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Research Article

Homoclinic Orbits for Second-Order Hamiltonian Systems with Some Twist Condition

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We study the existence and multiplicity of homoclinic orbits for second-order Hamiltonian systems $\ddot{q} - L(t)q + \nabla_q W(t,q) = 0$, where $L(t)$ is unnecessarily positive definite for all $t \in \mathbb{R}$, and $\nabla_q W(t,q)$ is of at most linear growth and satisfies some twist condition between the origin and the infinity.

1. Introduction

Consider the following second-order non-autonomous Hamiltonian system

$$\ddot{q} - L(t)q + \nabla_q W(t,q) = 0, \tag{1.1}$$

where $L(t) \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function, $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla_q W(t,q)$ denotes the gradient of $W(t,q)$ with respect to q. As usual, we say that a nonzero solution $q(t)$ of (1.1) is homoclinic (to 0) if $q(t) \to 0$ and $\dot{q}(t) \to 0$ as $|t| \to \infty$.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. While it is well known that homoclinic solutions play an important role in analyzing the chaos of Hamiltonian systems, if a system has the transversely intersected homoclinic solutions, then it must be chaotic. If it has the smoothly connected homoclinic solutions, then it cannot stand the perturbation, its perturbed system probably produces chaotic phenomena. For the chaos theory, the readers can refer to [1–3] and the references therein for more details. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic solutions of Hamiltonian systems emanating from 0.

In the past years, the existence and multiplicity of homoclinic orbits for (1.1) have been extensively investigated in many papers via the variational methods. Most of them (see [4–13]) treated the case where $L(t)$ and $W(t,u)$ are either independent of t or periodic in t. In this kind of problem, the function $L(t)$ plays an important role. If $L(t)$ is neither a constant nor periodic, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [13], many results (see, e.g., [9, 14–22]) were obtained for the case where $L(t)$ is neither a constant nor periodic. Among them, except for [13, 16, 18, 20–22], all known results were obtained under the following assumption that $L(t)$ is positive definite for all $t \in \mathbb{R}$, that is,

$$(L(t)u,u) > 0, \quad \forall t \in \mathbb{R},\ u \in \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

In the present paper, we will study the existence and multiplicity of homoclinic orbits for (1.1) under the condition that $L(t)$ is coercive but unnecessarily positive definite for all $t \in \mathbb{R}$. More precisely, L satisfies the following conditions:

(L_1) There exists an $\alpha < 2$ such that

$$l(t)|t|^{\alpha-2} \longrightarrow \infty \quad \text{as } |t| \longrightarrow \infty, \tag{1.3}$$

where $l(t)$ is the smallest eigenvalue of $L(t)$, that is,

$$l(t) \equiv \inf_{|\xi|=1}(L(t)\xi,\xi). \tag{1.4}$$

Before presenting the conditions on the nonlinearity of (1.1), we note that in the recent paper [23], under a twisting of the nonlinearity between the origin and the infinity, the authors studied the existence and multiplicity of nontrivial solutions for nonlinear elliptic equations and also for nonlinear elliptic systems. Subsequently, this kind of twist conditions and the idea of the methods in [23] were also applied to first-order Hamiltonian systems in [24].

Inspired by these works, we will present some similar twist condition on the nonlinearity of (1.1) to those in [23, 24], which will be specified in what follows.

Here, we introduce some notations. Denote by $\mathcal{B}$ the set of all uniformly bounded symmetric $N \times N$ matric functions. That is to say, $B \in \mathcal{B}$ if and only if $B^T(t) = B(t)$ for all $t \in \mathbb{R}$ and $B(t)$ is uniformly bounded in t as the operator on $\mathbb{R}^N$. For any $B \in \mathcal{B}$, in the next section, we will define an index pair $(i(B), \nu(B))$, satisfying $0 \le i(B), \nu(B) < \infty$.

With this index, we can present the conditions on $W(t,q)$ and the nonlinearity $\nabla_q W(t,q)$ as follows. For notational simplicity, we set $B_0(t) = \nabla_q^2 W(t,0)$, and in what follows the letter c will be repeatedly used to denote various positive constants whose exact value is irrelevant. Besides, for two $N \times N$ symmetric matrices M_1 and M_2, $M_1 \le M_2$ means that $M_2 - M_1$ is semipositive definite.

(W_1) $W \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there exists a constant $c > 0$ such that

$$\left|\nabla_q^2 W(t,q)\right| \le c, \quad \forall (t,q) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.5}$$

(W_0) $\nabla_q W(t,0) \equiv 0$ and $B_0 \in \mathcal{B}$,

(W_∞) there exists some $R_0 > 0$ and continuous symmetric matrix functions $B_1, B_2 \in \mathcal{B}$ with $i(B_1) = i(B_2)$ and $\nu(B_2) = 0$ such that

$$B_1(t) \le \nabla_q^2 W(t,z) \le B_2(t), \quad \forall t \in \mathbb{R},\ |z| > R_0. \tag{1.6}$$

Our first result reads as follows.

Theorem 1.1. *Assume* (L_1), (W_1), (W_0), *and* (W_∞) *hold. If*

$$i(B_1) \notin [i(B_0), i(B_0) + \nu(B_0)], \tag{1.7}$$

then (1.1) *has at least one nontrivial homoclinic orbit. Moreover, if* $\nu(B_0) = 0$ *and* $|i(B_1) - i(B_0)| \ge N$, *the problem possesses at least two nontrivial homoclinic orbits.*

Condition (W_∞) is a two-side pinching condition near the infinity, learning from the idea of [23, 24], we can relax (W_∞) to condition $(W_\infty^\pm)$ as follows.

