R^n)^4 \times R \to R$ is a differentiable function, D_j denotes the partial differential with respect to the jth vector variable, and $r \in (0,\infty)$ is fixed. When they consider the almost periodicity in the sense of Corduneau [7], they obtain some results on the structure of the set of Bohr almost periodic solutions in the case that L is autonomous and convex. When they consider the almost periodicity in the sense of Besicovitch [8], they assume that D_kL satisfies a Lipschitiz condition and L is convex and obtain the existence of Besicovitch almost periodic solution by the least action principle.

A special case of the above equation is the system (1.1); our main purpose is to apply Minmax method to study the existence of quasi periodic solutions to the system (1.1), and we do not assume that $\nabla F(t,\cdot)$ is Lipschitzian, but we assume that F satisfies some growth conditions, then we obtain results of existence of quasi periodic solution to the system (1.1). Moreover, when we consider only a frequency $2\pi/T$, our results will cover the results of periodic solutions to the system (1.1).

The present paper is organized as follows. In Section 1 we review some notations and definitions of almost periodic functions. In Section 2, we state our main theorems. In Section 3, in order to prove our main results, we will state our basic lemmas. In Section 4, we prove our main results and give an example.

Now we give some notations and definitions of almost periodic functions.

Definition 1.1 (Fink [9]). A function $f(t)$ is said to be Bohr almost periodic, if for any $\epsilon > 0$, there is a constant $l_\epsilon > 0$, such that in any interval of length l_ϵ, there exists τ such that the inequality $|f(t+\tau) - f(t)| < \epsilon$ is satisfied for all $t \in R$.

Definition 1.2 (He [10]). $f \in C^0(R \times R^m, R^N)$ is so called almost periodic in t uniformly for $x \in R^m$ when, for each compact subset K in R^m, for each $\varepsilon > 0$, there exists $l > 0$, and for each $\alpha \in R$, there exists $\tau \in [\alpha, \alpha + l]$ such that

$$\sup_{t\in R}\sup_{x\in K}\left\| f(t+\tau, x) - f(t,x)\right\|_{R^N} < \varepsilon. \tag{1.6}$$

$AP^0(R^N)$ is the space of the Bohr almost periodic functions from R to R^N, endowed with the norm $\|x\|_\infty = \sup_{t\in R}|x(t)|$ and it is a Banach space.

$AP^1(R^N) = \{x \in AP^0(R^N) \bigcap C^1(R, R^N) \mid x'(t) \in AP^0(R^N)\}$; endowed with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$, it is a Banach space.

A fundamental property of almost periodic functions is that such functions have convergent means, that is, the following limit exists:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt. \tag{1.7}$$

The Fourier-Bohr coefficients of $AP^0(R^N)$ are the complex vectors

$$a(x,\lambda) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}x(t)dt, \tag{1.8}$$

and $\Lambda(x) = \{\lambda \in R \mid a(x,\lambda) \neq 0\}$ and it is a countable set,

when $p \in Z^+$, $B^P(R^N)$ is the completion of $AP^0(R^N)$ with respect to the norm

$$\|u\|_p = \left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|u|^p dt\right\}^{1/p}. \tag{1.9}$$

When $p = 2, B^2(R^N)$ is a Hilbert spaces and its norm $\|\cdot\|_2$ is associated to the inner product

$$\langle u, v\rangle_2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(u, v)dt, \tag{1.10}$$

the elements of these spaces $B^P(R^N)$ are called Besicovitch almost periodic functions.

We use the generalized derivative $\nabla u \in B^2(R^N)$ of $u \in B^2(R^N)$ defined by $\|\nabla u - (1/s)(u(\cdot + s) - u)\| \to 0(s \to 0)$, and we will identify the equivalence class u and its continuous representant

$$u(t) = \int_0^t \nabla u(t)dt + c. \tag{1.11}$$

Then we define $B^{1,2}(R^N) = \{u \in B^2(R^N) \mid \nabla u \in B^2(R^N)\}$, endowed with the norm

$$\|u\| = \left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(|u(t)|^2 + |\nabla u(t)|^2\right)dt\right\}^{1/2}. \tag{1.12}$$

Its norm is associated to the inner product $\langle u, v\rangle = \langle u, v\rangle_2 + \langle \nabla u, \nabla v\rangle_2$, and $B^{1,2}(R^N)$ is a Hilbert space.

