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Research Article

Containment Control of Multiagent Systems with Multiple Leaders and Noisy Measurements

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We consider the distributed containment control of multiagent systems with multiple stationary leaders and noisy measurements. A stochastic approximation type and consensus-like algorithm is proposed to solve the containment control problem. We provide conditions under which all the followers can converge both almost surely and in mean square to the stationary convex hull spanned by the leaders. Simulation results are provided to illustrate the theoretical results.

1. Introduction

In recent years, there has been an increasing interest in the coordination control of multiagent systems. This is partly due to broad applications of multiagent systems in many areas including consensus, formation control, flocking, distributed sensor networks, and attitude alignment of clusters of satellites [1]. As a critical issue for coordination control, consensus means that the group of agents reach a state of agreement through local communication. Up to now, a variety of consensus algorithms have been developed to deal with measurement delays [1–3], noisy measurements [4–6], dynamic topologies [7–9], random network topologies [10, 11], and finite-time convergence [12, 13].

Existing consensus algorithms mainly focus on leaderless coordination for a group of agents. However, in many applications envisioned, there might exist one or even multiple leaders in the agent network. The role of the leaders is to guide the group of agents, and the existence of the leaders is useful to increase the coordination effectiveness for an agent group.

In the case of single leader, the control goal is to let all the follower-agents converge to the state of the leader, which is commonly called a leader-following consensus problem. Such a problem has been studied extensively. Leader-following consensus with a constant leader was addressed, respectively, in [14, 15] for a group of first-order and second-order follower agent under dynamic topologies. A neighbor-based local controller together with a neighbor-based state-estimation rule was proposed in [16] to track an active leader whose velocity cannot be measured. Consensus with a time-varying reference state was studied in [17], and further studied in [18] accounting for bounded control effort. Leader-following consensus with time delays was reported in [19, 20]. In the presence of multiple leaders, the follower agents are to be driven to a given target location spanned by the leaders, which is called a containment control problem. In [21], hybrid control schemes were proposed to drive a collection of follower agents to a target area spanned by multiple stationary/moving leaders under fixed network topology. In [22], containment control with multiple stationary leaders and switching communication topologies was studied by means of LaSalle's Invariance Principle for switched systems. Containment control with multiple stationary/dynamic leaders was investigated in [23] for both fixed and switching topologies. The paper [24] considered the containment control problem for multiagent systems with general linear dynamic under fixed topology. However, it was assumed in these references concerning containment control that each agent can obtain the accurate information from its neighbors. This assumption is often impractical since information exchange within networks typically involves quantization, wireless channels, and/or sensing [25]. Therefore, it is important and meaningful to consider the containment control problem with noisy measurements. It is worthy to note that containment control of multiagent system with noisy measurements receives less attention.

In this paper, we are interested in the containment control problem for a group of agents with multiple stationary leaders and noisy measurements. By employing a stochastic approximation type and consensus-like algorithm, we show that all the follower-agents converge both almost surely and in mean square to the convex hull spanned by the stationary leaders as long as the communication topology contains a united spanning tree. The convergence analysis is given with the help of M-matrix theory and stochastic Lyapunov function.

The following notations will be used throughout this paper. For a given matrix A, A^T denotes its transpose; $\|A\|$ denotes its 2-norm; $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalues, respectively. A matrix A is said to be positive stable if all of its eigenvalues have positive real parts. For a given random variable ξ, $E[\xi]$ denotes its mathematical expectation.

2. Preliminaries

Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph, where $\mathcal{V} = \{1, \ldots, N\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix with nonnegative elements. An edge of $\mathcal{G}$ is denoted by (i, j), representing that the jth agent can directly receive information from the ith agent. The element a_{ij} associated with the edge is positive, that is, $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. A path in $\mathcal{G}$ is a sequence $i_0, i_1, \ldots, i_m$ of distinct nodes such that $(i_{j-1}, i_j) \in \mathcal{E}$ for $j = 1, \ldots, m$. A digraph $\mathcal{G}$ contains a spanning tree if there exists at least one node having a directed path to all other nodes.

The Laplacian matrix associated with $\mathcal{G}$ is defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i. \end{cases} \tag{2.1}$$

The definition of L clearly implies that L must have a zero eigenvalue corresponding to a eigenvector $\mathbf{1}$, where $\mathbf{1}$ is an all-one column vector with appropriate dimension. Moreover, zero is a simple eigenvalue of L if and only if $\mathcal{G}$ contains a spanning tree [8].