$(W_\infty^\pm)$ There exist some $R_0 > 0$ and a continuous symmetric matrix function $B_\infty \in \mathcal{B}$ with $\nu(B_\infty) = 0$ such that

$$\pm \nabla_q^2 W(t,q) \ge \pm B_\infty(t), \quad \forall t \in \mathbb{R},\ |q| > R_0. \tag{1.8}$$

The uniform boundary of $\nabla_q^2 W(t,q)$ displayed in condition (W_1) can also be relaxed as

(W_1^*) $W \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and there exists a constant $c > 0$ such that

$$|\nabla_q W(t,q)| \le c|q|, \quad \forall (t,q) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.9}$$

(W_1^{**}) For any $M > 0$, $\nabla_q^2 W(t,q)$ is bounded on $\mathbb{R} \times [-M, M]^n$.

On the other hand, we need some sharply twisted conditions than the above theorem, and we have the following theorems.

Theorem 1.2. *Assume* (L_1), (W_1^*), (W_1^{**}), (W_0), (W_∞^+) *(or* (W_∞^-)*), and* $\nu(B_0) = 0$ *hold. If* $i(B_\infty) \ge i(B_0) + 2$ *(or* $i(B_\infty) \le i(B_0) - 2$*), then* (1.1) *has at least one nontrivial homoclinic orbit.*

Theorem 1.3. *Suppose that* (L_1), (W_1^*), (W_1^{**}), (W_0), (W_∞^+) *(or* (W_∞^-)*), and* $\nu(B_0) = 0$ *are satisfied. If, in addition,* W *is even in* q *and* $i(B_\infty) \ge i(B_0) + 2$ *(or* $i(B_\infty) \le i(B_0) - 2$*), then* (1.1) *has at least* $|i(B_\infty) - i(B_0)| - 1$ *pairs of nontrivial homoclinic orbits.*

Remark 1.4. Note that the assumption $\nu(B_\infty) = 0$ in $(W_\infty^\pm)$ is not essential for our main results. For the case of (W_∞^+) with $\nu(B_\infty) \ne 0$, let $\widetilde{B}_\infty = B_\infty - \varepsilon I_{n\times n}$ with $\varepsilon > 0$ small enough, where

$I_{n\times n}$ is the identity map on $\mathbb{R}^N$, then $i(\widetilde{B}_\infty) = i(B_\infty)$ and $\nu(\widetilde{B}_\infty) = 0$, and hence (W_∞^+) holds for $\widetilde{B}_\infty$. Therefore, Theorems 1.2 and 1.3 still hold in this case. While for the case of (W_∞^-) with $\nu(B_\infty) \neq 0$, if we replace $i(B_\infty)$ by $i(B_\infty) + \nu(B_\infty)$ in Theorems 1.2 and 1.3, then similar results hold. Indeed, let $\widetilde{B}_\infty = B_\infty + \varepsilon I_{n\times n}$ with $\varepsilon > 0$ small enough such that $i(\widetilde{B}_\infty) = i(B_\infty) + \nu(B_\infty)$ and $\nu(\widetilde{B}_\infty) = 0$, then this case is also reduced to the case of (W_∞^-) for $\widetilde{B}_\infty$ with $\nu(\widetilde{B}_\infty) = 0$.

Remark 1.5. Choose $\overline{W}(t,q) = W(t,q) - W(t,0)$ instead of W in (1.1), then conditions (W_1^*), (W_1^{**}), (W_0), and (W_∞^+) *(or* (W_∞^-)) still hold for $\overline{W}$, so we can always assume $W(t,0) \equiv 0$.

2. Preliminaries

Denote by $\widetilde{A}$ the self-adjoint extension of the operator $-d^2/dt^2 + L(t)$ with domain $D(\widetilde{A}) \subset L^2 \equiv L^2(\mathbb{R},\mathbb{R}^N)$. Let $\{E(\lambda) : -\infty < \lambda < \infty\}$ and $|\widetilde{A}|$ be the spectral resolution and the absolute value of $\widetilde{A}$, respectively, and let $|\widetilde{A}|^{1/2}$ be the square root of $|\widetilde{A}|$ with domain $D(|\widetilde{A}|)$. Set $U = I - E(0) - E(-0)$, where I is the identity map on L^2. Then, U commutes with $\widetilde{A}$, $|\widetilde{A}|$, and $|\widetilde{A}|^{1/2}$, and $\widetilde{A} = U|\widetilde{A}|$ is the polar decomposition of $\widetilde{A}$. Let $E = D(|\widetilde{A}|^{1/2})$, and define on E the inner product and norm by

$$(u,v)_0 = \left(\left|\widetilde{A}\right|^{1/2}u, \left|\widetilde{A}\right|^{1/2}v\right)_2 + (u,v)_2,$$
$$\|u\|_0 = (u,u)_0^{1/2}, \tag{2.1}$$

where $(\cdot,\cdot)_{L^2}$ denotes the usual inner product on $L^2(\mathbb{R},\mathbb{R}^N)$. Then, E is a Hilbert space. It is easy to see that E is continuously embedded in $W^1(\mathbb{R},\mathbb{R}^N)$. In fact, we further have the following lemmas.

Lemma 2.1 (see [16], Lemma 2.2). *Suppose that L satisfies (L_1). Then, E is compactly embedded in $L^p(\mathbb{R},\mathbb{R}^N)$ with the usual norm $\|\cdot\|_{L^p}$ for any $1 \leq p \in (2/(3-\alpha),\infty]$.*

From [16], under the assumption (L_1) on L and by Lemma 2.1, we know that $\widetilde{A}$ possesses a compact resolvent. Therefore, the spectrum $\sigma(\widetilde{A})$ consists of only eigenvalues numbered in $\eta_1 \leq \eta_2 \leq \cdots \to \infty$ (counted with multiplicity), and the corresponding system of eigenfunctions $\{e_n : n \in \mathbb{N}\}$ $(\widetilde{A}e_n = \eta_n e_n)$ forms an orthogonal basis in L^2.