For convenience, we denote $\Lambda = \{(2k\pi/T_j) \mid k \in Z, j = 1,\ldots,p\}$ ($T_1,\ldots,T_p$ is rationally independent), $\Lambda(x)$ is the set of all Fourier exponents $\{\lambda_k\}$ of x, which is called the spectrum of $x; V = \{x \in B^{1,2}(R^N) \mid \Lambda(x) \subseteq \Lambda\}$, it is easily obtained that V is a linear subspace of $B^{1,2}(R^N)$ and V is a Hilbert space.

2. Main Theorems

In this section, we state our main results. First, we give the following list of assumptions on F:

(f_1) $F(t,\cdot) \in C^1(R \times R^N, R)$, and $F(t,\cdot)$ is almost periodic in t uniformly for $x \in R^N$,

(f_2) $\nabla F(t,\cdot)$ is almost periodic in t uniformly for $x \in R^N$,

(f_3) for any $\lambda \in R \setminus \Lambda$, $x \in V$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\nabla F(t,x)e^{-i\lambda t}dt = 0. \tag{2.1}$$

Theorem 2.1. *Suppose F satisfies (f_1)–(f_3), the functional $I : V \to R$, defined by*

$$I(x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{2}|\nabla x|^2 + F(t,x)\right]dt \tag{2.2}$$

is continuously differentiable on V, and $I'(x)$ is defined by

$$I'(x)h = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\nabla x \nabla h + \nabla F(t,x)h]dt. \tag{2.3}$$

Moreover, if x is a critical point of I in V, then

$$\nabla F(t,x) = \nabla(\nabla x). \tag{2.4}$$

Definition 2.2. When x satisfies (2.4) in Theorem 2.1, we say that x is a weak solution of (1.1).

Theorem 2.3. *Suppose that F satisfies (f_1)–(f_3), and*

(f_4) *there exists $g \in L^1_{\text{loc}}(R)$, for a.e. $t \in R$ and all $x \in R^N$, such that*

$$|\nabla F(t,x)| \le g(t), \tag{2.5}$$

(f_5) *There exists*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t,x)dt \longrightarrow +\infty \quad as\ |x| \longrightarrow \infty. \tag{2.6}$$

Then (1.1) *has at least a quasi periodic solution.*

Theorem 2.4. *Suppose that F satisfies (f_1)–(f_4), and*

(f_6) *One has*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F(t,x)dt \longrightarrow -\infty \quad as\ |x| \longrightarrow \infty. \tag{2.7}$$

Then (1.1) *has at least a quasi periodic solution by saddle point theorem.*

Remark 2.5. When V only contains a frequency $2\pi/T$, $F(t,x)$ is periodic in t with periodic T, which means that (f_3) is satisfied; our results cover some results in [11].

3. Basic Lemmas

To apply critical point theory to study the quasi periodic solution of (1.1), we will state our basic lemmas, which will be used in the proofs of our main results.

Lemma 3.1. *If $x \in V$, then*

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{i\lambda_k t} \in AP^0\left(R^N\right), \qquad \|x\|_\infty \le (C+1)\|x\|. \tag{3.1}$$

Proof. For any $x(t) \in V$,

$$x(t) \sim \sum_{k=-\infty}^{+\infty} a_k e^{i\lambda_k t}, \qquad \nabla x(t) \sim \sum_{k=-\infty}^{+\infty} i a_k \lambda_k e^{i\lambda_k t}, \quad \lambda_k \in \Lambda. \tag{3.2}$$

It is easily obtained that there exists a constant $C > 0$, such that

$$\begin{aligned} &\sum_{k=-\infty, \lambda_k \neq 0}^{k=+\infty} \frac{1}{\lambda_k^2} \le C^2, \\ |a_0| + \sum_{k=-\infty,\, \lambda_k \neq 0}^{k=+\infty} |a_k| &\le \|x\|_2 + \left(\sum_{k=-\infty,\, \lambda_k \neq 0}^{k=+\infty} \frac{1}{\lambda_k^2} \right)^{1/2} \left(\sum_{k=-\infty,\, \lambda_k \neq 0}^{k=+\infty} |\lambda_k a_k|^2 \right)^{1/2} \\ &\le \|x\|_2 + C\|\nabla x\|_2 \\ &\le (C+1)\|x\|. \end{aligned} \tag{3.3}$$

So

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{i\lambda_k t} \in AP^0(R^N), \qquad \|x\|_\infty \le (C+1)\|x\|. \tag{3.4}$$