In the present paper, we consider a multiagent system consisting of n follower agents and k leader agents (just called followers and leaders for simplicity, resp.). Denote the follower set and leader set by $\mathcal{V}_F = \{1, \ldots, n\}$ and $V_L = \{n+1, \ldots, n+k\}$, respectively. Then the communication topology between the $n+k$ agents can be described by a digraph $\overline{\mathcal{G}} = (\mathcal{V}, \mathcal{E}(\overline{\mathcal{G}}))$ with $\mathcal{V} = \mathcal{V}_F \cup \mathcal{V}_L$, and the communication topology between the n followers can be described by a digraph $\mathcal{G} = (\mathcal{V}_F, \mathcal{E}(\mathcal{G}))$. We say that $\overline{\mathcal{G}}$ contains a united spanning tree if, for any one of the n followers, there exists at least one leader that has a path to the follower.

Next, we shall recall some notations in convex analysis. A set $K \subset \mathbb{R}^m$ is said to be convex if $(1-\gamma)x + \gamma y \in K$ whenever $x \in K, y \in K$ and $0 < \gamma < 1$. For any set $S \subset \mathbb{R}^m$, the intersection of all convex sets containing S is called the convex hull of S, denoted by $\mathrm{co}(S)$. The convex hull of a finite set of points $x_1, \ldots, x_n \in \mathbb{R}^m$ is a polytope, denoted by $\mathrm{co}\{x_1, \ldots, x_n\}$. For $x \in \mathbb{R}^m$ and $S \subset \mathbb{R}^m$, define $\|x - S\| = \inf_{y\in S}\|x - y\|$.

2.1. Models

For agent i, denote its state at time t by $x_i(t) \in \mathbb{R}$, where $t \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. We assume that the k leaders are static, that is, $x_i(t) = x_i$, for all $i \in \mathcal{V}_L$. For convenience, we denote the convex hull formed by the leaders' states by $\mathrm{co}(\mathcal{V}_L)$.

Due to the existence of noise or disturbance, each follower can only receive noisy measurements of the states of its neighbors. We denote the resulting measurement by follower i of the jth agent's state by

$$y_{ij}(t) = x_j(t) + w_{ij}(t), \quad i \in \mathcal{V}_F,\ j \in \mathcal{N}_i,\ t \in \mathbb{Z}^+, \tag{2.2}$$

where $w_{ij}(t)$ is the additive noise. The underlying probability space is $(\Omega, \mathcal{F}, P)$. For each $t \in \mathbb{Z}^+$, the set of noises $\{w_{ij}(t), j \in \mathcal{N}_i \neq \phi\}$ is listed into a vector w_t in which the position of $w_{ij}(t)$ depends only on (i, j) and does not change with t. Similar to [25], we introduce the following assumption on the measurement noises.

(A1) The sequence $\{\mathrm{w}_t, t \in \mathbb{Z}^+\}$ satisfies that (i) $E[\mathrm{w}_t | \mathcal{F}_{t-1}] = 0$ for $t \geq 0$, where $\mathcal{F}_t$ denote the σ-algebras $\sigma(x(0), \mathrm{w}_k, k = 0, \ldots, t)$ with $\mathcal{F}_{-1} = \{\phi, \Omega\}$, and (ii) $\sup_{t\geq 0} E\|\mathrm{w}_t\|^2 < \infty$.

Definition 2.1. The followers are said to converge to the static convex hull $\mathrm{co}(\mathcal{V}_L)$ almost surely (a.s.) if $\lim_{t\to\infty}\|x_i(t) - \mathrm{co}(\mathcal{V}_L)\| = 0$ a.s., for all $i \in \mathcal{V}_F$.

Definition 2.2. The followers are said to converge in mean square to the static convex hull $\mathrm{co}(\mathcal{V}_L)$ if $\lim_{t\to\infty} E\|x_i(t) - \mathrm{co}(\mathcal{V}_L)\|^2 = 0$, for all $i \in \mathcal{V}_F$.

Each follower updates its state by the rule

$$x_i(t+1) = x_i(t) + a(t)\sum_{j=1}^{n+k} a_{ij}\left(y_{ij}(t) - x_i(t)\right), \quad i \in \mathcal{V}_F, \tag{2.3}$$

where $a(t) > 0$ is the step size. Here, the introduction of the step size is to attenuate the noises, which is often used in classical stochastic approximation theory [26]. We introduce the following assumption on the step size sequence:

(A2) $\sum_{t=0}^{\infty} a(t) = \infty$, $\sum_{t=0}^{\infty} a^2(t) < \infty$.