Let

$$n^- = \#\{i \mid \lambda_i < 0\}, \qquad n^0 = \#\{i \mid \lambda_i = 0\}, \qquad \overline{n} = n^- + n^0,$$
$$E^- = \operatorname{span}\{e_1,\ldots,e_{n^-}\}, \qquad E^0 = \operatorname{span}\{e_{n^-+1},\ldots,e_{\overline{n}}\}, \qquad E^+ = \overline{\operatorname{span}\{e_{\overline{n}+1,\ldots}\}}, \tag{2.2}$$

where the closure is taken with respect to the norm $\|\cdot\|_0$. Then, one has the orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ with respect to the inner product $(\cdot,\cdot)_0$. Now, we introduce on E the following inner product and norm:

$$(u,v)_E = \left(|\widetilde{A}|^{1/2}u, \left|\widetilde{A}\right|^{1/2}v\right)_{L^2} + \left(u^0,v^0\right)_{L^2},$$
$$\|u\|_E = (u,u)_E^{1/2}, \tag{2.3}$$

where $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ correspondingly. Clearly, norms $\|\cdot\|_E$ and $\|\cdot\|_0$ are equivalent (cf. [16]). From now on, we take E with inner product $(\cdot,\cdot)_E$ and norm $\|\cdot\|_E$ as our working space.

Remark 2.2. Note that the decomposition $E = E^- \oplus E^0 \oplus E^+$ with respect to the inner product $(\cdot,\cdot)_0$ is also orthogonal with respect to both inner products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_{L^2}$. In what follows, we always denote by $E = E^- \oplus E^0 \oplus E^+$ the orthogonal decomposition with respect to the inner products $(\cdot,\cdot)_E$ unless specified otherwise.

In view of Lemma 2.1 and the equivalence of the norms $\|\cdot\|_E$ and $\|\cdot\|_0$, there exists a constant $c_\infty > 0$ such that

$$\|q\|_{L^\infty} \le c_\infty \|q\|_E, \quad \forall q \in E. \tag{2.4}$$

Define the quadratic form a on E by

$$a(u,v) = \int_{\mathbb{R}} ((\dot{u},\dot{v}) + (L(t)u,v))dt, \quad \forall u,v \in E. \tag{2.5}$$

Then by definition, we have

$$a(u,u) = ((P^+ - P^-)u,u)_E = \|u^+\|_E^2 - \|u^-\|_E^2 \tag{2.6}$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$, where $P^\pm : E \to E^\pm$ are the respective orthogonal projections. Define the self-adjoint operators $A : E \to E$ and $K : L^2 \to E$ by

$$\begin{gathered} Au = u^+ - u^-, \quad \forall u \in E, \\ (Ku,v)_E = (u,v)_{L^2}, \quad \forall u \in L^2,\ v \in E, \end{gathered} \tag{2.7}$$

it is easy to check that K is a compact operator and $a(u,v) = (Au,v)_E$ for all u, v in E.

For any $B \in \mathcal{B}$, it is easy to see B determines a bounded self-adjoint operator on L^2, by $z(t) \mapsto B(t)z(t)$, for any $z \in L^2$, we still denote this operator by B, then $KB : E \subset L^2 \to E$ is a self-adjoint compact operator on E and satisfies

$$(KBu,v)_E = (Bu,v)_{L^2}, \quad \forall u,v \in E. \tag{2.8}$$

We decompose the space E as $E^-(B) \oplus E^0(B) \oplus E^+(B)$, so that $A - KB$ is negatively definite on $E^-(B)$, null on $E^0(B)$, and positively definite on $E^+(B)$. From the definition of A and the compactness of KB, we know that $E^-(B)$ and $E^0(B)$ are both finite dimensional. Denote

$$\begin{gathered} i(B) = \dim\big(E^-(B)\big), \\ \nu(B) = \dim\big(E^0(B)\big). \end{gathered} \tag{2.9}$$

Define the functionals Ψ and Φ on E by

$$\begin{aligned}\Psi(u) &= \int_{\mathbb{R}} W(t,u)dt,\\ \Phi(u) &= \frac{1}{2}a(u,u) - \Psi(u)\\ &= \frac{1}{2}\int_{\mathbb{R}} ((\dot{u},\dot{u}) + (L(t)u,u))dt - \Psi(u).\end{aligned} \tag{2.10}$$

Combining this with Lemma 2.1, we know that Ψ and Φ are both well defined. Furthermore, we have the following:

Proposition 2.3. *Let* (L_1) *and* (W_1) *be satisfied. Then,* $\Psi \in C^2(E,\mathbb{R})$*, and hence* $\Phi \in C^2(E,\mathbb{R})$*. Moreover, from the similarly argument in [13], one has all critical points of* Φ *on* E *are homoclinic orbits of* (1.1).

Lemma 2.4. *Let* (L_1)*,* (W_1)*, and* (W_∞) *be satisfied. Then, one has*

(1) Φ *satisfies (PS) condition,*

(2) $H_j(E,\Phi;\mathbb{R}) \cong \delta_{j,r}\mathbb{R},\ j = 0,1,\ldots$ *for* $-a \in \mathbb{R}$ *large enough, where* $r = i(B_1)$.