□

Lemma 3.2. *For any $\{x_k\} \subset V$, if the sequence $\{x_k\}$ converges weakly to x, then $\{x_k\}$ converges uniformly to x on any compact subset of R.*

Proof. By Lemma 3.1, the injection of V into $C(R, R^N)$, with its natural norm $\|\cdot\|_\infty$, is continuous. Since $\{x_k\} \rightharpoonup x$ in V, it follows that $\{x_k\} \rightharpoonup x$ in $C(R, R^N)$. By Banach-Steinhaus theorem, $\{x_k\}$ is bounded in V, and hence in $C(R, R^N)$, we need to show that the sequence $\{x_k\}$ is equiuniformly continuous, for any $x_k(t) \in V$,

$$x_k(t) = \sum_{m=-\infty}^{+\infty} a_m e^{i\lambda_m t}, \tag{3.5}$$

let

$$x_{kj}(t) = \sum_{m=-\infty,\ \lambda_m \in \left\{ \left(2n\pi/T_j\right) | n \in Z \right\}}^{+\infty} a_m e^{i\lambda_m t}, \tag{3.6}$$

then

$$x_k(t) = x_{k1}(t) + x_{k2}(t) + \cdots + x_{kp}(t). \tag{3.7}$$

Denoting

$$T_{\min} = \min\{T_1, T_2, \ldots, T_p\}, \qquad T_{\max} = \max\{T_1, T_2, \ldots, T_p\}, \tag{3.8}$$

for $0 \le t - s \le T_{\min}$, we have

$$\begin{aligned}
&|x_k(t) - x_k(s)| \\
&\quad\le \int_s^t |x_k'(\tau)|d\tau \le \int_s^t |x_{k1}'(\tau)|d\tau + \int_s^t |x_{k2}'(\tau)|d\tau + \cdots + \int_s^t |x_{kn}'(\tau)|d\tau \\
&\quad\le (t-s)^{1/2}\left\{\int_s^t |x_{k1}'(\tau)|^2 d\tau\right\}^{1/2} + (t-s)^{1/2}\left\{\int_s^t |x_{k2}'(\tau)|^2 d\tau\right\}^{1/2} \\
&\qquad + \cdots + (t-s)^{1/2}\left\{\int_s^t |x_{kn}'(\tau)|^2 d\tau\right\}^{1/2} \\
&\quad\le (t-s)^{1/2}\left\{\left[\int_s^{s+T_1} |x_{k1}'(\tau)|^2 d\tau\right]^{1/2} + \left[\int_s^{s+T_2} |x_{k2}'(\tau)|^2 d\tau\right]^{1/2} + \cdots + \left[\int_s^{s+T_p} |x_{kn}'(\tau)|^2 d\tau\right]^{1/2}\right\} \\
&\quad\le (t-s)^{1/2}\{T_1\|x_k'\|_2 + T_2\|x_k'\|_2 + \cdots + T_p\|x_k'\|_2\} \le (t-s)^{1/2} p T_{\max}\|x_k\| \le C_1(t-s)^{1/2}.
\end{aligned} \tag{3.9}$$

By Arzela-Ascoli theorem, $\{x_k\}$ is relatively compact on any compact of R. By the uniqueness of the weak limit, every uniformly convergent subsequence of $\{x_k\}$ converges to x on any compact of R. □

Lemma 3.3. *If $x \in V$ and*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t) = 0, \tag{3.10}$$

then there exists $C > 0$, such that

$$\|x\|_\infty^2 \le C^2 \|\nabla x\|_2^2. \tag{3.11}$$

Proof. Since, by Lemma 3.1, x has the Fourier expansion

$$x(t) = \sum_{k=-\infty,\ \lambda_k \neq 0}^{+\infty} a_k e^{i\lambda_k t}. \tag{3.12}$$

The Cauchy-Schwarz inequality and Parseval equality imply that

$$
\begin{aligned}
|x(t)|^2 &\leq \left(\sum_{k=-\infty,\ \lambda_k\neq 0}^{+\infty} |a_k|\right)^2 \\
&\leq \left(\sum_{k=-\infty,\ \lambda_k\neq 0}^{+\infty} \frac{1}{\lambda_k^2}\right)\left(\sum_{k=-\infty,\ \lambda_k\neq 0}^{+\infty} |\lambda_k a_k|^2\right) \\
&\leq C^2\|\nabla x\|_2^2.
\end{aligned}
\tag{3.13}
$$

□

Lemma 3.4 (saddle point theorem). *Let X be a real Banach space, $X = X_1 \bigoplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(X, R)$ satisfies the PS condition and*

