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Research Article

Some Convexity Properties of Certain General Integral Operators

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The main object of the present paper is to discuss some extensions of certain integral operators and to obtain their order of convexity. Several other closely related results are also considered.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions defined in the open unit disk of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by S the subclass of $\mathcal{A}$ consisting of all univalent functions in U. A function $f(z) \in S$ is *starlike function of order* α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U) \tag{1.1}$$

for some α $(0 \le \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of the functions which are starlike of order α in U. For $\alpha = 0$, we obtain the class of starlike functions, denoted by S^*.

A function $f(z) \in S$ is *convex of order* α if it satisfies

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in U) \tag{1.2}$$

for some α $(0 \le \alpha < 1)$. We denote by $K(\alpha)$ the subclass of $\mathcal{A}$ consisting of the functions which are convex of order α in U. For $\alpha = 0$, we obtain the class of convex functions, denoted by K.

A function $f(z) \in \mathcal{A}$ is in the class $R(\alpha)$ if

$$\operatorname{Re}\big(f'(z)\big) > \alpha, \quad (z \in U). \tag{1.3}$$

Frasin and Jahangiri introduced in [1] the family $B(\mu, \alpha),\ \mu \ge 0,\ 0 \le \alpha < 1$ consisting of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1 - \alpha, \quad (z \in U). \tag{1.4}$$

For $\mu = 0$ we have $B(0, \alpha) \equiv R(\alpha)$, and for $\mu = 1$ we have $B(1, \alpha) \equiv S^*(\alpha)$.

In this paper, we will obtain the order of convexity of the following general integral operators:

$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left(te^{f_i(t)}\right)^{1/\gamma} \mathrm{d}t, \tag{1.5}$$

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(f_i(t)\right)^{\beta_i - 1} \mathrm{d}t, \tag{1.6}$$

$$F(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\beta_i} \mathrm{d}t, \tag{1.7}$$

where the functions $f_i(t)$ are in $B(\mu_i, \alpha_i)$ for all $i = 1, 2, \ldots, n$.

In order to prove our main results, we recall the following lemma.

Lemma 1.1 (see [2, General Schwarz Lemma]). *Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$\left|f(z)\right| \le \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R). \tag{1.8}$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m, \tag{1.9}$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 0$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M_i$ ($M_i \geq 1$, $z \in U$) for all $i = 1, 2, \ldots, n$, then the integral operator*

$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left(te^{f_i(t)} \right)^{1/\gamma} dt \tag{2.1}$$

is in $K(\delta)$, where

$$\delta = 1 - \frac{1}{|\gamma|} \left[n + \sum_{i=1}^n (2 - \alpha_i) M_i^{\mu_i} \right] \tag{2.2}$$

and $1/|\gamma| < 1/(n + \sum_{i=1}^n (2 - \alpha_i) M_i^{\mu_i})$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Let $f_i \in \mathcal{A}$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 0$, $0 \leq \alpha_i < 1$. We have from (1.5) that

$$H_{n,\gamma}(z) = \int_0^z t^{n/\gamma} e^{(1/\gamma) \sum_{i=1}^n f_i(t)} \mathrm{d}t, \qquad H'_{n,\gamma}(z) = z^{n/\gamma} e^{(1/\gamma) \sum_{i=1}^n f_i(z)}. \tag{2.3}$$

Also

$$H''_{n,\gamma}(z) = \frac{1}{\gamma} \left(z^n e^{\sum_{i=1}^n f_i(z)} \right)^{(1/\gamma)-1} \cdot z^{n-1} \cdot e^{\sum_{i=1}^n f_i(z)} \left(n + z \sum_{i=1}^n f'_i(z) \right). \tag{2.4}$$

Then

$$\frac{H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} = \frac{1}{\gamma} \left(\frac{n}{z} + \sum_{i=1}^n f'_i(z) \right) \tag{2.5}$$

and, hence,

$$\begin{aligned} \left| \frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} \right| &= \frac{1}{|\gamma|} \left| n + z \sum_{i=1}^n f'_i(z) \right| \leq \frac{1}{|\gamma|} \left(\sum_{i=1}^n \left| 1 + z f'_i(z) \right| \right) \\ &\leq \frac{1}{|\gamma|} \sum_{i=1}^n \left[1 + \left| f'_i(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot \left| \left(\frac{f_i(z)}{z} \right)^{\mu_i} \right| \cdot |z| \right]. \end{aligned} \tag{2.6}$$