Proof. Assume $\{u_n\} \subset E$ with $\Phi'(u_n) \to 0$ as $n \to \infty$. That is,

$$\|Au_n - K\nabla_q W(t,u_n)\|_E \longrightarrow 0. \tag{2.11}$$

First, we prove $\{u_n\}$ is bounded in E. For each $\varepsilon \in (0,1)$, define $C_n \in \mathcal{B}$ by

$$C_n(t) = \begin{cases} \displaystyle\int_0^1 \nabla_q^2 W(t,su_n)ds, & |u_n(t)| \geq \dfrac{R_0}{\varepsilon},\\ B_1(t), & |u_n(t)| < \dfrac{R_0}{\varepsilon}.\end{cases} \tag{2.12}$$

It is easy to verify that $\{C_n\}$ satisfies

$$B_1(t) - \varepsilon(B_1(t) + c\cdot I) \leq C_n(t) \leq B_2(t) + \varepsilon(c\cdot I - B_2(t)), \quad \forall t \in \mathbb{R}, \tag{2.13}$$

where c is the constant in condition (W_1) and I is the identity map on $\mathbb{R}^N$. Since $B_1 \leq B_2$, $i(B_1) = i(B_2)$, and $\nu(B_1) = \nu(B_2) = 0$, we can choose ε small enough, such that for each $n \in \mathbb{N}^+$, satisfying $i(C_n) = i(B_1)$ and $\nu(C_n) = 0$. Thus $A - KC_n$ is reversible on E and there is a constant $\delta > 0$, such that

$$\|(A - KC_n)u\|_E \geq \delta\|u\|_E, \quad \forall u \in E,\ n \in \mathbb{N}^+. \tag{2.14}$$

On the other hand, for $b \in (0,1)$, there is a constant $c > 0$ depending on b, such that for each $n \in \mathbb{N}^+$,

$$\left|\nabla_q W(t, u_n(t)) - C_n u_n(t)\right| \le c|u_n(t)|^b, \quad \forall t \in \mathbb{R}. \tag{2.15}$$

Choose $b > (\alpha - 1)/(3 - \alpha)$ in (2.15), we have

$$\begin{aligned} \left\|(Au_n - K\nabla_q W(t, u_n)) - (A - KC_n)u_n\right\|_E^2 &= \|K(\nabla_u W(t, u_n) - C_n u_n)\|_E^2 \\ &\le \|\nabla_u W(t, u_n) - C_n u_n\|_{L^2}^2 \\ &\le c\int_{\mathbb{R}} \frac{|\nabla_u W(t, u_n) - C_n u_n|}{|u_n|^b}|u_n|^{1+b}dt \\ &\le c\|u_n\|_{L^{1+b}}^{1+b}. \end{aligned} \tag{2.16}$$

As we claimed in the part of introduction, in (2.15) and (2.16), the letter c denotes different positive constants whose exact value is irrelevant. Thus, from (2.11), (2.14), (2.16), and Lemma 2.1, we have $\{u_n\}$ in bounded in E. Thus, there exists a subsequence $\{u_{n_k}\}$ such that $\{K\nabla_q W(t, u_{n_k})\}$ is convergent in E. By the definition of Φ, we have

$$u_{n_k}^+ - u_{n_k}^- - K\nabla_q W(t, u_{n_k}) = \Phi'(u_{n_k}) \longrightarrow 0 \quad \text{in } E, \tag{2.17}$$

which yields the convergence of $\{u_{n_k}^+\}$ and $\{u_{n_k}^-\}$ in E. Additionally, passing to a subsequence, if necessary, $\{u_{n_k}^0\}$ is convergent in E since $\dim(E^0) < \infty$. Thus, $\{u_{n_k}\}$ is convergent and the (PS) condition is verified. By Lemma 5.1 in Chapter II of [25], we have

$$H_q(X, (a)_\alpha; \mathbb{R}) \cong \delta_{q,r}\mathbb{R}, \qquad q = 0, 1, \ldots, \tag{2.18}$$

for $-\alpha \in \mathbb{R}$ large enough. □

In order to prove Theorems 1.2 and 1.3, we need the following lemma which is similar to Lemma 3.4 in [23] and Lemma 3.3 in [24].

Lemma 2.5. *Assume* (W_1^*), (W_1^{**}), *and* (W_∞^+) *hold, then there exists a sequence of functions:*

$$W_m \in C^2\left(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\right), \quad m \in \mathbb{N}, \tag{2.19}$$

satisfying the following properties:

(1) *there exists an increasing sequence of real numbers* $R_m \to \infty$ $(m \to \infty)$ *such that*

$$W_m(t, q) \equiv W(t, q), \quad \forall t \in \mathbb{R}, \ |q| \le R_m; \tag{2.20}$$

(2) *for each $m \in \mathbb{N}$, there is a $c_m > 0$ such that*

$$\left|\nabla_q^2 W_m(t,q)\right| \le c_m, \quad \forall t \in \mathbb{R},\ q \in \mathbb{R}^N, \tag{2.21}$$

$$\nabla_q^2 W_m(t,q) \ge B_{-\varepsilon} := B_\infty - \varepsilon I_{n\times n}, \quad \forall t \in \mathbb{R},\ |q| \ge R_0 \tag{2.22}$$

for $\varepsilon > 0$ small enough;

(3) *there exist some $c > 0$ such that*

$$|\nabla_q W_m(t,q)| \le c|q|, \quad \forall (t,q) \in \mathbb{R}\times\mathbb{R}^N,\ m \in \mathbb{N}; \tag{2.23}$$

(4) *for each $m \in \mathbb{N}$, there exists some $C_m > 0$ and a constant γ with $\gamma I_{n\times n} > B_\infty$, $\nu(\gamma I_{n\times n}) = 0$ such that*

$$|\nabla_q W_m(t,q) - \gamma q| < C_m, \quad \forall (t,q) \in \mathbb{R}\times\mathbb{R}^N, \tag{2.24}$$

where $I_{n\times n}$ is the identity map on $\mathbb{R}^N$.