(I_1) *there exist constants $\sigma, \rho > 0$, such that $I_{\partial B_\rho \bigcap X_1} \leq \sigma$;*

(I_2) *there exists a constant $\omega > \sigma$, such that $I_{X_2} \geq \omega$.*

Then I possesses a critical value $c \geq \omega$ and

$$
c = \inf_{h\in\Gamma} \max_{u\in B_\rho \bigcap X_1} I(h(u)),
\tag{3.14}
$$

where $\Gamma = \{h \in C(\overline{B_\rho} \bigcap X_1, X) | h|_{\partial B_\rho \bigcap X_1} = id\}$.

4. The Proof of Main Results

In this section, we prove the main results stated in Section 2.

Proof of Theorem 2.1. First step: we show that I has at every point x a directional derivative $I'(x) \in V^*$ given by (2.3).

It follows easily from Lemma 3.1 and (f_3), for any $x \in V$ that we have

$$
F(t, x(t)) \in AP^0(R), \quad \nabla x \in B^2(R^N).
\tag{4.1}
$$

So I is everywhere finite on V. For x, h fixed in V, $\lambda \in [-1, 1]$, let us define

$$
\begin{aligned}
G(\lambda, t) &= \frac{1}{2}|\nabla x + \lambda\nabla h|^2 + F(t, x + \lambda h), \\
\varphi(\lambda) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} G(\lambda, t) = I(x + \lambda h).
\end{aligned}
\tag{4.2}
$$

There exists $\theta_t \in [0, 1]$, such that

$$
\begin{aligned}
&\frac{1}{\lambda}\left[\varphi(\lambda) - \varphi(0)\right] - \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\nabla F(t, x)h + \nabla x\nabla h]dt \\
&\quad = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\lambda|\nabla h|^2 + (\nabla F(t, x(t) + \theta_t\lambda h(t))h - \nabla F(t, x(t))h)\right]dt.
\end{aligned}
\tag{4.3}
$$

For x, h are fixed in V, there exists $M > 0$, such that $|x(t)| \leq M, |h(t)| \leq M$. Since $\nabla F(t,x)$ is almost periodic in t uniformly for $x \in R^N$, we have that $\nabla F(t,x)$ is uniformly continuous on $R \times K$, where $K = \{x \in R^N \mid |x| \leq 2M\}$ is compact subset in R^N, so

$$\lim_{\lambda \to 0} \frac{1}{\lambda} [\varphi(\lambda) - \varphi(0)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\nabla F(t,x)h + \nabla x \nabla h] dt. \tag{4.4}$$

Moreover, by Lemma 3.1,

$$\begin{aligned} (I'(x), h) &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\nabla F(t,x)h + \nabla x \nabla h] dt \\ &\leq c_1 \|h\|_\infty + c_2 \|\nabla h\|_2 \\ &\leq c_3 \|h\|. \end{aligned} \tag{4.5}$$

So I has, at x, a Gâteaux derivative $I'(x) \in (V)^*$.

Second step: we show that the mapping

$$I' : V \longrightarrow V^*, \qquad x \longrightarrow I'(x) \tag{4.6}$$

is continuous.

For any $\epsilon > 0$, x is fixed in V and $\|x\| \leq M$, let $y \in V$ with $\|x - y\| \leq \delta_0 < \epsilon/2$, by Lemma 3.1, it is easily obtained $|x(t)| \leq (C+1)M$ and $|y(t)| \leq (C+1)(M+\delta_0)$. Since $\nabla F(t,x)$ is uniformly continuous on $R \times K_1$, $K_1 = \{x \in R^N \mid |x| \leq (C+1)(M+\delta_0)\}$ is compact subset in R^N, then there exists δ_1, such that $|x(t) - y(t)| \leq \delta_1$ and we have

$$\nabla F(t, x(t)) - \nabla F\big(t, y(t)\big) \leq \frac{\epsilon}{2(C+1)}. \tag{4.7}$$

We denote $\delta^* = \min\{\delta_0, \delta_1/(C+1)\}$, then, for all $h \in V$ and $\|h\| \leq 1$, such that $\|x - y\| \leq \delta^*$, we have

$$\begin{aligned} \left|I'(x)h - I'(y)h\right| &\leq \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[\left(\nabla F(t,x) - \nabla F\big(t,y\big)\right)h + \left(\nabla x - \nabla y\right)\nabla h \right] dt \right| \\ &\leq \frac{\epsilon}{2(C+1)} \|h\|_\infty + \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left|\nabla x - \nabla y\right|^2 \right\}^{1/2} \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\nabla h|^2 \right\}^{1/2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned} \tag{4.8}$$

The above inequality holds, which implies the continuity of I' so that I is Fréchet differentiable on V.

