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Research Article

On a Periodic System of Difference Equations

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Here we give a very short and elegant proof of an extension of two recent results on global periodicity of two systems of difference equations in a work of Elsayed et al. 2012.

1. Introduction

For a solution $(x_n^{(1)}, \dots, x_n^{(k)})$, $n \geq -l$, $l \in \mathbb{N}$, of the system of difference equations

$$x_n^{(j)} = f_j\left(x_{n-1}^{(1)}, \dots, x_{n-1}^{(k)}, \dots, x_{n-l}^{(1)}, \dots, x_{n-l}^{(k)}\right), \quad j = \overline{1, k}, \ n \in \mathbb{N}_0, \tag{1.1}$$

is said that it is eventually periodic with period $\widehat{p}$, if there is an $n_1 \geq -l$ such that

$$\left(x_{n+\widehat{p}}^{(1)}, \dots, x_{n+\widehat{p}}^{(k)}\right) = \left(x_n^{(1)}, \dots, x_n^{(k)}\right), \quad \text{for } n \geq n_1. \tag{1.2}$$

If $n = -l$, then for the solution is said that it is periodic with period $\widehat{p}$. If all well-defined solutions of system (1.1) are eventually periodic with period $\widehat{p}$, it is said that the system is $\widehat{p}$-periodic.

For some periodic difference equations and systems see, for example, [1–11] and the references therein. Periodic systems are essentially subclasses of the systems which can be solved explicitly, an area that has again attracted attention recently (see, e.g., [12–21]). For some classical results in the topic see, for example, [22].

In the recent paper [23] are formulated two results on global periodicity of particular systems of difference equations. Namely, they gave some tedious straightforward calculations which suggest that all the solutions with nonzero real initial values of the following systems of difference equations

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}}, \qquad y_{n+1} = \frac{x_{n-p}y_{n-p}}{x_{n-q}y_{n-q}}, \tag{1.3}$$

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}z_{n-p}}, \qquad y_{n+1} = \frac{x_{n-p}y_{n-p}z_{n-p}}{x_{n-q}y_{n-q}z_{n-q}}, \qquad z_{n+1} = \frac{x_{n-q}y_{n-q}z_{n-q}}{x_{n-r}y_{n-r}z_{n-r}} \tag{1.4}$$

are periodic with periods $q+1$ and $r+1$, respectively, without giving complete proofs of these statements.

2. Main Result

Here we prove a result which naturally extends the results on systems (1.3) and (1.4), and give a very short and elegant proof of it.

Theorem 2.1. *Assume that k, l, $s \in \mathbb{N}$, $k \geq 2$, functions $g_j : (\mathbb{R} \setminus \{0\})^l \to \mathbb{R} \setminus \{0\}$, $j = \overline{1,k}$, satisfy the following condition*

$$\prod_{j=1}^{k} g_j(t_1, t_2, \ldots, t_l) = 1, \tag{2.1}$$

for every $(t_1, t_2, \ldots, t_l) \in (\mathbb{R} \setminus \{0\})^l$.

Consider the following system of difference equations

$$\begin{aligned} x_n^{(j)} &= g_j\left(\prod_{i=1}^{k} x_{n-1}^{(i)}, \ldots, \prod_{i=1}^{k} x_{n-l}^{(i)}\right), \quad j = \overline{1, k-1}, \\ x_n^{(k)} &= g_k\left(\prod_{i=1}^{k} x_{n-1}^{(i)}, \ldots, \prod_{i=1}^{k} x_{n-l}^{(i)}\right)\left(\prod_{i=1}^{k} x_{n-s}^{(i)}\right)^{-1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.2}$$

Then every solution $(x_n^{(1)}, \ldots, x_n^{(k)})$ of system (2.2) *whose initial values are arbitrary nonzero real numbers is eventually periodic with period* 2*s*.

Proof. Since $x_n^{(j)} \neq 0$ for every $-\max\{l, s\} \le n \le -1$ and $j = \overline{1, k}$, functions g_j, $j = \overline{1, k}$, map the set $(\mathbb{R} \setminus \{0\})^l$ to $\mathbb{R} \setminus \{0\}$ and

$$h(t_1, \ldots, t_k) = \left(\prod_{i=1}^{k} t_i\right)^{-1}, \tag{2.3}$$

map the set $(\mathbb{R} \setminus \{0\})^k$ to $\mathbb{R} \setminus \{0\}$, from (2.2) with $n = 0$, we see that $x_0^{(j)} \neq 0$ for every $j = \overline{1, k}$.

Assume we have proved that $x_i^{(j)} \neq 0$ for $-\max\{l, s\} \le i \le n_0$ and every $j = \overline{1, k}$. Then from this, (2.2) and since functions g_j, $j = \overline{1, k}$, map the set $(\mathbb{R} \setminus \{0\})^l$ to $\mathbb{R} \setminus \{0\}$, and h map the set $(\mathbb{R} \setminus \{0\})^k$ to $\mathbb{R} \setminus \{0\}$, we get $x_{n_0+1}^{(j)} \neq 0$ for every $j = \overline{1, k}$. So, by induction it follows that $x_n^{(j)} \neq 0$ for every $n \in \mathbb{N}_0$ and $j = \overline{1, k}$. Hence, every solution $(x_n^{(1)}, \ldots, x_n^{(k)})$ of system (2.2) whose initial values are arbitrary nonzero real numbers is well-defined.