Proof. Define $\eta : [0,\infty) \to \mathbb{R}$ by

$$\eta(t) = \begin{cases} 0, & 0 \le t < 1, \\ \dfrac{2}{9}(t-1)^3 - \dfrac{1}{9}(t-1)^4, & 1 \le t < 2, \\ 1 - \dfrac{128}{9(12+t^2)}, & 2 \le t < \infty. \end{cases} \tag{2.25}$$

It is easy to see that $\eta \in C^2([0,\infty),\mathbb{R})$. Choose a sequence $\{R_m\}$ of positive numbers such that $R_0 < R_1 < R_2 < \cdots < R_m < \cdots \to \infty$ as $m \to \infty$. For each $m \in \mathbb{N}$, let $\eta_m(t) = \eta(t/R_m)$ and

$$\widetilde{W}_m(t,q) = (1 - \eta_m(|q|))W(t,q) + \frac{\gamma}{2}\eta_m(|q|)|q|^2, \quad m \in \mathbb{N}. \tag{2.26}$$

As in [23, 24], we can easily check that $\widetilde{W}_m$ satisfies (2.20), (2.22) (with $\varepsilon = 0$), (2.23) and (2.24) for each $m \in \mathbb{N}$.

Define

$$W_m(t,q) = \rho\left(\frac{|q| - T_m}{T_m}\right)\widetilde{W}_m(t,q) + \frac{\gamma}{2}\left[1 - \rho\left(\frac{|q| - T_m}{T_m}\right)\right]|q|^2, \tag{2.27}$$

where $\rho \in C^2(\mathbb{R}, [0,1])$ is a cut-off function with $\rho(s) \equiv 1$ for $s \le 0$ and $\rho(s) \equiv 0$ for $s \ge 1$. If we choose T_m large enough, then W_m will satisfy (2.20)–(2.24). □

Remark 2.6. Similar to Remark 1.4, we can choose $\varepsilon > 0$ small enough in (2.22) such that $E^-(B_{-\varepsilon}) = E^-(B_\infty)$ and $E^0(B_{-\varepsilon}) = E^0(B_\infty) = \{0\}$, that is, $i(B_{-\varepsilon}) = i(B_\infty)$ and $\nu(B_{-\varepsilon}) = \nu(B_\infty) = 0$.

Remark 2.7. For the case of (W_∞^-), the sequence $\{W_m\}$ constructed in (2.26) will also satisfy (2.20), (2.21), (2.23), (2.24) (with $\gamma I_{n\times n} < B_\infty$ and $\nu(\gamma I_{n\times n}) = 0$) and

$$\nabla_q^2 W_m(t,q) \le B_\varepsilon := B_\infty + \varepsilon I_{n\times n}, \quad \forall t \in \mathbb{R},\ |q| > R_0, \tag{2.28}$$

with $\varepsilon > 0$ small enough, therefore, $i(B_\varepsilon) = i(B_\infty)$ and $\nu(B_\varepsilon) = \nu(B_\infty) = 0$.

Define

$$\begin{aligned} \Psi_m(u) &= \int_{\mathbb{R}} W_m(t,u)dt, \quad \forall u \in E, \\ \Phi_m(u) &= \frac{1}{2}a(u,u) - \Psi_m(u), \quad \forall u \in E. \end{aligned} \tag{2.29}$$

Form Lemma 2.1 and (2.21) in Lemma 2.5, we have $\Phi_m \in C^2(E,\mathbb{R})$. Similarly, we can define $\Psi_\gamma(u) = 1/2(\gamma K u, u)_E$ and $\Phi_\gamma(u) = 1/2a(u,u) - \Psi_\gamma(u)$ for all u in E, and it follows that $\Phi_\gamma \in C^2(E,\mathbb{R})$.

Lemma 2.8. *For each $m \in \mathbb{N}$, Φ_m satisfies the (PS) condition and the critical-point set $\mathcal{K}_m = \{z \in E \mid \Phi_m'(z) = 0\}$ is a compact set.*

Proof. For any $m \in \mathbb{N}$, assume $\{u_n\} \subset E$ with $\Phi_m'(u_n) \to 0$ as $n \to \infty$:

$$\|\Phi_m'(u_n) - \Phi_\gamma''(0)u_n\|_E = \|K(\nabla_q W_m(t,u_n) - \gamma u_n)\|_E \le \|\nabla_q W_m(t,u_n) - \gamma u_n\|_{L^2}. \tag{2.30}$$

By (2.23) and (2.27), there hold

$$\begin{aligned} &\nabla_q W_m(t,q) - \gamma q \equiv 0 \quad \text{for } t \in \mathbb{R},\ |q| > 2T_m, \\ &|\nabla_q W_m(t,q) - \gamma q| < c|q|, \quad \forall (t,q) \in \mathbb{R}\times\mathbb{R}^N, \end{aligned} \tag{2.31}$$

for some $c > 0$. Choose $b \in (0,1)$ with $b > (\alpha-1)/(3-\alpha)$, then there exists a constant $c > 0$ such that

$$\begin{aligned} \|\nabla_q W_m(t,u_n) - \gamma u_n\|_{L^2}^2 &= \int (\nabla_q W_m(t,u_n) - \gamma u_n, \nabla_q W_m(t,u_n) - \gamma u_n)dt \\ &\le c\int \frac{|\nabla_q W_m(t,u_n) - \gamma u_n|}{|u_n|^b}|u_n|^{1+b}dt \\ &\le c\|u_n\|_{L^{1+b}}^{1+b}. \end{aligned} \tag{2.32}$$

Since $\alpha < 2$, $\nu(\gamma I_{n\times n}) = 0$ and $\Psi_\gamma''(0)$ has bounded inverse, from Lemma 2.1 and (2.30), (2.32), we have $\{u_n\}$ is bounded in E. From the similar argument in Lemma 2.4, $\{u_n\}$ has a convergent subsequence and the (PS) condition is verified. From the same reason, we can also prove that $\mathcal{K}_{\mathrm{m}}$ is compact set. □

From Lemma II.5.1 in [25], by standard argument, we have the following.