If x is a critical point of I in V, for all $h \in V$, we have

$$I'(x)h = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\nabla F(t,x)h + \nabla x \nabla h]dt = 0, \tag{4.9}$$

by (f_3), then for all $h \in AP^1(R^N)$, we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\nabla F(t,x)h + \nabla x \nabla h]dt = 0. \tag{4.10}$$

Since $AP^1(R^N)$ is dense in $B^{1,2}(R^N)$, we have $DI(x)h = 0$, for all $h \in B^{1,2}(R^N)$; therefore, $I'(x) = 0$, and then we obtain (2.4) by using Blot [12]. The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.3. By Theorem 2.1, I is continuously differentiable on V. Next we will prove that I is weakly lower semicontinuous on V.

By Lemma 3.2, if $\{x_k\} \subset V$ converges weakly to x, then $\{x_k\}$ converges uniformly to x on any compact of R.

Since $x_k(t) \in AP^0(R^n)$, and $F(t,\cdot)$ is almost periodic in t uniformly for $x \in R^n$, then $F(t, x_k(t))$ is almost periodic, and $F(t, x_k(t))$ converges uniformly to $F(t, x(t))$ on any compact of R.

Let

$$\begin{aligned} F(t, x_k(t)) &= a_{k0} + \sum_{j=-\infty,\ \lambda_j \neq 0}^{+\infty} a_{kj} e^{i\lambda_j t}, \\ F(t, x(t)) &= a_0 + \sum_{j=-\infty,\ \lambda_j \neq 0}^{+\infty} a_j e^{i\lambda_j t}. \end{aligned} \tag{4.11}$$

Then it is easily obtained that

$$a_{k0} \longrightarrow a_0, \tag{4.12}$$

moreover,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t, x_k(t))dt = a_{k0}, \tag{4.13}$$

so

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t, x_k(t))dt \longrightarrow \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t, x(t))dt. \tag{4.14}$$

Moreover, $(1/2)|\nabla u(t)|^2$ is convex and continuous, so I is weakly lower semi-continuous.

For $x \in V$, we have $x = \overline{x} + \widetilde{x}$, where

$$\overline{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt, \tag{4.15}$$

then,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \widetilde{x}(t) = 0,\\
I(x) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{2}|\nabla x|^2 dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \{F(t,\overline{x}) + [F(t,x(t)) - F(t,\overline{x})]\}dt\\
&= \frac{1}{2}\|\nabla x\|_2^2 + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\{F(t,\overline{x}) + \int_0^1 (\nabla F(t,\overline{x} + s\widetilde{x}(t)), \widetilde{x}(t))ds\right\}dt\\
&\geq \frac{1}{2}\|\nabla x\|_2^2 - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)dt\|\widetilde{x}(t)\|_\infty + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,\overline{x})dt\\
&\geq \frac{1}{2}\|\nabla x\|_2^2 - c\|\nabla x\|_2 + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(t,\overline{x})dt.
\end{aligned} \tag{4.16}$$

As $\|x\| \longrightarrow \infty$ if and only if

$$\left(|\overline{x}|^2 + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\nabla x(t)|^2 dt\right)^{1/2} \longrightarrow \infty, \tag{4.17}$$

the above inequality and $(\mathbf{f}_5)$ imply that

$$I(x) \longrightarrow +\infty \quad \text{as } \|x\| \longrightarrow \infty. \tag{4.18}$$

Since V is a Hilbert space and I is weakly lower semi-continuous, the proof of Theorem 2.3 is completed. □

Proof of Theorem 2.4. Let $V = V^+ \bigoplus V^-$, V^+ denote the subspace of functions with mean value zero in V, and V^- denote the subspace of constant functions in V. By Theorem 2.1, we know the functional

$$I(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left\{\frac{1}{2}|\nabla x|^2 + F(t,x(t))\right\}dt \tag{4.19}$$

is continuously differentiable on V.