Let $w(t) = (w_1(t), \ldots, w_n(t))^T$ with $w_i(t) = \sum_{j=1}^{n+k} a_{ij} w_{ij}(t)$ and

$$B = \begin{bmatrix} a_{1,n+1} + \cdots + a_{1,n+k} & & \\ & \ddots & \\ & & a_{n,n+1} + \cdots + a_{n,n+k} \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} a_{1,n+1} & \cdots & a_{1,n+k} \\ \cdots & \cdots & \cdots \\ a_{n,n+1} & \cdots & a_{n,n+k} \end{bmatrix}. \tag{2.4}$$

Then (2.3) can be rewritten in the vector form

$$x_F(t+1) = x_F(t) - a(t)(L+B)x_F(t) + a(t)\widetilde{B}x_L + a(t)w(t), \tag{2.5}$$

where L is the Laplacian matrix associated with $\mathcal{G}$, $x_F(t) = (x_1(t), \ldots, x_n(t))^T, x_L = (x_{n+1}, \ldots, x_{n+k})^T$.

3. Main Results

We begin by introducing some definitions and lemmas concerning M matrix, which will be used to obtain our main result.

Definition 3.1 (See [27]). Let $Z_n = \{A = [a_{ij}] \in \mathbb{R}^{n\times n} | a_{ij} \le 0, i \ne j\}$. Then a matrix A is called an M matrix if $A \in Z_n$ and A is positive stable.

Lemma 3.2 (See [27]). *Assuming that $A \in \mathbb{Z}_n$, A is an M matrix if and only if A is nonsingular and A^{-1} is a nonnegative matrix.*

Definition 3.3 (See [28]). A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a weakly chained diagonally dominant (w.c.d.d.) matrix if A is diagonally dominant, that is,

$$\begin{gathered} |a_{ii}| \ge \sum_{i=1, j\ne i}^{n} \left|a_{ij}\right|, \quad i = 1, 2, \ldots, n, \\ J(A) = \left\{ i \mid |a_{ii}| > \sum_{i=1, j\ne i}^{n} \left|a_{ij}\right| \right\} \ne \phi, \end{gathered} \tag{3.1}$$

where ϕ is the empty set and for each $i \notin J(A)$, there is a sequence of nonzero elements of A of the form $a_{i,i_1}, a_{i_1,i_2}, \ldots, a_{i_r,j}$ with $j \in J(A)$.

Lemma 3.4 (See [29]). *Let $A \in Z_n$ and A be a w.c.d.d. matrix, then A is an M matrix.*

For simplicity, denote that $H = L + B$, where L is the Laplacian matrix associated with $\mathcal{G}$. The following lemmas are given for H.

Lemma 3.5. *H is positive stable if $\overline{\mathcal{G}}$ contains a united spanning tree.*

Proof. Denote that $\mathcal{O} = \{j \in \mathcal{V}_F \mid (i,j) \in \mathcal{E}(\overline{\mathcal{G}}), i \in \mathcal{V}_L\}$. That is, $\mathcal{O}$ denotes the set of nodes whose neighbors include one of the leaders. Then $\mathcal{O} \neq \phi$, and, for each $i \in \mathcal{V}_F, i \notin \mathcal{O}$, there is a path $ji_1 \cdots i_r i$ with $j \in \mathcal{O}$ since $\overline{\mathcal{G}}$ contains a united spanning tree. In other words, there is a sequence of nonzero elements of the form $h_{i,i_r}, \ldots, h_{i_1,j}$ with $j \in \mathcal{O}$. Noting that $h_{ii} \geq \sum_{j \neq i} |h_{ij}|$, for all $i \in \mathcal{V}_F$ and $h_{ii} > \sum_{j \neq i} |h_{ij}|$, for all $i \in \mathcal{O}$, we know that H is a w.c.d.d. matrix. Invoking Lemma 3.4, H is an M matrix by noting that $H \in \mathbb{Z}_n$; that is, H is positive stable. □

Lemma 3.6. *If $\overline{\mathcal{G}}$ contains a united spanning tree, then $H^{-1}\widetilde{B}$ is a stochastic matrix.*

Proof. By Lemmas 3.2 and 3.5, H^{-1} is a nonnegative matrix. Note that $H\mathbf{1} = B\mathbf{1} = \widetilde{B}\mathbf{1}$ by noting that $L\mathbf{1} = 0$. It follows that $H^{-1}\widetilde{B}\mathbf{1} = \mathbf{1}$, which implies the conclusion. □

We also need the following lemmas to derive our main results.