Applying the General Schwarz lemma, we have $|f_i(z)/z| \leq M_i$, for all $i = 1, 2, \ldots, n$. Therefore, from (2.6), we obtain

$$\left| \frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} \right| \leq \frac{1}{|\gamma|} \sum_{i=1}^n \left[1 + \left| f'_i(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot M_i^{\mu_i} \right], \quad (z \in U). \tag{2.7}$$

From (1.4) and (2.7), we see that

$$\left|\frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)}\right| \leq \frac{1}{|\gamma|}\left[n+\sum_{i=1}^{n}(2-\alpha_i)M_i^{\mu_i}\right] = 1-\delta. \tag{2.8}$$

□

Letting $\mu_i = 0$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $f_i(z) \in \mathcal{A}$ be in the class $R(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. Then the integral operator defined in* (1.5) *is in $K(\delta)$, where*

$$\delta = 1 - \frac{1}{|\gamma|}\left(3n - \sum_{i=1}^{n}\alpha_i\right) \tag{2.9}$$

and $1/|\gamma| < 1/(3n - \sum_{i=1}^{n}\alpha_i)$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Letting $\mu_i = 1$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $f_i \in \mathcal{A}$ be in the class $S^*(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.5) *is in $K(\delta)$, where*

$$\delta = 1 - \frac{1}{|\gamma|}\left[n + M\left(2n - \sum_{i=1}^{n}\alpha_i\right)\right] \tag{2.10}$$

and $1/|\gamma| < 1/(n + M(2n - \sum_{i=1}^{n}\alpha_i)), \gamma \in \mathbb{C} \setminus \{0\}$.

Letting $\alpha_i = \delta = 0$, $\mu_i = 1$, and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let $f_i(z) \in \mathcal{A}$ be starlike functions in U for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.5) *is convex in U, where $1/|\gamma| = 1/(n(2M+1))$, $\gamma \in \mathbb{C} \setminus \{0\}$.*

Theorem 2.5. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M_i$ $(M_i \geq 1,\ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator*

$$G_n(z) = \int_0^z \prod_{i=1}^{n}(f_i(t))^{\beta_i - 1}\mathrm{d}t \tag{2.11}$$

is in $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n}\left|\beta_i - 1\right|(2-\alpha_i)M_i^{\mu_i - 1} \tag{2.12}$$

and $\sum_{i=1}^{n}|\beta_i - 1|(2-\alpha_i)M_i^{\mu_i-1} < 1$, $\beta_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$.

Proof. Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$. It follows from (1.6) that

$$\frac{G_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \frac{(\beta_i - 1) f_i'(z)}{f_i(z)}, \tag{2.13}$$

and, hence,

$$\begin{aligned} \left| \frac{zG_n''(z)}{G_n'(z)} \right| &\leq \sum_{i=1}^{n} |\beta_i - 1| \left| \frac{z f_i'(z)}{f_i(z)} \right| \\ &\leq \sum_{i=1}^{n} |\beta_i - 1| \cdot \left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot \left| \left(\frac{f_i(z)}{z} \right)^{\mu_i - 1} \right|. \end{aligned} \tag{2.14}$$

Applying the General Schwarz lemma, we have $|f_i(z)/z| \leq M_i$, $(z \in U)$ for all $i = 1, 2, \ldots, n$. Therefore, from (2.14), we obtain

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| \leq \sum_{i=1}^{n} |\beta_i - 1| \cdot \left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot M_i^{\mu_i - 1}, \quad (z \in U). \tag{2.15}$$

From (1.4) and (2.15), we see that

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| \leq \sum_{i=1}^{n} |\beta_i - 1| \cdot (2 - \alpha_i) \cdot M_i^{\mu_i - 1} = 1 - \delta. \tag{2.16}$$

This completes the proof. □

Letting $\delta = 0$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.5, we have the following corollary.

Corollary 2.6. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1, z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.6) *is convex function in U, where*

$$\sum_{i=1}^{n} |\beta_i - 1| (2 - \alpha_i) M^{\mu_i - 1} = 1, \quad \beta_i \in \mathbb{C} \ \forall i = 1, 2, \ldots, n. \tag{2.17}$$

Letting $\mu_i = 1$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.5, we have the following corollary.

Corollary 2.7. *Let $f_i(z)$ be in the class $S^*(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1, \ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.6) *is in $K(\delta)$, where*

$$\delta = 1 - \sum_{i=1}^{n} |\beta_i - 1| \cdot (2 - \alpha_i) \tag{2.18}$$

and $\sum_{i=1}^{n} |\beta_i - 1| \cdot (2 - \alpha_i) < 1$, $\beta_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$.