Lemma 2.9. *For any $m \in \mathbb{N}$, there is an $a_m \in \mathbb{R}$ with $-a_m$ large enough such that*

$$H_q\big(E, (\Phi_m)_{a_m}; \mathbb{R}\big) = \delta_{qr}\mathbb{R}, \tag{2.33}$$

where $r = \dim(E_\gamma^-) = i(\gamma I_{n\times n})$.

Note that θ is an isolated critical point of Φ_m since $\nu(B_0) = 0$. For each $m \in \mathbb{N}$, let $\mathcal{K}_m^* = \mathcal{K}_m \setminus \{\theta\}$, then $\mathcal{K}_m^*$ is also compact since $\mathcal{K}_m$ is compact. Then we have

Lemma 2.10. *For any $\epsilon, \mu > 0$ small enough, for each $m \in \mathbb{N}$, there exists a functional $\widetilde{\Phi}_m$ such that*

(1) $\|\Phi_m - \widetilde{\Phi}_m\|_{C^2} < \epsilon$,

(2) $\Phi(z) = \widetilde{\Phi}_m$, $z \notin N_{2\mu}(\mathcal{K}_m^*)$,

(3) $\Phi_m''(z) = \widetilde{\Phi}_m''(z)$, $z \in N_\mu(\mathcal{K}_m^*)$,

where $N_\mu(\mathcal{K}_m^*) = \{z \in X \mid \operatorname{dist}(z, \mathcal{K}_m^*) < \mu\}$. Moreover, $\widetilde{\Phi}_m$ satisfies the (PS) condition and has only a finite number of critical points, all nontrivial critical points of $\widetilde{\Phi}_m$ lie in $N_\mu(\mathcal{K}_m^*)$ and are nondegenerate.

Proof. We follow the idea of [26], since $\mathcal{K}_m^*$ is a compact subset of E, for every $\mu > 0$, there exists a C^∞ function $l : E \to [0,1]$, with all its derivatives bounded and

$$\begin{aligned} l(z) &= 1, \quad \forall z \in N_\mu(\mathcal{K}_m^*), \\ l(z) &= 0, \quad \forall z \in E \setminus N_{2\mu}(\mathcal{K}_m^*). \end{aligned} \tag{2.34}$$

Let $M = \sup_{z\in N_{2\mu}(\mathcal{K}_m^*)}\{\|z\|\}$, $C_l = \|l(z)\|_{C^2}$, $\delta = \inf_{z\in N_{2\mu}(\mathcal{K}_m^*)\setminus N_\mu(\mathcal{K}_m^*)}\{\|\Phi_m'(z)\|\} > 0$. We use the Sard-Smale Theorem to find $y \in E$, with $\|y\| < \min\{\epsilon/C_l(2+2M), \delta/2(C_l(1+2M))\}$, and $-y$ is a regular value for Φ_m'. For any $z_0 \in N_{2\mu}(\mathcal{K}_m^*)$, the functional is defined by

$$\widetilde{\Phi}_m(z) = \Phi_m(z) + l(z)\langle y, z - z_0\rangle. \tag{2.35}$$

By the fact $\|y\| < \epsilon/C_l(2+2M)$ and the definition of $l(z)$, it is easy to check that conclusions (1), (2), and (3) hold. Since $\|y\| < \delta/2C_l(1+2M)$ and $-y$ is a regular value for Φ_m, then all nontrivial critical points of $\widetilde{\Phi}_m$ are nondegenerate and lie in $N_\mu(\mathcal{K}_m^*)$.

In order to prove that $\widetilde{\Phi}_m$ satisfies the (PS) condition for each $m \in \mathbb{N}$, assume there is a sequence $\{z_n\} \subset E$ such that $\widetilde{\Phi}_m'(z_n) \to 0$, $(n \to \infty)$. From the definition of $\widetilde{\Phi}_m$, we have $\|\widetilde{\Phi}_m'(z)\| > \delta/2$ for all $z \in N_{2\mu}(\mathcal{K}_m^*) \setminus N_\mu(\mathcal{K}_m^*)$. So $z_n \in (E \setminus N_{2\mu}(\mathcal{K}_m^*)) \cup N_\mu(\mathcal{K}_m^*)$, when n is large enough. From the definition of $\widetilde{\Phi}_m$ and the proof in Lemma 2.8, we know that $\widetilde{\Phi}_m$ satisfies the (PS) condition and hence has a finite number of critical points. □

3. Proof of the Main Results

From Lemma 2.4, Theorem 1.1 is a direct consequence of Theorem 5.1 and Corollary 5.2 in chapter II of [25].

Proof of Theorem 1.2. We first consider the case of (W_∞^+). We divide the proof into two steps and follow the ideas of [24].

Step 1. We claim that Φ_m has a nontrivial critical point z_m with its Morse index satisfying

$$m^-(z_m) \le i(B_0) + 1. \tag{3.1}$$

Note that $z = 0$ is a critical point of Φ_m. The Morse index of 0 for Φ_m is $i(B_0)$, since $\gamma I_{n\times n} \ge B_\infty$,

$$i(\gamma I_{n\times n}) \ge i(B_\infty) > i(B_0) + 1. \tag{3.2}$$

If Φ_m has only finite critical points, consider $(i(B_0) + 1)$th Morse inequality:

$$\sum_{p=0}^{q}(-1)^{q-p} M_p(a_m, b_m, \Phi_m) \ge \sum_{p=0}^{q}(-1)^{q-p}\beta_p(a_m, b_m, \Phi_m), \tag{3.3}$$

where $q = i(B_0) + 1$, and b_m is large enough such that $\mathcal{K}_m \subset \Phi_m^{-1}[a_m, b_m]$.

By Lemmas 2.8 and 2.9, we have

$$\beta_p(a_m, b_m, \Phi_m) = \operatorname{rank}\big(H_p\big(E, (\Phi_m)_{a_m}\big)\big) = \delta_{pr}, \tag{3.4}$$

where $r = i(\gamma I_{n\times n})$. Since $i(\gamma I_{n\times n}) > i(B_0) + 1$, the right side of the inequality is equal to 0. If Φ_m has no nontrivial critical point with its Morse index less than $i(\mathrm{B}_0) + 1$, the left side of the inequality is equal to -1, it is a contradiction.