For any $v \in V^+$,

$$\begin{aligned} I(v) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}|\nabla v|^2 dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{F(t,0)+[F(t,v(t))-F(t,0)]\}dt \\ &= \frac{1}{2}\|\nabla v\|_2^2 + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(t,0)dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_0^1(\nabla F(t,sv(t)),v(t))ds\,dt \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - c_1 + \|v\|_\infty \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t)dt \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - c_1 - C\|\nabla v\|. \end{aligned} \tag{4.20}$$

So we see that

$$\inf_{V^+} I > -\infty, \tag{4.21}$$

by (f_6), there exists $R > 0$, such that

$$\sup_{S_R^-} I < \inf_{V^+} I, \tag{4.22}$$

where $S_R^- = \{u \in V^- \mid |u| = R\}$, so ($\mathrm{I}_1$) and ($\mathrm{I}_2$) of Lemma 3.4. are satisfied.

Finally, we show that $(PS)_c$ condition holds, that is, each sequence $\{x_k\}$ in V such that $I(x_k) \to c$ and $\nabla I(x_k) \to 0$ contains a convergent subsequence.

Letting $x_k = \overline{x}_k + \tilde{x}_k$ with $\overline{x}_k = \lim_{T\to\infty}(1/2T)\int_{-T}^{T}x_k(t)dt$, since $\nabla I(x_k) \to 0$, there exists some k_0 such that $|\langle \nabla I(x_k), h\rangle| \leq \|h\|$ for all $k \geq k_0$ and $h \in V$; we obtain, for $k > k_0$,

$$|\langle \nabla I(x_k), \tilde{x}_k\rangle| = \left|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(|\nabla x_k(t)|^2 + (\nabla F(t,x_k(t)),\tilde{x}_k(t))\Big)dt\right| \leq \|\tilde{x}_k\| \tag{4.23}$$

and hence

$$\|\tilde{x}_k\| \leq C_1, \quad k \geq k_0 \tag{4.24}$$

because of Lemma 3.3. Now $I(x_k) \to c$, hence there exists C_2, such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}|\nabla x_k|^2 dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\{F(t,\overline{x}_k)+[F(t,x_k(t))-F(t,\overline{x}_k)]\}dt \geq C_2, \tag{4.25}$$

by using (4.24), we obtain

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F(t,\overline{x}_k)dt \geq C_3, \tag{4.26}$$

and then $|\overline{x}_k| \leq C_4, k \in N$. By (4.24), thus $\{x_k\}$ is bounded in V and hence contains a subsequence, relabeled $\{x_k\}$ which weakly converges to some $x \in V$. Now, the equality

$$\begin{aligned}\langle \nabla I(x_k) - \nabla I(x), x_k - x\rangle = &\|\nabla x_k - \nabla x\|_2^2 \\ &+ \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(\nabla F(t, x_k(t)) - \nabla F(t, x(t)), x_k(t) - x(t))dt\end{aligned} \tag{4.27}$$

holds, and Lemma 3.2 implies that $\|\nabla x_k - \nabla x\|_2 \to 0$ as $k \to \infty$, so $\|x_k - x\| \leq \|x_k - x\|_2 + \|\nabla x_k - \nabla x\|_2 \to 0$, and the $(PS)_c$ condition holds, then the proof of Theorem 2.4 is completed by saddle point theorem.

Example 4.1. Consider the scalar problem:

$$\ddot{x}(t) = P\left[\sin(x - b\,\mathrm{sgn}\,x) + \sin(b\,\mathrm{sgn}\,x)\right] + h(t), \tag{4.28}$$

where $0 < b < 2\pi$ and $b \neq \pi$. P is a projection operator from $AP^0(R)$ to V_1, and $V_1 = \{x \in B^{1,2}(R) \mid \Lambda(x) \subseteq \Lambda\}$. $h \in V_1$ and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} h(t)dt = 0. \tag{4.29}$$

In this case, $F(t, x) = P[(\sin b)|x| - \cos(|x| - b)] + h(t)x$, and hence, when $0 < b < \pi$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(t, x)dt = |x|\sin b - \cos(|x| - b) \to +\infty, \tag{4.30}$$

if $|x| \to \infty$. So it is easy to check that the conditions (f_1)–(f_5) are satisfied, then (4.28) has at least a quasi periodic solution by using Theorem 2.3.

When $\pi < b < 2\pi$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(t, x)dt = |x|\sin b - \cos(|x| - b) \to -\infty, \tag{4.31}$$

if $|x| \to \infty$. So it is easy to check that the conditions (f_1)–(f_4) and (f_6) are satisfied, then (4.28) has at least a quasi periodic solution by using Theorem 2.4. □

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