Multiplying all the equations in (2.2), then using condition (2.1) and the above proven fact that $\prod_{i=1}^{k} x_n^{(i)} \neq 0$, for every $n \ge -\max\{l, s\}$, we get

$$\prod_{j=1}^{k} x_n^{(j)} = \frac{\prod_{j=1}^{k} g_j\left(\prod_{j=1}^{k} x_{n-1}^{(i)}, \ldots, \prod_{j=1}^{k} x_{n-l}^{(i)}\right)}{\prod_{i=1}^{k} x_{n-s}^{(i)}} = \frac{1}{\prod_{i=1}^{k} x_{n-s}^{(i)}}, \tag{2.4}$$

from which it follows that

$$\prod_{j=1}^{k} x_n^{(j)} = \prod_{j=1}^{k} x_{n-2s}^{(j)}, \tag{2.5}$$

that is, the sequence $\prod_{j=1}^{k} x_n^{(j)}$ is eventually periodic with period $2s$. In particular, the sequences $\prod_{j=1}^{k} x_{n-q}^{(j)}$ are eventually periodic with period $2s$, for every $q \in \mathbb{N}$.

This implies that the sequences

$$\begin{gathered} g_j\left(\prod_{i=1}^{k} x_{n-1}^{(i)}, \ldots, \prod_{i=1}^{k} x_{n-l}^{(i)}\right), \quad j = \overline{1, k}, \\ \left(\prod_{i=1}^{k} x_{n-s}^{(i)}\right)^{-1} \end{gathered} \tag{2.6}$$

are eventually periodic with period $2s$, which along with (2.2) implies that the sequences $(x_n^{(j)})_{n \ge -\max\{l,s\}}$ are eventually periodic with period $2s$, for $j = \overline{1, k}$, from which the theorem follows. □

Now we present the main results in paper [23] as two corollaries.

Corollary 2.2. *Assume that* (x_n, y_n), $n \ge -\max\{p, q\}$, *is an arbitrary solution of system* (1.3) *whose initial values are arbitrary nonzero real numbers. Then the solution is eventually periodic with period* $2q + 2$.

Proof. Note that for system (1.3), $l = p+1$, $s = q+1$, $g_1(t_1,\dots,t_{p+1}) = 1/t_{p+1}$ and $g_2(t_1,\dots,t_{p+1}) = t_{p+1}$, so that

$$g_1(t_1,\dots,t_{p+1})g_2(t_1,\dots,t_{p+1}) = 1 \quad \text{for } (t_1,\dots,t_{p+1}) \in (\mathbb{R}\setminus\{0\})^{p+1}. \tag{2.7}$$

Hence, all the conditions of Theorem 2.1 are fulfilled from which the corollary follows. □

Corollary 2.3. *Assume that* (x_n, y_n, z_n), $n \ge -\max\{p,q,r\}$, *is an arbitrary solution of system* (1.4) *whose initial values are arbitrary nonzero real numbers. Then the solution is eventually periodic with period* $2r+2$.

Proof . For system (1.4), we have that $l = \max\{p+1, q+1\}$, $s = r+1$,

$$g_1(t_1,\dots,t_l) = \frac{1}{t_{p+1}}, \qquad g_2(t_1,\dots,t_l) = \frac{t_{p+1}}{t_{q+1}}, \qquad g_3(t_1,\dots,t_l) = t_{q+1}, \tag{2.8}$$

so that

$$g_1(t_1,\dots,t_l)g_2(t_1,\dots,t_l)g_3(t_1,\dots,t_l) = 1 \quad \text{for } (t_1,\dots,t_l) \in (\mathbb{R}\setminus\{0\})^{l}. \tag{2.9}$$

Hence, all the conditions of Theorem 2.1 are fulfilled from which the corollary follows. □

The next corollary, which corresponds to case $k = 4$ in Theorem 2.1, is obtained similarly, so we omit the proof.

Corollary 2.4. *Assume that* (x_n, y_n, z_n, w_n), $n \ge -\max\{p,q,r,s\}$, *is an arbitrary solution of the following system*

$$\begin{aligned} x_{n+1} &= \frac{1}{x_{n-p}y_{n-p}z_{n-p}w_{n-p}}, \qquad & y_{n+1} &= \frac{x_{n-p}y_{n-p}z_{n-p}w_{n-p}}{x_{n-q}y_{n-q}z_{n-q}w_{n-q}}, \\ z_{n+1} &= \frac{x_{n-q}y_{n-q}z_{n-q}w_{n-q}}{x_{n-r}y_{n-r}z_{n-r}w_{n-r}}, \qquad & w_{n+1} &= \frac{x_{n-r}y_{n-r}z_{n-r}w_{n-r}}{x_{n-s}y_{n-s}z_{n-s}w_{n-s}}, \end{aligned} \tag{2.10}$$

$n \in \mathbb{N}_0$, *whose initial values are arbitrary nonzero real numbers. Then the solution is eventually periodic with period* $2s+2$.

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