If Φ_m has infinitely many critical points. Assume for any $z \in \mathcal{K}_m^*$,

$$m^-(z) > i(B_0) + 1, \tag{3.5}$$

then from Lemma 2.10, we can choose μ small enough such that

(1) $0 \notin N_{2\mu}(\mathcal{K}_m^*)$, so 0 is also an isolated critical point of $\widetilde{\Phi}_m$ and has the same Morse index $i(B_0)$;

(2) for any $z \in N_\mu(\mathcal{K}_m^*)$, $m^-(z)$ is the dimension of the negative subspace of $\Phi''(z)$, satisfying $m^-(z) > i(B_0) + 1$. (Because Φ_m is C^2 continuous, we can assume this.)

From (3) in Lemma 2.10, if z is a nontrivial critical point of $\widetilde{\Phi}_m$, the Morse index $m^-_{\widetilde{\Phi}_m}(z)$ satisfies

$$m^-_{\widetilde{\Phi}_m}(z) > i(B_0) + 1. \tag{3.6}$$

Then choose $\tilde{a}_m$ satisfying $N_{2\mu}(\mathcal{K}_m^*) \cap (\Phi_m)_{\tilde{a}_m} = \emptyset$, that is, $(\Phi_m)_{\tilde{a}_m} = (\widetilde{\Phi}_m)_{\tilde{a}_m}$. So,

$$H_q\big(E, (\Phi_m)_{\tilde{a}_m}; \mathbb{R}\big) = H_q\Big(E, (\widetilde{\Phi}_m)_{\tilde{a}_m}; \mathbb{R}\Big) = \delta_{qr}\mathbb{R}. \tag{3.7}$$

Then, $\widetilde{\Phi}_m$ will not satisfy the $(i(B_0) + 1)$th Morse inequality. It is a contradiction.

Step 2. We show that $\{z_m\}$ is bounded in $L^\infty(\mathbb{R},\mathbb{R}^N)$, so from the definition of Φ_m, z_m is a nontrivial critical point of Φ for m large enough.

We prove it indirectly, assume $\|z_m\|_{L^\infty} \to \infty$, from (2.4), we have $\|z_m\|_E \to \infty$. Denote $v_m = z_m/\|z_m\|_E$. Passing to a subsequence, we assume that for some $v \in E$,

$$v_m \rightharpoonup v \quad \text{in } E, \qquad v_m \longrightarrow v \quad \text{in } L^2, \qquad v_m \longrightarrow v \quad \text{a. e. in } \mathbb{R}. \tag{3.8}$$

Since z_m satisfies $\ddot{z}_m(t) - L(t)z_m(t) + \nabla_q W_m(t, z_m) = 0$, we have

$$\|z_m\|_E \le c\|z_m\|_{L^2}, \quad \text{for some } c > 0, \tag{3.9}$$

which implies $\|v_m\|_{L^2} \ge 1/c$ for each $m \in \mathbb{N}$ and thus $\|v\|_{L^2} \ge 1/c$.

For each $m \in \mathbb{N}$, let $h_m(t) = \nabla_q W_m(t, z_m)/\|z_m\|_E$. By (W_0), (2.20), and (2.23), there holds $|h_m(t)| \le c|v_m(t)|$ for all $t \in \mathbb{R}$. Assume $h_m(t) \rightharpoonup h(t)$ in L^2, then $|h(t)| \le c|v(t)|$, almost everywhere in $\mathbb{R}$. By standard argument, we have

$$\ddot{v} - L(t)v + h(t) = 0, \quad \forall t \in \mathbb{R}. \tag{3.10}$$

By Lemma 3.1 in [23], we have $v(t) \neq 0$ almost everywhere in $\mathbb{R}$, which implies $z_m(t) \to \infty$ almost everywhere in $\mathbb{R}$. For any $u \in E^-(B_{-\varepsilon}) = E^-(B_\infty)$,

$$\left(\Phi_m''(z_m)u, u\right) = \int_{\mathbb{R}} \left(|\dot{u}|^2 - (L(t)u, u) - \nabla_q^2 W_m(t, z_m(t))u^2\right)dt. \tag{3.11}$$

By (W_1^{**}), (2.20), and (2.22), $\{\nabla_q^2 W_m\}$ is uniformly bounded from below. Using Fatou's Lemma, we have

$$\liminf_{m\to\infty} \int_{\mathbb{R}} \nabla_q^2 W_m(t, z_m)u^2 dt \ge \int_{\mathbb{R}} \liminf_{m\to\infty} \nabla_q^2 W_m(t, z_m)u^2 dt \ge \int_{\mathbb{R}} B_{-\varepsilon}(t)u^2 dt, \tag{3.12}$$

which implies

$$\limsup_{m\to\infty} \left(\Phi_m''(z_m)u, u\right) \le \int_{\mathbb{R}} \left(|\dot{u}|^2 - (L(t)u, u)\right)dt - \int_{\mathbb{R}} B_{-\varepsilon}(t)u^2 dt < 0, \tag{3.13}$$

which contradicts to $m^-(z_m) \le i(B_0) + 1 < i(B_{-\varepsilon})$. So $\{z_m\}$ is bounded in $L^\infty(\mathbb{R},\mathbb{R}^N)$, and Φ has a nontrivial critical point.