Lemma 3.7 (See [30]). *Let $\{u(k), k = 0, 1, \ldots\}$, $\{\alpha(k), k = 0, 1, \ldots\}$ and $\{q(k), k = 0, 1, \ldots\}$ be real sequence, satisfying that $0 < q(k) \leq 1, \alpha(k) \geq 0, k = 0, 1, \ldots, \sum_{k=0}^{\infty} q(k) = \infty, \alpha(k)/q(k) \to 0, k \to \infty$, and*

$$u(k+1) \leq \left(1 - q(k)\right)u(k) + \alpha(k). \tag{3.2}$$

Then $\limsup_{k\to\infty} u(k) \leq 0$. *In particular, if $u(k) \geq 0, k = 0, 1, \ldots$, then $u(k) \to 0$ as $k \to \infty$.*

Lemma 3.8 (See [30]). *Consider a sequence of nonnegative random variables $\{V(t)\}_{t\geq 0}$ with $E\{V(0)\} < \infty$. Let*

$$E\{V(t+1) \mid V(t), \ldots, V(1), V(0)\} \leq (1 - c_1(t))V(t) + c_2(t), \tag{3.3}$$

where

$$\begin{gathered} 0 \leq c_1(t) \leq 1, \quad c_2(t) \geq 0, \quad \forall t, \\ \sum_{t=0}^{\infty} c_2(t) < \infty, \qquad \sum_{t=0}^{\infty} c_1(t) = \infty, \\ \lim_{t\to\infty} \frac{c_2(t)}{c_1(t)} = 0. \end{gathered} \tag{3.4}$$

Then, $V(k)$ almost surely converges to zero, that is,

$$\lim_{t\to\infty} V(t) = 0 \ a.s. \tag{3.5}$$

Now, we can present our main results.

Theorem 3.9. *Assume that (A1) and (A2) hold. All the followers converge almost surely to* $co(\mathcal{V}_L)$ *if* $\overline{\mathcal{G}}$ *contains a united spanning tree.*

Proof. Let $\delta(t) = x_F(t) - H^{-1}\widetilde{B}x_L$. Then from (2.5), we have

$$\delta(t+1) = (I - a(t)H)\delta(t) + a(t)w(t). \tag{3.6}$$

From Lemma 3.5 and Lyapunov theorem, there is a positive definite matrix P such that

$$PH + H^T P = I. \tag{3.7}$$

Choose a Lyapunov function

$$V(t) = \delta^T(t)P\delta(t). \tag{3.8}$$

From (3.6), we have

$$\begin{aligned} V(t+1) &= \delta^T(t)\Big[P - a(t)I + a^2(t)H^T PH\Big]\delta(t) \\ &\quad + 2a(t)w^T(t)P(I - a(t)H)\delta(t) + a^2(t)w^T(t)Pw(t) \\ &\le \left[1 - a(t)\frac{1}{\lambda_{\max}(P)} + a^2(t)\frac{\lambda_{\max}(H^T PH)}{\lambda_{\min}(P)}\right]V(t) \\ &\quad + 2a(t)w^T(t)P(I - a(t)H)\delta(t) + a^2(t)w^T(t)Pw(t). \end{aligned} \tag{3.9}$$

Taking the expectation of the above, given $\{V(s) : s \le t\}$, yields

$$E\{V(t+1) \mid V(s) : s \le t\} \le \left[1 - a(t)\frac{1}{\lambda_{\max}(P)} + a^2(t)\frac{\lambda_{\max}(H^T PH)}{\lambda_{\min}(P)}\right]V(t) + C_1 a^2(t), \tag{3.10}$$

for some constant $C_1 > 0$, where we have used the fact that $E[w^T(t)P(I - a(t)H)\delta(t)] = 0$ by noting (A1).

By (A2), there exists a $t_0 > 0$ such that $a(t) \le \min\{\lambda_{\min}(P)/2\lambda_{\max}(P)\lambda_{\max}(H^T PH), \lambda_{\max}(P)\}$ for all $t \ge t_0$. Thus, we have

$$E\{V(t+1) \mid V(s) : s \le t\} \le \left[1 - a(t)\frac{1}{2\lambda_{\max}(P)}\right]V(t) + C_1 a^2(t), \quad \forall t \ge t_0. \tag{3.11}$$

Again by (A2), it is clear that the conditions in Lemma 3.8 hold. Therefore,

$$\lim_{t\to\infty} V(t) = 0 \text{ a.s.} \tag{3.12}$$

On the other hand, it follows from Lemma 3.6 that $H^{-1}\widetilde{B}x_L \in \mathrm{co}(\mathcal{V}_L)$. This together with (3.12) implies the conclusion. □

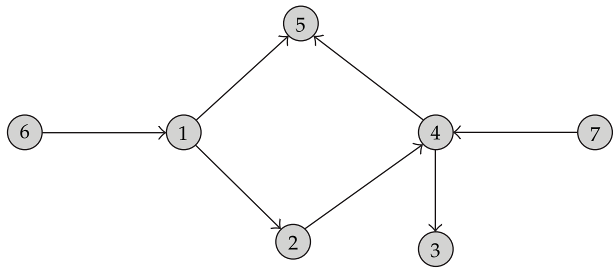

Figure 1: The communication topology $\overline{\mathcal{G}}$.