Letting $n=1,\ \mu_i=1,\ M_i=M$, and $\alpha_i=\delta=0$ for all $i=1,2,\ldots,n$ in Theorem 2.5, we have the following corollary.

Corollary 2.8. *Let $f(z)$ be a starlike function in $\mathcal{U}$. If $|f(z)|\le M$ $(M\ge 1,\ z\in\mathcal{U})$, then the integral operator $\int_0^z f(t)^{\beta-1}\mathrm{d}t$ is convex in $\mathcal{U}$, where $|\beta-1|=1/2,\ \beta\in\mathbb{C}$.*

Theorem 2.9. *Let $f_i(z)$ be in the class $\mathcal{B}(\mu_i,\alpha_i)$, $\mu_i\ge 1$, $0\le\alpha_i<1$ for all $i=1,2,\ldots,n$. If $|f_i(z)|\le M_i$ $(M_i\ge 1,\ z\in\mathcal{U})$ for all $i=1,2,\ldots,n$, then the integral operator*

$$F(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\beta_i}dt \tag{2.19}$$

is in $\mathcal{K}(\delta)$, where

$$\delta=1-\sum_{i=1}^n|\beta_i|\cdot\left[(2-\alpha_i)M_i^{\mu_i-1}+1\right] \tag{2.20}$$

and $\sum_{i=1}^n|\beta_i|\cdot[(2-\alpha_i)M_i^{\mu_i-1}+1]<1$, $\beta_i\in\mathbb{C}$ for all $i=1,2,\ldots n$.

Proof. Let $f_i(z)$ be in the class $\mathcal{B}(\mu_i,\alpha_i)$, $\mu_i\ge 1$, $0\le\alpha_i<1$. It follows from (1.7) that

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^n\beta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right). \tag{2.21}$$

So, from (2.21), we have

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)}\right|&\le\sum_{i=1}^n|\beta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)\\&\le\sum_{i=1}^n|\beta_i|\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\cdot\left|\left(\frac{f_i(z)}{z}\right)^{\mu_i-1}\right|+1\right).\end{aligned} \tag{2.22}$$

Applying the General Schwarz lemma, we have $|f_i(z)/z|\le M_i$, $(z\in\mathcal{U})$ for all $i=1,2,\ldots,n$. Therefore, from (2.22), we obtain

$$\left|\frac{zF''(z)}{F'(z)}\right|\le\sum_{i=1}^n|\beta_i|\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\cdot M_i^{\mu_i-1}+1\right),\quad(z\in\mathcal{U}). \tag{2.23}$$

From (1.4) and (2.23), we see that

$$\left|\frac{zF''(z)}{F'(z)}\right|\le\sum_{i=1}^n|\beta_i|\left[(2-\alpha_i)M_i^{\mu_i-1}+1\right]=1-\delta. \tag{2.24}$$

This completes the proof. □

Letting $\delta = 0$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.9, we have the following corollary.

Corollary 2.10. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.7) *is convex function in U, where*

$$\sum_{i=1}^{n} \left|\beta_i\right| \left[(2 - \alpha_i) M^{\mu_i - 1} + 1\right] = 1, \quad \beta_i \in \mathbb{C}\ \forall i = 1, 2, \ldots, n. \tag{2.25}$$

Letting $\mu_i = 1$ and $M_i = M$ for all $i = 1, 2, \ldots, n$ in Theorem 2.9, we have the following corollary.

Corollary 2.11. *Let $f_i(z) \in \mathcal{A}$ be in the class $S^*(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \ldots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, \ldots, n$, then the integral operator defined in* (1.7) *is in $K(\delta)$, where*

$$\delta = 1 - \sum_{i=1}^{n} \left|\beta_i\right| (3 - \alpha_i) \tag{2.26}$$

and $\sum_{i=1}^{n} |\beta_i|(3 - \alpha_i) < 1$, $\beta_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$.

Letting $n = 1$, $\mu_i = 1$, $M_i = M$, and $\alpha_i = \delta = 0$ for all $i = 1, 2, \ldots, n$ in Theorem 2.9, we have the following corollary.

Corollary 2.12. *Let $f(z) \in \mathcal{A}$ be a starlike function in U. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$, then the integral operator $\int_0^z (f(t)/t)^{\beta} \mathrm{d}t$ is convex in U, where $|\beta| = 1/3$, $\beta \in \mathbb{C}$.*

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