In order to prove the case of (W_∞^-), we choose γ with $\gamma I_{n\times n} \le B_\infty$ and $\nu(\gamma I_{n\times n}) = 0$ in Lemma 2.5, then, using the similar argument as in Step 1 for the case of (W_∞^+), we can also prove that there exists a nontrivial critical point z_m of Φ_m satisfying

$$m^-(z_m) + \nu(z_m) \ge i(B_\infty) + 1. \tag{3.14}$$

As in Step 2 for the case of (W_∞^+), we will show that $\{z_m\}$ is bounded in $L^\infty(\mathbb{R},\mathbb{R}^N)$. We prove it indirectly, assume $\|z_m\|_{L^\infty} \to \infty$, $z_m(t) \to \infty$ almost everywhere in $\mathbb{R}$. We claim there exists $m_0 \in \mathbb{N}$ such that if $m > m_0$, then for any $u \in E^+(B_\infty) \setminus \{0\}$,

$$(\Phi_m''(z_m)u,u)_E > 0, \tag{3.15}$$

which implies $m^-(z_m) + \nu(z_m) \le i(B_\infty)$. If (3.15) is false, then there exist $m_j \to \infty$ and $u_j \in E^+(B_\infty)$ with $\|u_j\|_E = 1$ such that $(\Phi_m''(z_{m_j})u_j,u_j)_E \le 0$, which can be rewritten as

$$(Au_j,u_j)_E - (K\nabla_q^2 W_{m_j}(t,z_{m_j})u_j,u_j)_E \le 0, \tag{3.16}$$

$$1 - 2\|u_j^-\|_E^2 \le (K\nabla_q^2 W_{m_j}(t,z_{m_j})u_j,u_j)_E. \tag{3.17}$$

Passing to a subsequence if necessary, we assume that $u_j \to u$ in L^2 and almost everywhere in $\mathbb{R}$ for some $u \in E^+(B_\infty) = E^+(B_\varepsilon)$ with $\|u\|_E \le 1$. Since

$$\liminf_{j\to\infty}\int_{\mathbb{R}} -\nabla_q^2 W_{m_j}\left(t,z_{m_j}\right)u_j^2dt \ge \int_{\mathbb{R}} \liminf_{j\to\infty} -\nabla_q^2 W_{m_j}\left(t,z_{m_j}\right)u_j^2dt \ge -(B_\varepsilon u,u)_{L^2}, \tag{3.18}$$

we see that

$$\limsup_{j\to\infty}(K\nabla_q^2 W_{m_j}(t,z_{m_j})u_j,u_j)_E \le (B_\varepsilon u,u)_{L^2}. \tag{3.19}$$

From (3.17) and (3.19), we have

$$1 - 2\left\|u^-\right\|_E^2 \le (B_\varepsilon u,u)_{L^2}, \tag{3.20}$$

If $u = 0$, it is a contradiction, or we have

$$1 - 2\left\|u^-\right\|_E^2 \le (B_\varepsilon u,u)_{L^2} < (Au,u)_E = \|u^+\|_E^2 - \left\|u^-\right\|_E^2 \le 1 - 2\left\|u^-\right\|_E^2, \tag{3.21}$$

which is also a contradiction. Thus, $\{z_m\}$ is bounded in $L^\infty(\mathbb{R},\mathbb{R}^N)$ and Φ has a nontrivial critical point. By Proposition 2.3, (1.1) has a nontrivial homoclinic orbit. The proof is completed. □

The proof of Theorem 1.3 is similar to the proof of Theorem 1.2. The difference is in Step 1, instead of Morse theory we make use of minimax arguments for multiplicity of critical points.

Let X be a Hilbert space and assume $\phi \in C^2(X,\mathbb{R})$ is an even functional, satisfying the (PS) condition and $\phi(0) = 0$. Denote $S_c = \{u \in X \mid \|u\| = c\}$.

Lemma 3.1 (see [27], Corollary 10.19). *Assume Y and Z are subspaces of X satisfying* $\dim Y = j > k = \operatorname{codim} Z$. *If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf \phi(S_r \cap Z) \geq \alpha, \qquad \sup \phi(S_R \cap Y) \leq 0, \tag{3.22}$$

then ϕ has $j - k$ pairs of nontrivial critical points $\{\pm x_1, \pm x_2, \ldots, \pm x_{j-k}\}$, so that $\mu(u_i) \leq k + i$, for $i = 1, 2, \ldots, j - k$.

First, we consider the case of (W_∞^+), since W is even, we have W_m is also even, and it satisfies Lemma 2.5. Let $Y = E^-(B_\infty)$, and $Z = E^+(B_0)$, and we have $\dim Y = i(B_\infty)$, $\operatorname{codim} Z = i(B_0)$, $\dim Y > \operatorname{codim} Z$. Then, it is easy to prove that Φ_m satisfies Lemma 3.1 for R and $1/r$ large enough. So, Φ_m has $l := i_A(B_\infty) - i_A(B_0)$ pairs of nontrivial critical points:

$$\{\pm x_1, \pm x_2, \ldots, \pm x_l\}, \tag{3.23}$$

and $l - 1$ pairs of them satisfying

$$m^-(x_i) \leq i(B_0) + i < i(B_\infty), \qquad i = 1, 2, \ldots, l - 1. \tag{3.24}$$

Then, we can complete the proof. In order to prove the case of (W_∞^-), we need the following.

Lemma 3.2 (see [25], Corollary II 4.1). *Assume Y and Z are subspaces of X satisfying* $\dim Y = j > k = \operatorname{codim} Z$. *If there exist $r > 0$, and $\alpha > 0$ such that*

$$\inf \phi(Z) > -\infty, \qquad \sup \phi(S_r \cap Y) \leq -\alpha, \tag{3.25}$$

then ϕ has $j - k$ pairs of nontrivial critical points $\pm u_1, \pm u_2, \ldots, \pm u_{j-k}$ so that $\mu(u_i) + \nu(u_i) \geq k + i - 1$ for $i = 1, 2, \ldots, j - k$.

The proof is similar to the case of (W_∞^+), we omit it here.

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