Theorem 3.10. *Assume that (A1) and (A2) hold. All the followers converge in mean square to* $co(\mathcal{V}_L)$ *if* $\overline{\mathcal{G}}$ *contains a united spanning tree.*

Proof. Following the notations in the proof of Theorem 3.9, taking the expectation of (3.9), we have

$$E[V(t+1)] \leq \left[1 - a(t)\frac{1}{\lambda_{\max}(P)} + a^2(t)\frac{\lambda_{\max}(H^T P H)}{\lambda_{\min}(P)}\right] E[V(t)] + C_1 a^2(t), \tag{3.13}$$

for some constant $C_1 > 0$. By a similar argument to the proof of (3.11), we can obtain that

$$E[V(t+1)] \leq \left[1 - a(t)\frac{1}{2\lambda_{\max}(P)}\right] E[V(t)] + C_1 a^2(t), \quad \forall t \geq t_0. \tag{3.14}$$

By applying Lemma 3.7, we have

$$\lim_{t\to\infty} E[V(t)] = 0. \tag{3.15}$$

It follows that $\lim_{t\to\infty} E\|\delta(t)\|^2 = 0$, that is, $\lim_{t\to\infty} E\|x_F(t) - H^{-1}\widetilde{B}x_L\|^2 = 0$ which implies the conclusion by noting that $H^{-1}\widetilde{B}x_L \in \mathrm{co}(\mathcal{V}_L)$. □

Remark 3.11. In the case of single leader, by Theorems 3.9 and 3.10, it is easy to show that the states of the followers converge both almost surely and in mean square to that of the leader if the node representing the leader has a path to all other nodes.

4. Simulations

In this section, an example is provided to illustrate the theoretical results. Consider a multiagent system consisting of five followers (labeled by $1,\ldots,5$) and two leaders (labeled by $6,7$), and the communication topology is given as in Figure 1. For simplicity, we assume that $\overline{\mathcal{G}}$ has 0-1 weights. The variance of the i.i.d zero mean Gaussian measurement noises is $\sigma^2 = 0.01$, and the step size $a(k) = 1/(k+1), k \geq 0$. It is clear that $\overline{\mathcal{G}}$ contains a united spanning tree, and Assumptions (A1) and (A2) hold. The state trajectories of the agents are shown in

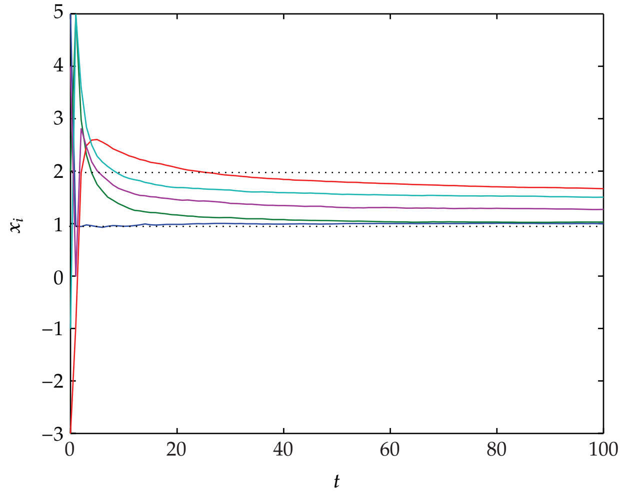

Figure 2: The state trajectories of the agents. The solid and dotted lines denote, respectively, the trajectories of the followers and the leaders.

Figure 2. It can be seen that the states of the followers converge to the convex hull spanned by the leaders.

5. Conclusion

In this paper, a containment control problem for a multiagent system with multiple stationary leaders and noisy measurements is investigated. A stochastic approximation type and consensus-like algorithm are proposed to solve the containment control problem. It is shown that the states of the followers converge both almost surely and in mean square to the convex hull spanned by the multiple stationary leaders as long as the communication topology contains a united spanning tree.

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